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On implicit discretization of prescribed-time differentiator

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***Abstract*—An implicit Euler discretization scheme of the prescribed-time converging observer from [1] for a second order system is given, which preserves all main properties of the continuous-time counterpart, and can be recursively applied on any interval of time. In addition, the estimation error stays bounded in the presence of bounded measurement noise. The efficiency of the suggested differentiator is illustrated through numeric experiments.**

I. Introduction

Estimation of the state vector for a dynamical system through noisy measurements of a part of its components in real time is a well-known issue, which has many popular and well established solutions [1], [2], [3], [4], [5], [6], [7]. Nevertheless, it is still an active area of research. Estimation of derivatives of a noisy signal can be presented as a state estimation problem [8], [9], [10], [11].

The main performance characteristics demanded from an observer or a differentiator include (but not limited to): the time of convergence of the estimate to the true value, asymptotic precision in the perturbation-free case (the steady-state error), measurement noise sensitivity and the implementation complexity. The existence of multiple solutions is related with the fact that it is difficult to design an observer that can outperform others in all valuable characteristics.

A recent popular observer solution is based on the concept of prescribed-time convergence [1] (see also a recent survey [12]), when an estimator with time-varying gains is designed in a way to guarantee that for any initial conditions, in the noise-free scenario, the estimation error vanishes exactly at the specified time instant, and this zero-settling property of the error is independent in the matched perturbations, which appear in the last equation (*i.e.*, exact estimation of derivatives in a prescribed time). However, the noise dependence of estimation error for this differentiator is not so advantageous, and special requirements are frequently imposed on the output perturbations in order to guarantee boundedness of the estimation error in the presence of measurement noise. Moreover, formally, the definition of solutions in such an observer after the settling time is problematic, and the authors do not consider the error behavior after the convergence (since the observer gains take infinite values at this instant of time). All these drawbacks are unusual and complicate the applicability of this observer.

Inspired by [13], where implicit discretization of a hyper-exponentially (asymptotically) converging differentiator was studied, in this note we are going to analyze an implicit discretization scheme for the prescribed-time observer of the second order. We show that such a discretization preserves the uniform convergence of the continuous-time counterpart, while being robust with respect to the measurement noise and avoiding the high-gain implementation problem. Moreover, the respective state estimates are made well-posed on the infinite horizon. It is illustrated by simulations that the obtained discrete-time differentiator has very advantageous performance qualities even in the case of a slow sampling.

The paper is organized as follows. The brief preliminaries are given in Section II. The problem statement is introduced in Section III. The properties of the considered differentiator in continuous time are recalled in Section IV. The properties of its implicit Euler discretization are investigated in Section V. The results of numeric simulation of the presented differentiator are shown in Section VI.

Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ is the set of integer numbers, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ is used for the Euclidean norm on $\mathbb{R}^n$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ define the norm $\|d\|_\infty = \text{ess}\ \sup_{t \in \mathbb{R}_+} \|d(t)\|$, and the set of d with the property $\|d\|_\infty < +\infty$ we further denote as $\mathcal{L}_\infty^m$ (the set of essentially bounded measurable functions).
- For a sequence $d_k \in \mathbb{R}^m$ with $k \in \mathbb{Z}_+$ define its norm by $|d|_\infty = \sup_{k \in \mathbb{Z}_+} \|d_k\|$ and the set of d with $|d|_\infty < +\infty$ we denote by l_∞^m.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and it is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity.
- A finite series of integers $1, 2, ..., n$ is denoted by $\overline{1,n}$, and $\{\overline{1,n}\} = \{1, 2, ..., n\}$.
- Denote the identity matrix of dimension $n \times n$ by I_n.
- $\text{diag}\{g\}$ represents a diagonal matrix of dimension $n \times n$ with a vector $g \in \mathbb{R}^n$ on the main diagonal.

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- The relation $P \prec 0$ ($P \succeq 0$) means that a symmetric matrix $P \in \mathbb{R}^{n\times n}$ is negative (positive semi-) definite, $\lambda_{\min}(P)$ denotes the minimal eigenvalue of such a matrix P.
- Denote $\mathbf{e} = \exp(1)$.

II. Preliminaries

The standard stability notions are used throughout and their definitions can be found in [14].

A. Uniform prescribed-time stability

Consider a non-autonomous differential equation:

$$dx(t)/dt = f(t, x(t), d(t)),\ t \geq t_0,\ t_0 \in \mathbb{R}_+, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $d(t) \in \mathbb{R}^m$ is the vector of external disturbances and $d \in \mathcal{L}_\infty^m$; $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function with respect to x, d and piecewise continuous with respect to t, $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}_+$. A solution of the system (1) for an initial condition $x_0 \in \mathbb{R}^n$ at time instant $t_0 \in \mathbb{R}_+$ and some $d \in \mathcal{L}_\infty^m$ is denoted as $X(t, t_0, x_0, d)$, and we assume that f ensures existence and uniqueness of solutions $X(t, t_0, x_0, d)$ at least locally in forward time.

The following definition is inspired by [1], and it is specified for a control-free system (1) while also using the same prescribed-time-based terminology that the settling-time instant has been assigned at the designer will.

Definition 1. For given $T > 0$ and a set $\mathbb{D} \subset \mathcal{L}_\infty^m$, the system (1) is called *uniformly prescribed-time stable* (uPTS) if there exist $\sigma_1, \sigma_2 \in \mathcal{K}$ such that for all $x_0 \in \mathbb{R}^n \setminus \{0\}$, $t_0 \in \mathbb{R}_+$ and $d \in \mathbb{D}$:

$$\|X(t, t_0, x_0, d)\| \leq \max\{\sigma_1(\|x_0\|), \sigma_2(\|d\|_\infty)\},$$
$$0 < \|X(t, t_0, x_0, 0)\|$$

for all $t \in [t_0, t_0 + T)$, and

$$\lim_{t \to t_0 + T} \|X(t, t_0, x_0, d)\| = 0.$$

It is important to highlight that the boundedness of solutions is claimed on a finite interval $[t_0, t_0 + T)$ only, and the solutions of (1) may be undefined for $t > T$. Hence, a uPTS system may do not demonstrate a Lyapunov stable behavior, and it is a variant of short-time stability (frequently also called finite-time one) as in [15]. In comparison with the concept of fixed-time stability [16], an important feature of a prescribed-time stable system is that the settling time is the same for all initial conditions out of the origin in the disturbance-free setting. In this definition the uniformity of convergence is understood in double meaning: as independence in the initial time t_0 and in the input $d \in \mathbb{D}$. Despite it is assumed that $\mathbb{D} \subset \mathcal{L}_\infty^m$, any other suitable class of inputs can be considered. The initial time t_0 can also be fixed being not arbitrary in $\mathbb{R}_+$ for many applications (the most common case is $t_0 = 0$).

A simple scalar example of a uPTS system (1) for $t_0 = 0$ is

$$\dot{x}(t) = -\frac{T}{T-t}x(t) + d(t),\ t \in [0, T),\ T > 0,$$

with $x(t), d(t) \in \mathbb{R}$ (see also [17]), whose solutions admit an estimate:

$$|x(t)| \leq \left(\frac{T-t}{T}\right)^T |x(0)| + \iota(t)\|d\|_\infty,\ t \in [0, T)$$

for any $x(0) \in \mathbb{R}$ and $d \in \mathcal{L}_\infty^1$, where

$$\iota(t) = \begin{cases} \frac{T\left(\frac{T-t}{T} - \left(\frac{T-t}{T}\right)^T\right)}{T-1} & \text{if } T \neq 1, \\ (1-t)\ln\frac{1}{1-t} & \text{if } T = 1. \end{cases}$$

Hence, $\sigma_1(s) = s$ and $\sigma_2(s) = \iota_{\max}s$ with

$$\iota_{\max} = \begin{cases} \frac{T^{(1-T)^{-1}} - T^{(1-T)^{-T}}}{1-T^{-1}} & \text{if } T \neq 1, \\ \mathbf{e}^{-1} & \text{if } T = 1. \end{cases}$$

B. Auxiliary property

The following block matrix inversion formula is used in the sequel:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BSCA^{-1} & -A^{-1}BS \\ -SCA^{-1} & S \end{bmatrix},$$
$$S = (D - CA^{-1}B)^{-1},$$

where A, B, C and D are matrices of any appropriate dimensions, A and S should be nonsingular.

III. Problem statement

Assume that a continuously differentiable signal $\phi(t) \in \mathbb{R}$ is measured with a noise $v(t) \in \mathbb{R}$:

$$y(t) = \phi(t) + v(t),$$

where $y(t) \in \mathbb{R}$, $v \in \mathcal{L}_\infty^1$, and $\phi : \mathbb{R}_+ \to \mathbb{R}$ has the second derivative $\ddot{\phi}(t) = \frac{d^2\phi(t)}{dt^2}$ with $\ddot{\phi} \in \mathcal{L}_\infty^1$ (without a known constant upper bound).

It is required to estimate the first derivative $\dot{\phi}(t) = \frac{d\phi(t)}{dt}$ of the signal ϕ with a prescribed time of convergence $T > 0$ and robustly with respect to the perturbation v.

The problem can be equivalently stated as the state estimation for the system

$$\dot{x}(t) = Ax(t) + b\ddot{\phi}(t),\ y(t) = Cx(t) + v(t), \qquad (2)$$

where $x(t) \in \mathbb{R}^2$ is the state, $x(0) = [\phi(0)\ \dot{\phi}(0)]^\top$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and $\ddot{\phi} \in \mathcal{L}_\infty^1$ corresponds to an unknown external input. And further in this work an observer for this system will be considered, which estimates $x(t)$ having uPTS estimation error for $\ddot{\phi} \in \mathbb{D} = \mathcal{L}_\infty^1$ while $\|v\|_\infty = 0$, and possessing a bounded estimation error for $v \in \mathcal{L}_\infty^1$.

IV. Differentiator in continuous time

Following [1], [18], take the state observer for (2) in the form:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + D(t)L(y(t) - C\hat{x}(t)), \quad (3)$$
$$D(t) = \operatorname{diag}\{[\varrho(t) \quad \varrho^2(t)]^\top\},$$

where $\hat{x}(t) \in \mathbb{R}^2$ is the estimate of the state $x(t)$, $L = [L_1\ L_2]^\top \in \mathbb{R}^2$ is the observer gain that will be selected later, $\varrho(t) = \frac{T}{T-t}$ is a strictly growing and unbounded function of time for $t \in [0, T)$.

Remark 1. Any integrable strictly growing and unbounded (for $t \to T$) function of time $\varrho(t)$ can be used in (3).

Remark 2. As it is common for the prescribed-time converging systems [12], the right-hand side of (3) is defined on a finite interval of time $[0, T)$, while also possessing a finite limit as $t \to T$ in spite of escaping the gain $\varrho(t)$ to infinity at $t = T$. Due to this, for $t \geq T$, frequently, another stabilization or estimation algorithm is applied, since $D(t)$ is not yet defined. Such a commutation is natural due to well-posedness of the system at $t = T$, and next for small regulation or observation errors other solutions can be used. However, such a switching does not take into account the advantageous uniformity of convergence realizable by prescribed-time controllers or estimators, which may be difficult to ensure through other methods. In this work, we will later consider another approach in the discrete-time setting by replacing $t \in \mathbb{R}_+$ with $\operatorname{mod}(t, T) \in [0, T)$. In such a case $\varrho(\operatorname{mod}(t, T))$ periodically ranges from 1 till $+\infty$ while t passes from iT to $(i+1)T$, correspondingly, for any $i \in \mathbb{Z}_+$.

Repeating the arguments of [1], [18], we get the following statement for the considered uPTS scenario:

Theorem 1. *Let there exist $P = P^\top \in \mathbb{R}^{2\times 2}$, $U \in \mathbb{R}^2$, $\gamma_1 > 0$ and $\gamma_2 > 0$ such that the linear matrix inequalities are verified:*

$$P \succ 0, \quad \begin{bmatrix} Q_{11} & Pb & -U \\ b^\top P & -\gamma_1 & 0 \\ -U^\top & 0 & -\gamma_2 \end{bmatrix} \preceq 0,$$
$$Q_{11} = \tilde{A}^\top P + P\tilde{A} - UC - C^\top U^\top + P,$$
$$\tilde{A} = A - \mathit{diag}\{[0 \quad T^{-1}]\}.$$

Then for $L = P^{-1}U$ and any $e(0) \in \mathbb{R}^2$, $\ddot{\phi}, v \in \mathcal{L}_\infty^1$ in (2), (3)*:*

$$\sqrt{\lambda_{\min}(P)} \begin{bmatrix} |e_1(t)| \\ |e_2(t)| \end{bmatrix} \leq \begin{bmatrix} 1 \\ \varrho(t) \end{bmatrix} (\varrho^{-T/2}(t)\sqrt{e(0)^\top P e(0)} + \sqrt{\gamma_1 \tilde{\iota}(t)}\|\ddot{\phi}\|_\infty + \sqrt{\gamma_2}\|v\|_\infty)$$

for all $t \in [0, T)$.

Remark 3. For $\|v\|_\infty = 0$ the result of the theorem implies uPTS of the estimation error $e(t)$ in (2), (3) with $\ddot{\phi} \in \mathcal{L}_\infty^1$ provided that $T > 2$ (note that any arbitrary time of convergence can be assigned by substituting $\varkappa t \to t$ in ϱ for any $\varkappa > 1$). This property means that (3) is a prescribed-time exact differentiator (as it is the super-twisting algorithm [3] in finite time), and the class of second derivatives $\ddot{\phi} \in \mathcal{L}_\infty^1$, for which the uniformity of the estimates is kept, can be enlarged by ones growing *not faster than*

$$\varrho(t)\sqrt{\gamma_1 \tilde{\iota}(t)} = \sqrt{T\gamma_1} \begin{cases} \sqrt{\frac{\varrho^{-2}(t) - \varrho^{2-T}(t)}{T-4}} & \textit{if } T \neq 4, \\ \varrho^{-1}(t)\sqrt{\ln \varrho(t)} & \textit{if } T = 4. \end{cases}$$

Clearly, the given linear matrix inequalities are always feasible (they are similar to ones used for a Luenberger observer design). Moreover, due to the form of Δ (or $\tilde{A}$), the second element of L can be selected zero, which is unusual for a differentiator design.

The upper bound on the estimation error of (2), (3) calculated in Theorem 1 implies that the gain of $|e_2(t)|$ in v is proportional to $\varrho(t)$, hence it becomes infinite at $t = T$. It is a serious drawback for a non-vanishing noise v often met in practical implementation, calling to sacrifice the precision by stopping the growth of observer/controller gains slightly before the prescribed time instant T. Let us investigate what happens after discretization of (3).

V. Differentiator in discrete time

Note that (3) is modeled by a linear time-varying system with an external known input $y(t)$. Since the time-varying gain $D(t)$ is strictly growing to infinity, the explicit Euler discretization cannot be used for all $t \in [0, T]$, however, the implicit one can be effectively applied [19]. Let $h > 0$ be constant discretization step, denote by $t_k = hk$ for $k \in \mathbb{Z}_+$ the discretization time instants, and slightly loosing generality assume that there exists $N \in \mathbb{Z}_+$ such that $T = Nh$, then application of the implicit Euler discretization method to (3) gives for $k \in \{\overline{0, N-1}\}$:

$$\hat{x}_{k+1} = Z(t_{k+1})(\hat{x}_k + hD(t_{k+1})Ly_{k+1}) \quad (4)$$
$$= Z(t_{k+1})\hat{x}_k + hF(t_{k+1})y_{k+1},$$
$$Z(t) = (I_2 - h(A - D(t)LC))^{-1} = \frac{K(t)}{O(t)},$$
$$F(t) = Z(t)D(t)L,$$
$$K(t) = \begin{bmatrix} 1 & h \\ -L_2 h\varrho^2(t) & 1 + L_1 h\varrho(t) \end{bmatrix},$$
$$O(t) = L_2 h^2 \varrho^2(t) + L_1 h\varrho(t) + 1,$$

where $\hat{x}_k$ is an approximation of $\hat{x}(t_k)$ and $y_k = y(t_k)$. As we can conclude from these expressions, the discrete state transition matrix $Z(t)$ is nonsingular for all $t \in [0, T]$ since

$$O^{-1}(t) = \frac{(T-t)^2}{k(t)},$$
$$k(t) = t^2 - T(L_1 h + 2)t + T^2(L_2 h^2 + L_1 h + 1)$$

and solution of the equation $k(\boldsymbol{t}) = 0$ gives

$$\boldsymbol{t} = T \begin{bmatrix} 1 + h\frac{L_1 - \sqrt{L_1^2 - 4L_2}}{2} \\ 1 + h\frac{L_1 + \sqrt{L_1^2 - 4L_2}}{2} \end{bmatrix}$$

that implies $\boldsymbol{t}_i > T$ for $i = 1, 2$ provided that $L_2 > 0$ (note that this is an additional restriction compared to Theorem 1). In addition, $Z(t)$ is elementwise bounded for all $t \in [0, T]$ with

$$Z(T) = \begin{bmatrix} 0 & 0 \\ -h^{-1} & 0 \end{bmatrix},\ k(T) = L_2 T^2 h^2,$$

but $O^{-1}(T) = 0$. Moreover, the input gain matrix, which evaluates the noise sensitivity of (4),

$$F(t) = \frac{\varrho(t)}{O(t)} \begin{bmatrix} L_2 h \varrho(t) + L_1 \\ L_2 \varrho(t) \end{bmatrix}$$

is also elementwise bounded with

$$h \lim_{t \to T} F(t) = \begin{bmatrix} 1 \\ h^{-1} \end{bmatrix}.$$

Consider a discrete-time model representing the solutions of (2), which can be approximated using the same implicit Euler method:

$$x_{k+1} = Z(t_{k+1}) \Big(x_k + hb\ddot{\phi}_{k+1} + hD(t_{k+1})L(y_{k+1} - v_{k+1}) \Big),$$
$$y_k = Cx_k + v_k,$$

for $k \in \mathbb{Z}_+$, where x_k should approach to $x(t_k)$ as h converges to zero, $\ddot{\phi}_k = \ddot{\phi}(t_k)$ and $v_k = v(t_k)$ (formally this perturbation v_k is different from the one used in (2) since it should also include the discretization error, but with a light ambiguity in notation we will continue to use the same symbol). In this work we will assume that v_k and $\ddot{\phi}_k$ take finite values with bounded norms as before, *i.e.*, $v, \ddot{\phi} \in l_\infty^1$.

To analyze the properties of (4) we will consider the discretization error $e_k = x_k - \hat{x}_k$, whose stability and convergence rate have been evaluated in Theorem 1 for the continuous-time scenario. The direct computations show that

$$e_{k+1} = Z(t_{k+1}) \Big(e_k + hb\ddot{\phi}_{k+1} - hD(t_{k+1})Lv_{k+1} \Big). \quad (5)$$

To investigate stability and the rate of convergence in (5), we will consider first a finite interval of time with $k \in \{\overline{0, N-2}\}$, and next possible extensions for $k \in \mathbb{Z}_+$.

A. Analysis for $k \in \{\overline{0, N-2}\}$

Let us define a time-varying Lyapunov function candidate (the same was used before):

$$V_k = e_k^\top \Pi_k e_k,\ \Pi_k = \Gamma(t_{k+1}) P \Gamma(t_{k+1}),\ \forall k \in \{\overline{0, N-2}\},$$

where $P = P^\top \succ 0$ is as in Theorem 1 (more precise requirements will be defined below). Note that for $k = N-1$, by definition $t_{k+1} = T$, hence, $\Gamma(T) = \text{diag}\{[1 \quad 0]^\top\}$ and $\Gamma(T)P\Gamma(T)$ is a singular matrix, then such a choice of Π_k is admissible for $k \in \{\overline{0, N-2}\}$ only.

For time-varying parameters $\alpha_k > 0$, $\gamma_k > 0$ and $\sigma_k > 0$ determined later for $k \in \{\overline{0, N-2}\}$ we obtain

$$V_{k+1} - \alpha_k V_k = \begin{bmatrix} e_k \\ \ddot{\phi}_{k+1} \\ v_{k+1} \end{bmatrix}^\top Q_k \begin{bmatrix} e_k \\ \ddot{\phi}_{k+1} \\ v_{k+1} \end{bmatrix} + \gamma_k \ddot{\phi}_{k+1}^2 + \sigma_k v_{k+1}^2$$

for

$$Q_k = \begin{bmatrix} I_2 \\ hb^\top \\ -hL^\top D(t_{k+1}) \end{bmatrix} Z^\top(t_{k+1}) \Pi_{k+1} Z(t_{k+1})$$
$$\times \begin{bmatrix} I_2 \\ hb^\top \\ -hL^\top D(t_{k+1}) \end{bmatrix}^\top - \begin{bmatrix} \alpha_k \Pi_k & 0 & 0 \\ 0 & \gamma_k & 0 \\ 0 & 0 & \sigma_k \end{bmatrix}.$$

We need to find the restrictions on P and the gains α_k, γ_k, σ_k such that $Q_k \preceq 0$. To this end, using the Schur complement and multiplying the obtained matrix from right and left sides on $\text{diag}\{I_2, Z^{-1}(t_{k+1})\Gamma^{-1}(t_{k+1}), 1, 1\}$ and its transpose, respectively, the latter property is equivalent to

$$\tilde{Q}_k \succeq 0,\ \tilde{Q}_k = \begin{bmatrix} \tilde{Q}_{11}^k & \tilde{Q}_{12}^k \\ (\tilde{Q}_{12}^k)^\top & \tilde{Q}_{22}^k \end{bmatrix},$$
$$\tilde{Q}_{11}^k = \begin{bmatrix} \Pi_{k+1}^{-1} & \Gamma^{-1}(t_{k+1}) \\ \Gamma^{-1}(t_{k+1}) & \alpha_k R(t_{k+1}) \end{bmatrix},$$
$$\tilde{Q}_{12}^k = h \begin{bmatrix} Z(t_{k+1})b & -F(t_{k+1}) \\ 0 & 0 \end{bmatrix},\ \tilde{Q}_{22}^k = \begin{bmatrix} \gamma_k & 0 \\ 0 & \sigma_k \end{bmatrix},$$

where

$$R(t_{k+1}) = \Gamma^{-1}(t_{k+1}) Z^{-\top}(t_{k+1}) \Pi_k Z^{-1}(t_{k+1}) \Gamma^{-1}(t_{k+1}).$$

Noting that

$$R(t) = \Gamma^{-1}(t) \left(I_2 - h(A - D(t)LC) \right)^\top \Gamma(t) P$$
$$\times \Gamma(t) \left(I_2 - h(A - D(t)LC) \right) \Gamma^{-1}(t)$$

and recalling that

$$\Gamma(t)(A - D(t)LC)\Gamma^{-1}(t) = \varrho(t)(A - LC),$$

we obtain

$$R(t) = P - h\varrho(t) \left((A - LC)^\top P + P(A - LC) \right)$$
$$+ h^2 \varrho^2(t) (A - LC)^\top P (A - LC),$$

which implies that this matrix is positive definite under the restrictions of Theorem 1 for any $t \geq 0$.

To formulate the conditions implying $\tilde{Q}_k \succeq 0$, first, let us investigate the restrictions for $\tilde{Q}_{11}^k \succ 0$. Calculating the Schur complement of $\tilde{Q}_{11}^k$ we get an inequality:

$$S_k \succ 0,$$
$$S_k = \alpha_k R(t_{k+1}) - \Gamma^{-1}(t_{k+1}) \Gamma(t_{k+2}) P \Gamma(t_{k+2}) \Gamma^{-1}(t_{k+1}),$$

and since

$$P \succ \Gamma^{-1}(t_{k+1}) \Gamma(t_{k+2}) P \Gamma(t_{k+2}) \Gamma^{-1}(t_{k+1})$$

for $k \in \{\overline{0, N-2}\}$, the property $\tilde{Q}_{11}^k \succ 0$ follows the conditions of Theorem 1 for $\alpha_k = a$ or $\alpha_k = \frac{a}{h\varrho(t_{k+1})}$ for some $a \geq 1$. If $\alpha_k = \frac{a}{h^2 \varrho^2(t_{k+1})}$ with $a > 0$, then an auxiliary linear matrix inequality should be verified:

$$a(A - LC)^\top P (A - LC) - P \succ 0,$$

and, obviously, always there is $a > 0$ such that it is true. Returning back to verification of $\tilde{Q}_k \succeq 0$, and having $\tilde{Q}_{11}^k \succ$

0 we can also use the Schur complement of $\tilde{Q}_k$ to check the desired property:

$$\tilde{Q}_{22}^k - (\tilde{Q}_{12}^k)^\top (\tilde{Q}_{11}^k)^{-1} \tilde{Q}_{12}^k \succeq 0,$$

then denote $T_k = (\tilde{Q}_{11}^k)^{-1}$, which can be calculated using the block inversion formula given in the preliminaries with the first block element

$$T_k^{11} = \Pi_{k+1} + \Pi_{k+1} \Gamma^{-1}(t_{k+1}) S_k^{-1} \Gamma^{-1}(t_{k+1}) \Pi_{k+1},$$

leading to

$$(\tilde{Q}_{12}^k)^\top (\tilde{Q}_{11}^k)^{-1} \tilde{Q}_{12}^k = h^2 \times \begin{bmatrix} b^\top Z^\top(t_{k+1}) T_k^{11} Z(t_{k+1}) b & -b^\top Z^\top(t_{k+1}) T_k^{11} F(t_{k+1}) \\ -F^\top(t_{k+1}) T_k^{11} Z(t_{k+1}) b & F^\top(t_{k+1}) T_k^{11} F(t_{k+1}) \end{bmatrix}$$

It is straightforward to check that $Z(t)b = \frac{1}{O(t)} \begin{bmatrix} h \\ L_1 h \varrho(t) + 1 \end{bmatrix}$ is of order $\varrho^{-1}(t)$ for any $t \geq 0$, while it has been discussed that $F(t)$ is globally bounded for all $t \geq 0$. For $\alpha_k = \frac{a}{h^2 \varrho^2(t_{k+1})}$ with $a > 0$, T_k^{11} is also bounded. Therefore, there exists $g > 0$ and $s > 0$ such that for $\gamma_k = \frac{gh^2}{\varrho^2(t_{k+1})}$ and $\sigma_k = sh^2$ the property $\tilde{Q}_k \succeq 0$ is verified, which leads to the estimate:

$$V_{k+1} \leq \frac{a}{h^2 \varrho^2(t_{k+1})} V_k + \frac{gh^2 |\ddot{\phi}|_\infty^2}{\varrho^2(t_{k+1})} + sh^2 |v|_\infty^2$$

for all $k \in \{\overline{0, N-2}\}$. Therefore, the following result can be formulated:

Theorem 2. *Let there exist $P = P^\top \in \mathbb{R}^{2\times 2}$, $U \in \mathbb{R}^2$ such that the linear matrix inequalities are verified:*

$$P \succ 0, \ A^\top P + PA - C^\top U^\top - UC \prec 0.$$

Then for $L = UP^{-1}$ there exist $a > 0$, $g > 0$ and $s > 0$ such that for any $e_0 \in \mathbb{R}^2$ in (5)*:*

$$\lambda_{\min}(P) \|e_k\|^2 \leq \varrho^2(t_{k+1}) [\prod_{i=0}^{k-1} \frac{a}{h^2 \varrho^2(t_{i+1})} e_0^\top P e_0 + gh^2 |\ddot{\phi}|_\infty^2 \sum_{i=0}^{k-1} \frac{1}{\varrho^2(t_{i+1})} \prod_{\ell=0}^{k-i-2} \frac{a}{h^2 \varrho^2(t_{\ell+1})} + sh^2 |v|_\infty^2 \sum_{i=0}^{k-1} \prod_{\ell=0}^{k-i-2} \frac{a}{h^2 \varrho^2(t_{\ell+1})}]$$

for all $k \in \{\overline{0, N-2}\}$.

Proof. Under the introduced restrictions,

$$V_k \leq \prod_{i=0}^{k-1} \frac{a}{h^2 \varrho^2(t_{i+1})} V_0 + gh^2 |\ddot{\phi}|_\infty^2 \sum_{i=0}^{k-1} \frac{1}{\varrho^2(t_{i+1})} \prod_{\ell=0}^{k-i-2} \frac{a}{h^2 \varrho^2(t_{\ell+1})} + sh^2 |v|_\infty^2 \sum_{i=0}^{k-1} \prod_{\ell=0}^{k-i-2} \frac{a}{h^2 \varrho^2(t_{\ell+1})}$$

for all $k \in \{\overline{0, N-2}\}$, which gives the required estimate. □

As we can see from the obtained results, the convergence in the initial error is close to prescribed-time one and the dependence in $\ddot{\phi}$ becomes infinitesimal at $k = N-2$, while the noise gain admits a static linear bound.

B. Analysis for $k \geq N-1$

For $k = N-1$, $t_{k+1} = T$ then all eigenvalues of the matrix $Z(T)$ are zero, hence, there is $\bar{P} = \bar{P}^\top \succ 0$ such that $Z^\top(T) \bar{P} Z(T) \prec \beta \bar{P}$ for any given $\beta \in (0,1)$. If we would like to extend the analysis to $k > N-1$ we need to introduce a definition of $Z(t)$ and $F(t)$ in (4) for $t > T$. A possible approach is just to take

$$Z(t) = Z(T), \ F(t) = F(T), \ \forall t > T.$$

In such a case let us modify our time-varying Lyapunov function $V_k = e_k^\top \Pi_k e_k$ as follows:

$$\Pi_k = \begin{cases} \Gamma(t_{k+1}) P \Gamma(t_{k+1}) & \text{if } k \in \{\overline{0, N-2}\} \\ \bar{P} & \text{if } k \geq N-1 \end{cases},$$

then it has been already proven an accelerated convergence of the estimation error for $k \in \{\overline{0, N-2}\}$ in Theorem 2, while for $k \geq N-1$ we get the error dynamics (note that $Z(T)b = 0$)

$$e_{k+1} = Z(T) e_k - hF(T) v_{k+1},$$

which is independent in $\ddot{\phi}$. Moreover, in the noise-free case $e_N = 0$ and we recover the prescribed-time convergence. However, it is easy to see that the observer (4) in such a case corresponds to the Euler approximation of the derivative:

$$\begin{bmatrix} \hat{x}_{1,k+1} \\ \hat{x}_{2,k+1} \end{bmatrix} = Z(T) \begin{bmatrix} \hat{x}_{1,k} \\ \hat{x}_{2,k} \end{bmatrix} + hF(T) y_{k+1} = \begin{bmatrix} y_{k+1} \\ h^{-1}(y_{k+1} - \hat{x}_{1,k}) \end{bmatrix},$$

whose bad sensitivity to noise is well-known [8].

Another approach is to recursively apply the observer (4) always staying onto the interval $k \in \{\overline{0, N-2}\}$ for the values of matrices Z and F:

$$\hat{x}_{k+1} = Z(\tilde{t}_{k+1}) \hat{x}_k + hF(\tilde{t}_{k+1}) y_{k+1}, \quad \tilde{t}_{k+1} = \operatorname{mod}(t_{k+1}, T) \tag{6}$$

for all $k \in \mathbb{Z}_+$. Then the dependence on initial conditions will be eliminated for $t_k \leq T$ and for $t_k > T$ such an observer will filter the noise and estimate the derivatives following the advantageous performance evaluated in Theorem 2. Note that in the continuous-time noise-free case, a similar modification can be applied to (3) extending its application to all $t \geq 0$, but in the presence of an arbitrary bounded noise v the estimate $\hat{x}$ may become anyway infinite while $\operatorname{mod}(t, T) = 0$.

Let us illustrate the efficiency of this scheme in simulations.

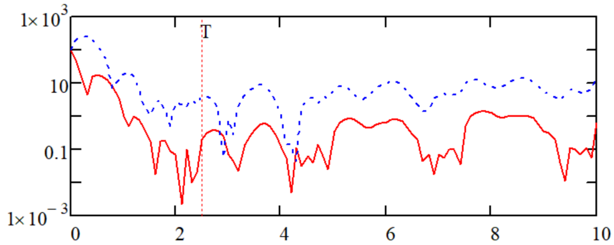

Figure 1. The results of estimation, $|e_{1,k}|$ (red solid line) and $|e_{2,k}|$ (blue dash line) versus time t_k, $h = 0.1$

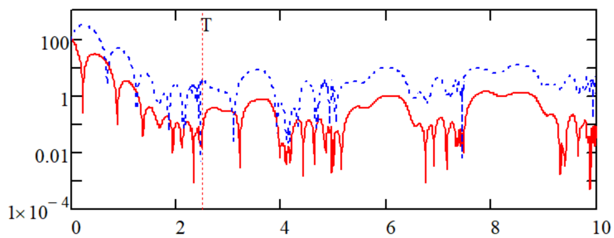

Figure 2. The results of estimation, $|e_{1,k}|$ (red solid line) and $|e_{2,k}|$ (blue dash line) versus time t_k, $h = 0.01$

VI. SIMULATIONS

For simulations, let us take

$$\phi(t) = t^3 - 100t + \sin(5t) + 100$$

and the noise

$$v(t) = 0.1\left(\text{rnd}(1) + \sin(15t)\right),$$

where $\text{rnd}(1)$ denotes a uniformly distributed in the interval $[0, 1]$ random number. Since

$$\dot{\phi}(t) = 3t^2 - 100 + 5\cos(5t),$$
$$\ddot{\phi}(t) = 6t - 25\sin(5t),$$

the perturbation $\ddot{\phi}$ is an unbounded and linearly growing function of time, which blocks the application of many existing differentiation solutions, whose utilization is based on assumption of boundedness of $\ddot{\phi}$ (*e.g.*, the linear high-gain observer [2] or the super-twisting differentiator [3]), which is however not an issue for the considered uPTS observer, as it is explained in Remark 3. The linear matrix inequalities from Theorem 2 are satisfied for

$$L = [5\ 20]^\top,$$

and take $T = 2.5$. The results of simulations are presented in figures 1 and 2 for $h = 0.1$ and $h = 0.01$, respectively. On the plots the absolute values of the errors, $|e_{1,k}|$ (red solid line) and $|e_{2,k}|$ (blue dash line), are shown versus the time, t_k, in logarithmic scale. As we can conclude from these results, the errors quickly converge to a neighborhood of zero (whose size is proportional to the noise amplitude) and stay there despite of the growing $\ddot{\phi}$.

VII. CONCLUSION

A new discretization scheme for a simple implementation of the prescribed time differentiator from [1] is proposed, which guarantees an accelerated rate of convergence of the estimation error, while remaining prescribed-time exact in the noise-free case. It has also certain robustness with respect to the measurement noise. The tuning rules are formulated using feasible linear matrix inequalities. Supporting simulation results demonstrate a good performance of the proposed discrete-time differentiator. Extension of this differentiation scheme to the case of a higher order observer can be considered as a direction of future research.

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