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An LMI-Based Observer Design Method for a Class of Nonlinear Systems

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***Abstract*— This paper deals with a new LMI-based observer design method for a class of nonlinear systems. Novel matrix multipliers are proposed to improve the feasibility of the LMI conditions existing in the literature. Two design procedures are proposed and both of them exploit a more general form of the matrix multiplier compared to the existing ones. The first method is based on the use of the standard Young relation jointly with a specific multiplier matrix, while the second one uses the LPV-based approach combined with a convenient Young inequality. The proposed LMIs contain additional numbers of decision variables as compared to the methods proposed in the literature, which add extra degrees of freedom thus improving the LMI feasibility. This is due to the use of a reformulated Lipschitz property and new matrix multipliers. Furthermore, the effectiveness of the proposed methodologies is highlighted through numerical comparisons.**

***Index Terms*— LMI-based nonlinear observer design, Lipschitz systems, $\mathcal{H}_\infty$ criterion.**

I. INTRODUCTION

The topic of state estimation for dynamical systems has become a centre of research during the last few decades. This is due to the fact that knowing the current state of the system is critical in many applications. Such real-time information is used for monitoring, decision-making, and controlling the systems. Installing sensors on physical systems is one of the methods used to capture real-time measurements. Since the quantity and quality of sensors are frequently limited in practice due to cost and physical constraints, state estimation tools play an important role in a wide range of applications [1], [2], [3].

State estimation approaches for linear systems have been extensively studied and proven to be quite reliable. The Kalman filter [4] and the Luenberger observer [5] were the first state estimation techniques designed for linear systems. In contrast to linear systems, designing observers for nonlinear systems remains a difficult task. As a result, a substantial amount of research has been conducted in this domain, and several methods for each type of system have been developed. Among these techniques, linear matrix inequality (LMI)-based methodologies have gained a lot of attention, and several outcomes are published in [1], [6] and [7].

Various LMI-based observer approaches for the nonlinear Lipschitz system have been presented in the literature. Some of these approaches depended on S-Procedure lemma [8], Riccati equations [9], and Young inequality [10]. Though each method provides a conservative LMI condition, there is scope for improvement. The motivation of the article is to develop an LMI-based $\mathcal{H}_\infty$ nonlinear observer with a matrix multiplier, which is inspired by [6], [11] and [12]. Further, the proposed approach is combined with the well-known LPV approach to deduce a less conservative LMI condition. The use of such a matrix multiplier adds extra decision variables and enhances an LMI condition from a feasibility point of view. Furthermore, a numerical example is used to demonstrate the efficiency and superiority of the proposed LMI techniques. To verify the observer performance, the proposed observer is implemented on the same example in MATLAB/Simulink.

The organisation of this letter is outlined as follows: Some preliminaries and background results required for nonlinear observer design are given in Section II. Further, Section III contains the system description and the problem formulation. The main contributions related to the development of the LMI conditions are illustrated in Section IV. The effectiveness of the new LMI conditions and the observer performance is shown in Section V. In Section VI, some conclusions are included.

Notation: Throughout the article, the following notations are used: $||e||$ and $||e||_{\mathcal{L}_2}$ describe the euclidean norm and the $\mathcal{L}_2$ norm of a vector e, respectively. Repeated blocks within a symmetric matrix are represented by $(\star)$. The transpose of matrix A is expressed as $A^\top$. For a matrix $A \in \mathbb{R}^{n\times n}$, $A > 0$ ($A < 0$) indicates that A is a positive definite matrix (a negative definite matrix). Similarly, a positive semi-definite matrix (a negative semi-definite matrix) is given by $A \geq 0$ ($A \leq 0$). $A = \text{block-diag}(A_1,\ldots,A_n)$ is a block-diagonal matrix having elements $A_1,\ldots,A_n$ in the diagonal. $\mathbb{I}$ denote an identity matrix. $e_s(i) = (\underbrace{0,\ldots,0,\overbrace{1}^{i^{\text{th}}},0,\ldots,0}_{s \text{ components}})^\top \in \mathbb{R}^s$, $s \geq 1$ is a vector of the canonical basis of $\mathbb{R}^s$. The term x_0 signifies the initial values of $x(t)$ at $t = 0$. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are minimum and maximum eigenvalues of matrix A.

II. PRELIMINARIES

This section contains the mathematical tools and background results which will be needed in the development of the main results.

Definition 1 ([6]): Let us consider the following two vectors:

$$a = \begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix}^\top,$$
$$b = \begin{pmatrix} b_1 & \ldots & b_n \end{pmatrix}^\top,$$

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Then, for all $i = 0,\ldots,n$, an auxiliary vector $a^{b_i} \in \mathbb{R}^n$ corresponding to a and b can be defined as:

$$a^{b_i} = \begin{cases} \begin{pmatrix} b_1 & \ldots & b_i & a_{i+1} & \ldots & a_n \end{pmatrix}^\top, & \text{for } i = 1,\ldots,n \\ a, & \text{for } i = 0. \end{cases} \tag{1}$$

Lemma 1 ([6]): Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a nonlinear function. Then, the following two statements are equivalent:

- ψ is globally Lipschitz with respect to its argument, i.e.,

$$||\psi(X) - \psi(Y)|| \le \psi_\psi ||X - Y||, \quad \forall X, Y \in \mathbb{R}^n \tag{2}$$

- For all, $i, j = 1,\ldots,n$, there exist functions $\psi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and constants $\psi_{ij_{\min}}$ and $\psi_{ij_{\max}}$ such that $\forall X, Y \in \mathbb{R}^n$,

$$\psi(X) - \psi(Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ij} \mathcal{H}_{ij}(X - Y), \tag{3}$$

where $\mathcal{H}_{ij} = e_n(i) e_n^\top(j)$, and $\psi_{ij} \triangleq \psi_{ij}(X_i^{Y_{i,j-1}}, X_i^{Y_{i,j}})$. The functions $\psi_{ij}(.)$ are globally bounded as follows:

$$\psi_{ij_{\min}} \le \psi_{ij} \le \psi_{ij_{\max}}. \tag{4}$$

Lemma 2 ([6]): For any two vectors $X, Y \in \mathbb{R}^n$ and a matrix $Z = Z^\top > 0 \in \mathbb{R}^{n\times n}$, the following matrix inequality holds:

$$X^\top Y + Y^\top X \le X^\top Z^{-1} X + Y^\top Z Y. \tag{5}$$

The authors of [6] proposed the subsequent new variant of Young inequality (5):

$$X^\top Y + Y^\top X \le \frac{1}{2}(X + ZY)^\top Z^{-1}(X + ZY). \tag{6}$$

Inequality (5) is used in various control problems with $Z = \varepsilon \mathbb{I}, \varepsilon > 0$. However, in this paper, both inequalities (5) and (6) are used with a new form of Z:

$$Z = \begin{bmatrix} Z_1 & Z_{a_2} & \ldots & Z_{a_n} \\ \star & Z_2 & \ldots & Z_{a_n} \\ \star & \star & \ddots & \vdots \\ \star & \star & \ldots & Z_n \end{bmatrix}, \tag{7}$$

where $Z_i = Z_i^\top > 0 \in \mathbb{R}^{\bar{n}\times\bar{n}}$ and $Z_{a_i} = Z_{a_i}^\top \ge 0 \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $\forall\, i \in \{1,\ldots,n\}$.

Lemma 3 ([12]): Let us define Ψ_a and Ψ_b as follows:

$$\Psi_a^\top = \begin{bmatrix} a_1 \mathbb{I}_n & a_2 \mathbb{I}_n & \ldots & a_n \mathbb{I}_n \end{bmatrix}, \tag{8}$$
$$\Psi_b^\top = \begin{bmatrix} b_1 \mathbb{I}_n & b_2 \mathbb{I}_n & \ldots & b_n \mathbb{I}_n \end{bmatrix}, \tag{9}$$

where $0 \le a_i \le b_i, \forall\, i \in \{1,\ldots,n\}$. Then, the following inequality holds:

$$\Psi_a^\top Z \Psi_a \le \Psi_b^\top Z \Psi_b, \tag{10}$$

for any matrix $Z = Z^\top > 0 \in \mathbb{R}^{n\times n}$, which has same structure as in (7).
The inequality (10) will be very useful in LMI formulation.

III. Problem formulation

The class of nonlinear systems with nonlinear outputs is described by

$$\begin{aligned} \dot{x} &= Ax + Gf(x) + Bu + E\omega, \\ y &= Cx + Fg(x) + D\omega, \end{aligned} \tag{11}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^s$ are state vectors, output measurements, and inputs of the system, respectively. $\omega \in \mathbb{R}^q$ represents the $\mathcal{L}_2$ bounded noise/disturbance present in the system dynamics and outputs. $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times s}, G \in \mathbb{R}^{n\times m}, E \in \mathbb{R}^{n\times q}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times q}, F \in \mathbb{R}^{p\times r}$ are known constant matrices. Both nonlinear functions $f(.) : \mathbb{R}^n \to \mathbb{R}^m$ and $g(.) : \mathbb{R}^n \to \mathbb{R}^r$ are assumed to be globally Lipschitz. The detailed form of $f(x)$ and $g(x)$ are as follows:

$$f(x) = \begin{bmatrix} f_1(\overbrace{H_1 x}^{\theta_i = H_i x}) \\ \vdots \\ f_m(\theta_m) \end{bmatrix}, \; g(x) = \begin{bmatrix} g_1(\overbrace{G_1 x}^{\Lambda_i = G_i x}) \\ \vdots \\ g_r(\Lambda_r) \end{bmatrix}, \tag{12}$$

where $H_i \in \mathbb{R}^{\bar{n}\times n}\, \forall i \in \{1,\ldots,m\}$ and $G_i \in \mathbb{R}^{\bar{p}\times n}\, \forall i \in \{1,\ldots,r\}$.

Now, Let us consider the following Luenberger observer form:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Gf(\hat{x}) + Bu + L(y - \hat{y}), \\ \hat{y} &= C\hat{x} + Fg(\hat{x}), \end{aligned} \tag{13}$$

where $\hat{x}$, $\hat{y}$ denote estimated states and observer outputs, respectively. $L \in \mathbb{R}^{n\times p}$ is the observer gain matrix. The estimation error of the observer is defined as $\tilde{x} = x - \hat{x}$. From (11) and (13), the estimation error dynamic is given by

$$\dot{\tilde{x}} = \underbrace{(A - LC)}_{\mathbb{A}} \tilde{x} + G\tilde{f}(x,\hat{x}) - LF\tilde{g}(x,\hat{x}) + \underbrace{(E - LD)}_{\mathbb{E}} \omega, \tag{14}$$

where $\tilde{f}(x,\hat{x}) = f(x) - f(\hat{x})$ and $\tilde{g}(x,\hat{x}) = g(x) - g(\hat{x})$. The objective is to compute the observer gain L such that

1) When $\omega = 0$, the estimation error dynamic (14) is converging towards zero at $t \to \infty$.
2) When $\omega \neq 0$, the estimation error dynamic (14) fulfills the $\mathcal{H}_\infty$ criterion [13]:

$$||\tilde{x}||_{\mathcal{L}_2^n} \le \sqrt{\mu ||\omega||^2_{\mathcal{L}_2^q} + \nu ||\tilde{x}_0||^2}, \tag{15}$$

where $\mu > 0$. The term $\sqrt{\mu}$ indicates the disturbance attenuation level, and $\nu > 0$ is to be estimated.

Since $f(x)$ and $g(x)$ are globally Lipschitz, then from Lemma 1, there exist functions $f_{ij} : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$, $g_{ij} : \mathbb{R}^{\bar{p}} \times \mathbb{R}^{\bar{p}} \to \mathbb{R}$, and known constants $f_{a_{ij}}$, $f_{b_{ij}}$, $g_{a_{ij}}$ and $g_{b_{ij}}$, such that

$$\tilde{f}(x,\hat{x}) = \sum_{i,j=1}^{m,\bar{n}} f_{ij} \mathcal{H}_{ij} H_i \tilde{x}, \text{ and } \tilde{g}(x,\hat{x}) = \sum_{i,j=1}^{r,\bar{p}} g_{ij} \mathcal{G}_{ij} G_i \tilde{x}, \tag{16}$$

where $f_{ij} \triangleq f_{ij}(\theta_i^{\hat{\theta}_{i,j-1}}, \theta_i^{\hat{\theta}_{i,j}})$ and $g_{ij} \triangleq g_{ij}(\Lambda_i^{\hat{\Lambda}_{i,j-1}}, \Lambda_i^{\hat{\Lambda}_{i,j}})$. The functions f_{ij}, g_{ij} hold: $f_{a_{ij}} \le f_{ij} \le f_{b_{ij}}$; $g_{a_{ij}} \le g_{ij} \le g_{b_{ij}}$.

Without loss of generality, let us assume that $f_{a_{ij}} = 0$ and $g_{a_{ij}} = 0$, i.e.,

$$0 \leq f_{ij} \leq f_{b_{ij}}, \tag{17}$$

$$0 \leq g_{ij} \leq g_{b_{ij}}. \tag{18}$$

From (16), the error dynamic (14) becomes:

$$\dot{\tilde{x}} = \mathbb{A}\tilde{x} + \sum_{i,j=1}^{m,\bar{n}} f_{ij} G\mathcal{H}_{ij} H_i \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} g_{ij} LF\mathcal{G}_{ij} G_i \tilde{x} + \mathbb{E}\omega. \tag{19}$$

Remark 1: In various practical applications, it is possible to have $f_{a_{ij}}, g_{a_{ij}} \neq 0$. In such cases, (19) is rewritten as:

$$\begin{aligned} \dot{\tilde{x}} = &\left(\mathbb{A} + \sum_{i,j=1}^{m,\bar{n}} f_{a_{ij}} G\mathcal{H}_{ij} H_i - \sum_{i,j=1}^{r,\bar{p}} g_{a_{ij}} LF\mathcal{G}_{ij} G_i\right)\tilde{x} \\ &+ \sum_{i,j=1}^{m,\bar{n}} \tilde{f}_{ij} G\mathcal{H}_{ij} H_i \tilde{x} - \sum_{i,j=1}^{r,\bar{p}} \tilde{g}_{ij} LF\mathcal{G}_{ij} G_i \tilde{x} + \mathbb{E}\omega, \end{aligned} \tag{20}$$

where $\tilde{f}_{ij} = f_{ij} - f_{a_{ij}}$, and $\tilde{g}_{ij} = g_{ij} - g_{a_{ij}}$. For the error dynamics (20), $\tilde{f}_{ij}$ and $\tilde{g}_{ij}$ fullfil (17) and (18), respectively.

Let us define a Lyapunov function to evaluate $\mathcal{H}_\infty$ stability of the error dynamics (19):

$$V(\tilde{x}) = \tilde{x}^\top P \tilde{x}, \text{ where } P = P^\top > 0.$$

By computing $\dot{V}(\tilde{x})$ along the trajectories of (19), we get

$$\begin{aligned} &\dot{V}(\tilde{x}) = \tilde{x}^\top(\mathbb{A}^\top P + P\mathbb{A})\tilde{x} + \tilde{x}^\top(P\mathbb{E})\omega + \omega^\top(\mathbb{E}^\top P)\tilde{x} + \\ &\tilde{x}^\top\Big[\Big(\sum_{i,j=1}^{m,\bar{n}} f_{ij} PG\mathcal{H}_{ij} H_i\Big) + \Big(\sum_{i,j=1}^{m,\bar{n}} f_{ij} PG\mathcal{H}_{ij} H_i\Big)^\top\Big]\tilde{x} - \\ &\tilde{x}^\top\Big[\Big(\sum_{i,j=1}^{r,\bar{p}} g_{ij} PLF\mathcal{G}_{ij} G_i\Big) + \Big(\sum_{i,j=1}^{r,\bar{p}} g_{ij} PLF\mathcal{G}_{ij} G_i\Big)^\top\Big]\tilde{x}. \end{aligned}$$

From [6], the $\mathcal{H}_\infty$ criterion (15) is satisfied if the following inequality holds:

$$\mathcal{W} \triangleq \dot{V}(\tilde{x}) + ||\tilde{x}||^2 - \mu||\omega||^2 \leq 0. \tag{21}$$

Remark 2: At $\omega = 0$, inequality (21) becomes: $\dot{V}(\tilde{x}) + ||\tilde{x}||^2 \leq 0$, and it yields the exponential stability condition $\dot{V}(\tilde{x}) \leq -\sigma V(\tilde{x})$, where $\sigma = \frac{1}{\lambda_{\max}(P)} > 0$.

Then,

$$\begin{aligned} \mathcal{W} = &\begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix}^\top \begin{bmatrix}\mathbb{A}^\top P + P\mathbb{A} + \mathbb{I}_n & P\mathbb{E}\\ \mathbb{E}^\top P & -\mu\mathbb{I}_q\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix} \\ &+ \begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix}^\top \Bigg[\sum_{i,j=1}^{m,\bar{n}}\Bigg(\begin{bmatrix}(PG\mathcal{H}_{ij})\\ 0\end{bmatrix}\begin{bmatrix}f_{ij}H_i & 0\end{bmatrix} \\ &+ \begin{bmatrix}f_{ij}H_i & 0\end{bmatrix}^\top \begin{bmatrix}(PG\mathcal{H}_{ij})\\ 0\end{bmatrix}^\top\Bigg)\Bigg]\begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix} \\ &+ \begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix}^\top \Bigg[\sum_{i,j=1}^{r,\bar{p}}\Bigg(\begin{bmatrix}(-PLF\mathcal{G}_{ij})\\ 0\end{bmatrix}\begin{bmatrix}g_{ij}G_i & 0\end{bmatrix} \\ &+ \begin{bmatrix}g_{ij}G_i & 0\end{bmatrix}^\top \begin{bmatrix}(-PLF\mathcal{G}_{ij})\\ 0\end{bmatrix}^\top\Bigg)\Bigg]\begin{bmatrix}\tilde{x}\\ \omega\end{bmatrix}. \end{aligned} \tag{22}$$

It follows that $\mathcal{W} \leq 0$ if the following inequality is true:

$$\mathbb{L}_1 + \mathbf{N}_1 + \mathbf{N}_2 \leq 0, \tag{23}$$

where

$$\mathbb{L}_1 = \begin{bmatrix}A^\top P + PA - R^\top C - C^\top R + \mathbb{I}_n & PE - R^\top D\\ (\star) & -\mu\mathbb{I}_q\end{bmatrix}, \tag{24}$$

$$\mathbf{N}_1 = \sum_{i,j=1}^{m,\bar{n}}\Bigg(\underbrace{\begin{bmatrix}(PG\mathcal{H}_{ij})\\ 0\end{bmatrix}}_{\mathbb{U}_{ij}^\top}\underbrace{f_{ij}\overbrace{\begin{bmatrix}H_i & 0\end{bmatrix}}^{\mathbb{H}_i}}_{\mathbb{V}_{ij}} + \mathbb{V}_{ij}^\top\mathbb{U}_{ij}\Bigg), \tag{25}$$

$$\mathbf{N}_2 = \sum_{i,j=1}^{r,\bar{p}}\Bigg(\underbrace{\begin{bmatrix}(-R^\top F\mathcal{G}_{ij})\\ 0\end{bmatrix}}_{\mathbb{M}_{ij}^\top}\underbrace{g_{ij}\overbrace{\begin{bmatrix}G_i & 0\end{bmatrix}}^{\mathbb{G}_i}}_{\mathbb{N}_{ij}} + \mathbb{N}_{ij}^\top\mathbb{M}_{ij}\Bigg), \tag{26}$$

and $R = L^\top P$.

Various LMI-based approaches have been developed in the literature [6], [7], [3], where each approach provides improved LMI conditions by using different mathematical tools. Despite advances in this area of LMI relaxations, the resulting LMIs remain conservative, then more enhancements are possible. In the sequel, two novel LMI techniques will be proposed. To this end, we exploit both of Lemma 2 and Lemma 3.

IV. New LMI-Based Design Procedures

This section focuses on the development of two LMI-based observer design techniques. First, Young inequality (5) and Lemma 3 are used to formulate a new LMI. Further, the variant of Young inequality (6) and the LPV-based technique are combined to derive less conservative LMI conditions.

Over the last few decades, LMI-based methods have been extensively studied to handle Lipschitz nonlinearities ([6], [7], [10]). The authors of [10] and [14] have used the global form of nonlinearities (i.e., $\tilde{f}(x,\hat{x}) = f(x) - f(\hat{x})$). However, the detailed form (12) was used in [6] and [7]. The use of nonlinearities in their detailed form enables the inclusion of additional decision variables in the LMI approach. In this paper, a new technique is proposed to handle the nonlinearities as compared to the one given in [6].

A. First LMI-based method

Let us consider the following notations to avoid cumbersome equations:

$$\mathbb{U} = \begin{bmatrix}\mathbb{U}_{11}^\top & \dots & \mathbb{U}_{1\bar{n}}^\top & \dots & \mathbb{U}_{m1}^\top & \dots & \mathbb{U}_{m\bar{n}}^\top\end{bmatrix}^\top, \tag{29}$$

$$\mathbb{V} = \begin{bmatrix}\mathbb{V}_{11}^\top & \dots & \mathbb{V}_{1\bar{n}}^\top & \dots & \mathbb{V}_{m1}^\top & \dots & \mathbb{V}_{m\bar{n}}^\top\end{bmatrix}^\top, \tag{30}$$

$$\mathbb{M} = \begin{bmatrix}\mathbb{M}_{11}^\top & \dots & \mathbb{M}_{1\bar{p}}^\top & \dots & \mathbb{M}_{r1}^\top & \dots & \mathbb{M}_{r\bar{p}}^\top\end{bmatrix}^\top, \tag{31}$$

$$\mathbb{N} = \begin{bmatrix}\mathbb{N}_{11}^\top & \dots & \mathbb{N}_{1\bar{p}}^\top & \dots & \mathbb{N}_{r1}^\top & \dots & \mathbb{N}_{r\bar{p}}^\top\end{bmatrix}^\top, \tag{32}$$

where $\mathbb{U}_{ij}$, $\mathbb{V}_{ij}$, $\mathbb{M}_{ij}$ and $\mathbb{N}_{ij}$ are defined in (25) and (26). With these notations, $\mathbf{N}_1$ and $\mathbf{N}_2$ are rewritten as

$$\mathbf{N}_1 = \mathbb{U}^\top\mathbb{V} + \mathbb{V}^\top\mathbb{U}, \tag{33}$$

$$\mathbf{N}_2 = \mathbb{M}^\top\mathbb{N} + \mathbb{N}^\top\mathbb{M}. \tag{34}$$

$$\mathbb{Z} = \begin{bmatrix} \mathbb{Z}_1 & \mathbb{Z}_{b_2} & \dots & \mathbb{Z}_{b_m} \\ \star & \mathbb{Z}_2 & \dots & \mathbb{Z}_{b_m} \\ & & \ddots & \vdots \\ \star & \star & \ddots & \vdots \\ \star & \star & \dots & \mathbb{Z}_m \end{bmatrix}, \text{ where } \mathbb{Z}_i = \begin{bmatrix} Z_{i1} & Z_{a_{i2}} & \dots & Z_{a_{i\bar{n}}} \\ \star & Z_{i2} & \dots & Z_{a_{i\bar{n}}} \\ & & \ddots & \vdots \\ \star & \star & \ddots & \vdots \\ \star & \star & \dots & Z_{i\bar{n}} \end{bmatrix} \forall i \in \{1,\dots,m\} \text{ and } \mathbb{Z}_{b_i} = \begin{bmatrix} Z_{b_{i1}} & Z_{c_{i2}} & \dots & Z_{c_{i\bar{n}}} \\ \star & Z_{b_{i2}} & \dots & Z_{c_{i\bar{n}}} \\ & & \ddots & \vdots \\ \star & \star & \ddots & \vdots \\ \star & \star & \dots & Z_{b_{i\bar{n}}} \end{bmatrix} \forall i \in \{2,\dots,m\}, \quad (27)$$

such that:

- $Z_{ij} = Z_{ij}^\top > 0 \in \mathbb{R}^{\bar{n}\times\bar{n}}\ \forall i \in \{1,\dots,m\}\ \&\ j \in \{1,\dots,\bar{n}\}$, $Z_{a_{ij}} = Z_{a_{ij}}^\top \geq 0 \in \mathbb{R}^{\bar{n}\times\bar{n}}\ \forall i \in \{1,\dots,m\}\ \&\ j \in \{1,\dots,\bar{n}\}$; $Z_{b_{ij}} = Z_{b_{ij}}^\top \geq 0 \in \mathbb{R}^{\bar{n}\times\bar{n}} \forall i \in \{2,\dots,m\}\ \&\ j \in \{1,\dots,\bar{n}\}$; $Z_{c_{ij}} = Z_{c_{ij}}^\top \geq 0 \in \mathbb{R}^{\bar{n}\times\bar{n}} \forall i \in \{2,\dots,m\}\ \&\ j \in \{2,\dots,\bar{n}\}$ such that $\mathbb{Z} > 0$.

$$\mathbb{S} = \begin{bmatrix} \mathbb{S}_1 & \mathbb{S}_{b_2} & \dots & \mathbb{S}_{b_r} \\ \star & \mathbb{S}_2 & \dots & \mathbb{S}_{b_r} \\ & & \ddots & \vdots \\ \star & \star & \ddots & \vdots \\ \star & \star & \dots & \mathbb{S}_r \end{bmatrix}, \text{ where } \mathbb{S}_i = \begin{bmatrix} S_{i1} & S_{a_{i2}} & \dots & S_{a_{i\bar{p}}} \\ \star & S_{i2} & \dots & S_{a_{i\bar{p}}} \\ & & \ddots & \vdots \\ \star & \star & \ddots & \vdots \\ \star & \star & \dots & S_{i\bar{p}} \end{bmatrix} \forall i \in \{1,\dots,r\} \text{ and } \mathbb{S}_{b_i} = \begin{bmatrix} S_{b_{i1}} & S_{c_{i2}} & \dots & S_{c_{i\bar{p}}} \\ \star & S_{b_{i2}} & \dots & S_{c_{i\bar{p}}} \\ & & \ddots & \vdots \\ \star & \star & \ddots & \vdots \\ \star & \star & \dots & S_{b_{i\bar{p}}} \end{bmatrix} \forall i \in \{2,\dots,r\}, \quad (28)$$

such that:

- $S_{ij} = S_{ij}^\top > 0 \in \mathbb{R}^{\bar{p}\times\bar{p}}\ \forall i \in \{1,\dots,r\}\ \&\ j \in \{1,\dots,\bar{p}\}$, $S_{a_{ij}} = S_{a_{ij}}^\top \geq 0 \in \mathbb{R}^{\bar{p}\times\bar{p}}\ \forall i \in \{1,\dots,r\}\ \&\ j \in \{1,\dots,\bar{p}\}$; $S_{b_{ij}} = S_{b_{ij}}^\top \geq 0 \in \mathbb{R}^{\bar{p}\times\bar{p}} \forall i \in \{2,\dots,r\}\ \&\ j \in \{1,\dots,\bar{p}\}$; $S_{c_{ij}} = S_{c_{ij}}^\top \geq 0 \in \mathbb{R}^{\bar{p}\times\bar{p}} \forall i \in \{2,\dots,r\}\ \&\ j \in \{2,\dots,\bar{p}\}$ so that $\mathbb{S} > 0$.

From the form of $\mathbb{V}_{ij}$ and $\mathbb{N}_{ij}$ described in (25) and (26), $\mathbb{V}$ and $\mathbb{N}$ can be written as:

$$\mathbb{V} = \begin{bmatrix} \mathbb{V}_{11}^\top & \dots & \mathbb{V}_{1\bar{n}}^\top & \dots & \mathbb{V}_{m1}^\top & \dots & \mathbb{V}_{m\bar{n}}^\top \end{bmatrix}^\top = \mathbb{H}\boldsymbol{\Phi}, \quad (35)$$

$$\mathbb{N} = \begin{bmatrix} \mathbb{N}_{11}^\top & \dots & \mathbb{N}_{1\bar{n}}^\top & \dots & \mathbb{N}_{m1}^\top & \dots & \mathbb{N}_{m\bar{n}}^\top \end{bmatrix}^\top = \mathbb{G}\boldsymbol{\Psi}, \quad (36)$$

where,

$$\mathbb{H} = \text{block-diag}(\underbrace{\mathbb{H}_1,\dots,\mathbb{H}_1}_{\bar{n}\text{ times}},\dots,\underbrace{\mathbb{H}_m,\dots,\mathbb{H}_m}_{\bar{n}\text{ times}}), \quad (37)$$

$$\boldsymbol{\Phi}^\top = \begin{bmatrix} f_{11}\mathbb{I} & \dots & f_{1\bar{n}}\mathbb{I} & \dots & f_{m1}\mathbb{I} & \dots & f_{m\bar{n}}\mathbb{I} \end{bmatrix}, \quad (38)$$

$$\mathbb{G} = \text{block-diag}(\underbrace{\mathbb{G}_1,\dots,\mathbb{G}_1}_{\bar{p}\text{ times}},\dots,\underbrace{\mathbb{G}_r,\dots,\mathbb{G}_r}_{\bar{p}\text{ times}}), \quad (39)$$

$$\boldsymbol{\Psi}^\top = \begin{bmatrix} g_{11}\mathbb{I} & \dots & g_{1\bar{p}}\mathbb{I} & \dots & g_{r1}\mathbb{I} & \dots & g_{r\bar{n}}\mathbb{I} \end{bmatrix}. \quad (40)$$

By implementing inequality (5) and using equations (35) and (36) on (33) and (34), we obtain:

$$\mathbf{N}_1 \leq \mathbb{U}^\top\mathbb{Z}^{-1}\mathbb{U} + \boldsymbol{\Phi}^\top\mathbb{H}^\top\mathbb{Z}\mathbb{H}\boldsymbol{\Phi}, \quad (41)$$

$$\mathbf{N}_2 \leq \mathbb{M}^\top\mathbb{S}^{-1}\mathbb{M} + \boldsymbol{\Psi}^\top\mathbb{G}^\top\mathbb{S}\mathbb{G}\boldsymbol{\Psi}, \quad (42)$$

where $\mathbb{Z}$ and $\mathbb{S}$ are defined in (27) and (28), respectively.

Consider $\boldsymbol{\Phi}_m$ and $\boldsymbol{\Psi}_m$ as follows:

$$\boldsymbol{\Phi}_m^\top = \begin{bmatrix} f_{b_{11}}\mathbb{I} & \dots & f_{b_{1\bar{n}}}\mathbb{I} & \dots & f_{b_{m1}}\mathbb{I} & \dots & f_{b_{m\bar{n}}}\mathbb{I} \end{bmatrix}, \quad (43)$$

$$\boldsymbol{\Psi}_m^\top = \begin{bmatrix} g_{b_{11}}\mathbb{I} & \dots & g_{b_{1p_1}}\mathbb{I} & \dots & g_{b_{r1}}\mathbb{I} & \dots & g_{b_{r\bar{p}}}\mathbb{I} \end{bmatrix}. \quad (44)$$

Since $\mathbb{Z} > 0$ and $\mathbb{S} > 0$, $\mathbb{H}^\top\mathbb{Z}\mathbb{H} \geq 0$ and $\mathbb{G}^\top\mathbb{S}\mathbb{G} \geq 0$. Then, from (17), (18) and Lemma 3, we have $\boldsymbol{\Phi}^\top\mathbb{H}^\top\mathbb{Z}\mathbb{H}\boldsymbol{\Phi} \leq \boldsymbol{\Phi}_m^\top\mathbb{H}^\top\mathbb{Z}\mathbb{H}\boldsymbol{\Phi}_m$ and $\boldsymbol{\Psi}^\top\mathbb{G}^\top\mathbb{S}\mathbb{G}\boldsymbol{\Psi} \leq \boldsymbol{\Psi}_m^\top\mathbb{G}^\top\mathbb{S}\mathbb{G}\boldsymbol{\Psi}_m$. Furthermore, (41) and (42) are reformulated as

$$\mathbf{N}_1 \leq \mathbb{U}^\top\mathbb{Z}^{-1}\mathbb{U} + \boldsymbol{\Phi}_m^\top\mathbb{H}^\top\mathbb{Z}\mathbb{H}\boldsymbol{\Phi}_m, \quad (45)$$

$$\mathbf{N}_2 \leq \mathbb{M}^\top\mathbb{S}^{-1}\mathbb{M} + \boldsymbol{\Psi}_m^\top\mathbb{G}^\top\mathbb{S}\mathbb{G}\boldsymbol{\Psi}_m. \quad (46)$$

Hence, inequality (23) is satisfied if

$$\begin{aligned} &\mathbb{L}_1 + \mathbb{U}^\top\mathbb{Z}^{-1}\mathbb{U} + \boldsymbol{\Phi}_m^\top\mathbb{H}^\top\mathbb{Z}\mathbb{H}\boldsymbol{\Phi}_m \\ &\quad + \mathbb{M}^\top\mathbb{S}^{-1}\mathbb{M} + \boldsymbol{\Psi}_m^\top\mathbb{G}^\top\mathbb{S}\mathbb{G}\boldsymbol{\Psi}_m \leq 0. \end{aligned} \quad (47)$$

Now, we are ready to state the following theorem.

Theorem 1: The estimation error $\tilde{x}$ satisfies $\mathcal{H}_\infty$ criterion (15) if there exist symmetric positive definite matrices $P \in \mathbb{R}^{n\times n}$, $Z_{ij} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $S_{ij} \in \mathbb{R}^{\bar{p}\times\bar{p}}$, a matrix $R \in \mathbb{R}^{p\times n}$ and symmetric positive semi-definite matrices $Z_{a_{ij}}, Z_{b_{ij}}, Z_{c_{ij}} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $S_{a_{ij}}, S_{b_{ij}}, S_{c_{ij}} \in \mathbb{R}^{\bar{p}\times\bar{p}}$ such that the following convex optimization problem is solvable:
$\min(\mu)$ subject to

$$\begin{bmatrix} \mathbb{L}_1 & \mathbb{U}^\top & \boldsymbol{\Phi}_m^\top\mathbb{H}^\top\mathbb{Z} & \mathbb{M}^\top & \boldsymbol{\Psi}_m^\top\mathbb{G}^\top\mathbb{S} \\ \star & -\mathbb{Z} & 0 & 0 & 0 \\ \star & \star & -\mathbb{Z} & 0 & 0 \\ \star & \star & \star & -\mathbb{S} & 0 \\ \star & \star & \star & \star & -\mathbb{S} \end{bmatrix} < 0, \quad (48)$$

where the matrices $\mathbb{L}_1, \mathbb{Z}, \mathbb{S}, \mathbb{U}, \mathbb{M}, \mathbb{H}, \mathbb{G}, \boldsymbol{\Phi}_m$ and $\boldsymbol{\Psi}_m$ are illustrated in (24), (27), (28), (29), (31), (37), (39), (43) and (44), respectively. The observer gain is computed as $L = P^{-1}R^\top$.

Proof: LMI (48) is derived by using Schur lemma on (47). Hence, the $\mathcal{H}_\infty$ criterion (15) is satisfied with minimum $\sqrt{\mu}$ obtained from the solution of LMI (48). ■

B. Second LMI method: LPV-based approach

In this section, an enhanced LMI approach is developed by combining the well-known LPV technique with the new variant of Young relation (6) and the proposed matrix multipliers.

The following inequalities are derived by applying inequality (6) on (33) and (34):

$$\mathbf{N}_1 \leq \frac{1}{2}\left[\left(\mathbb{U} + \mathbb{Z}\mathbb{H}\boldsymbol{\Phi}\right)^\top\mathbb{Z}^{-1}\left(\mathbb{U} + \mathbb{Z}\mathbb{H}\boldsymbol{\Phi}\right)\right], \quad (49)$$

$$\mathbf{N}_2 \leq \frac{1}{2}\left[\left(\mathbb{M} + \mathbb{S}\mathbb{G}\boldsymbol{\Psi}\right)^\top\mathbb{S}^{-1}\left(\mathbb{M} + \mathbb{S}\mathbb{G}\boldsymbol{\Psi}\right)\right], \quad (50)$$

where $\mathbb{Z} = \mathbb{Z}^\top > 0$ and $\mathbb{S} = \mathbb{S}^\top > 0$ are defined in (27) and (28), respectively.

Therefore, inequality (23) holds if

$$\begin{aligned} \mathbb{L}_1 &+ \frac{1}{2}\left[\left(\mathbb{U} + \mathbb{Z}\mathbb{H}\boldsymbol{\Phi}\right)^\top\mathbb{Z}^{-1}\left(\mathbb{U} + \mathbb{Z}\mathbb{H}\boldsymbol{\Phi}\right)\right] \\ &+ \frac{1}{2}\left[\left(\mathbb{M} + \mathbb{S}\mathbb{G}\boldsymbol{\Psi}\right)^\top\mathbb{S}^{-1}\left(\mathbb{M} + \mathbb{S}\mathbb{G}\boldsymbol{\Psi}\right)\right] \leq 0. \end{aligned} \quad (51)$$

From (17) and (18), each element inside Φ and Ψ are bounded and belong to convex sets $\mathcal{F}_m$ and $\mathcal{G}_r$, respectively. The sets $\mathcal{F}_m$ and $\mathcal{G}_r$ are defined as follows:

$$\mathcal{F}_m \triangleq \left\{\Phi : 0 \leq f_{ij} \leq f_{a_{ij}}, \forall i \in \{1,\ldots,m\} \,\&\, j \in \{1,\ldots,\bar{n}\}\right\},$$
$$\mathcal{G}_r \triangleq \left\{\Psi : 0 \leq g_{ij} \leq g_{a_{ij}}, \forall i \in \{1,\ldots,r\} \,\&\, j \in \{1,\ldots,\bar{p}\}\right\}.$$

The set of vertices of $\mathcal{F}_m$ and $\mathcal{G}_r$ are given by

$$\begin{aligned}\mathcal{F}_{H_m} &= \left\{\{\mathcal{F}_{11},\ldots,\mathcal{F}_{1\bar{n}},\ldots,\mathcal{F}_{m1},\ldots,\mathcal{F}_{m\bar{n}}\} : \mathcal{F}_{ij} \in [0, f_{b_{ij}}]\right\},\\ \mathcal{G}_{G_r} &= \left\{\{\mathcal{G}_{11},\ldots,\mathcal{G}_{1\bar{p}},\ldots,\mathcal{G}_{r1},\ldots,\mathcal{G}_{r\bar{p}}\} : \mathcal{G}_{ij} \in [0, g_{b_{ij}}]\right\}.\end{aligned} \tag{52}$$

Hence, the inequality (51) is rewritten as:

$$\begin{aligned}\mathbb{L}_1 &+ \left[\frac{1}{2}\big(\mathbb{U}+\mathbb{Z}\mathbb{H}\Phi\big)^\top \mathbb{Z}^{-1}\big(\mathbb{U}+\mathbb{Z}\mathbb{H}\Phi\big)\right]_{\forall\Phi\in\mathcal{F}_m}\\ &+ \left[\frac{1}{2}\big(\mathbb{M}+\mathbb{S}\mathbb{G}\Psi\big)^\top \mathbb{S}^{-1}\big(\mathbb{M}+\mathbb{S}\mathbb{G}\Psi\big)\right]_{\forall\Psi\in\mathcal{G}_r} \leq 0.\end{aligned} \tag{53}$$

Theorem 2: The estimation error $\tilde{x}$ satisfies $\mathcal{H}_\infty$ criterion (15) if there exist symmetric positive definite matrices $P \in \mathbb{R}^{n\times n}$, $Z_{ij} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $S_{ij} \in \mathbb{R}^{\bar{p}\times\bar{p}}$, symmetric positive semi-definite matrices $Z_{a_{ij}}, Z_{b_{ij}}, Z_{c_{ij}} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $S_{a_{ij}}, S_{b_{ij}}, S_{c_{ij}} \in \mathbb{R}^{\bar{p}\times\bar{p}}$ and a matrix $R \in \mathbb{R}^{p\times n}$, such that the following convex optimization problem is solvable:
$\min(\mu)$ subject to

$$\begin{bmatrix}\mathbb{L}_1 & \big(\mathbb{U}+\mathbb{Z}\mathbb{H}\Phi\big)^\top & \big(\mathbb{M}+\mathbb{S}\mathbb{G}\Psi\big)^\top\\ \star & -2\mathbb{Z} & 0\\ \star & \star & -2\mathbb{S}\end{bmatrix} < 0, \forall\Phi\in\mathcal{F}_m, \forall\Psi\in\mathcal{G}_r \tag{54}$$

where the matrices $\mathbb{L}_1, \mathbb{Z}, \mathbb{S}, \mathbb{U}, \mathbb{M}, \mathbb{H}, \mathbb{G}, \Phi$ and Ψ are illustrated in (24), (27), (28), (29), (31), (37), (39), (38) and (40), respectively. The observer gain is then obtained as $L = P^{-1}R^\top$.

Proof: The Schur complement of (53) yields the LMI (54). From convexity principal [8], the $\mathcal{H}_\infty$ criterion (15) is fulfilled by estimation error dynamics (19) if LMI (54) is solved for all $\Phi \in \mathcal{F}_m$ and $\Psi \in \mathcal{G}_r$. ∎

C. Some comments

The introduction of the matrices $\mathbb{Z}$ and $\mathbb{S}$ in LMI (48) and (54) allows the inclusion of additional numbers of decision variables. Hence, it is essential to calculate the exact number of decision variables and compare them to one obtained in the existing methods described in the literature.

Both LMIs contain the following number of decision variables:

$$\mathrm{N}_{dv_1} = np + \frac{n(n+1)}{2} + q + \mathcal{N}_{\mathrm{add}_1} + \mathcal{N}_{\mathrm{add}_2}, \tag{55}$$

where,

$$\begin{aligned}\mathcal{N}_{\mathrm{add}_1} &= \big(4m\bar{n} - 2m - 2\bar{n} + 1\big)\left(\frac{\bar{n}(\bar{n}+1)}{2}\right),\\ \mathcal{N}_{\mathrm{add}_2} &= \big(4r\bar{p} - 2r - 2\bar{p} + 1\big)\left(\frac{\bar{p}(\bar{p}+1)}{2}\right).\end{aligned} \tag{56}$$

$\mathcal{N}_{\mathrm{add}_1}$ and $\mathcal{N}_{\mathrm{add}_2}$ are the number of variables obtained from matrices $\mathbb{Z}$ and $\mathbb{S}$, respectively. Thus, LMI (48) and (54) have total $\mathrm{N}_{\mathrm{add}_1} = \mathcal{N}_{\mathrm{add}_1} + \mathcal{N}_{\mathrm{add}_2}$ additional number of decision variables. However, if we use block-diagonal matrix multipliers (similar to [6]) in the proposed LMIs, then we obtain the following number of decision variables:

$$\mathrm{N}_{dv_2} = np + \frac{n(n+1)}{2} + q + \underbrace{m\left(\frac{\bar{n}(\bar{n}+1)}{2}\right) + r\left(\frac{\bar{p}(\bar{p}+1)}{2}\right)}_{\mathrm{N}_{\mathrm{add}_2}}. \tag{57}$$

In (57), $\mathrm{N}_{\mathrm{add}_2}$ represents the number of variables obtained from block-diagonal matrices.

By comparing N_{dv_1} and N_{dv_2}, LMI (48) and (54) have more decision variables than the LMIs with block-diagonal matrices. From (55) and (57), we get $\mathrm{N}_{\mathrm{add}_2} \leq \mathrm{N}_{\mathrm{add}_1}$. It interprets that proposed matrices have more variables than the other matrices used in the literature. These additional variables may improve the feasibility of LMI. In the next section, the relaxation in LMI conditions due to the matrix multipliers is highlighted through an example.

V. Numerical Comparisons

The effectiveness of the proposed LMI is emphasized in this section with a numerical example. Consider a second-order system under the form of (11) with following parameters: $A = \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$, $G = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $B = \begin{bmatrix}1\\ 0\end{bmatrix}$, $C = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $E = \begin{bmatrix}-1\\ -1\end{bmatrix}$, $D = \begin{bmatrix}-1\\ -1\end{bmatrix}$ $F = \begin{bmatrix}1\\ 0\end{bmatrix}$, $f(x) = \begin{bmatrix}f_1(x)\\ f_2(x)\end{bmatrix} = \begin{bmatrix}sin(\theta x_1)\\ sin(\theta x_2)\end{bmatrix}$, $g(x) = sin(\lambda x_1)sin(\lambda x_2)$, $H_1 = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$, $H_2 = \begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}$ and $F_1 = \mathbb{I}_2$. Hence, $m = 2$, $\bar{n} = 2$, $r = 1$ and $\bar{p} = 2$.

The partial derivatives of f and g fulfill following inequalities:

$$-\theta \leq \frac{\partial f_1}{\partial x_1} \leq \theta, \; -\theta \leq \frac{\partial f_2}{\partial x_2} \leq \theta,$$
$$-\lambda \leq \frac{\partial g}{\partial x_1} \leq \lambda \text{ and } -\lambda \leq \frac{\partial g}{\partial x_2} \leq \lambda.$$

From Remark 1, we obtain:

$$\tilde{f}_{a_{11}} = 0; \tilde{f}_{b_{11}} = 2\theta; \tilde{f}_{a_{21}} = 0; \tilde{f}_{b_{21}} = 2\theta;$$
$$\tilde{g}_{a_{11}} = 0; \tilde{g}_{b_{11}} = 2\lambda; \tilde{g}_{a_{12}} = 0; \tilde{g}_{b_{12}} = 2\lambda;$$

Hence, system nonlinearities satisfy (17) and (18).

Further, we will consider the following cases to test the feasibility of LMIs:

1) **Case 1** : LMI (54) with proposed matrices,

$$\mathbb{Z} = \begin{bmatrix}Z_{11} & Z_{b_{21}}\\ Z_{b_{21}} & Z_{21}\end{bmatrix}; \; \mathbb{S} = \begin{bmatrix}S_{11} & S_{a_{12}}\\ S_{a_{12}} & S_{12}\end{bmatrix} \tag{58}$$

where $Z_{11} = Z_{11}^\top > 0, Z_{21} = Z_{21}^\top > 0, Z_{b_{21}} = Z_{b_{21}}^\top \geq 0 \in \mathbb{R}^{\bar{n}\times\bar{n}}$, and $S_{11} = S_{11}^\top > 0$, $S_{12} = Z_{12}^\top > 0, S_{a_{12}} = S_{a_{12}}^\top \geq 0 \in \mathbb{R}^{\bar{p}\times\bar{p}}$.

2) **Case 2** : LMI (48) with matrices defined in (58).

3) **Case 3** : LMI (48) with the matrices proposed in [11], i.e.,

$$\mathbb{Z} = \begin{bmatrix}Z_{11} & \alpha Z_{21}\\ \alpha Z_{21} & Z_{21}\end{bmatrix}; \; \mathbb{S} = \begin{bmatrix}S_{11} & \beta S_{12}\\ \beta S_{12} & S_{12}\end{bmatrix},$$

where $\alpha = 0.1, \beta = 0.1$, $Z_{11} = Z_{11}^{\top} > 0, Z_{21} = Z_{21}^{\top} > 0\mathbb{R}^{\bar{n}\times\bar{n}}$, and $S_{11} = S_{11}^{\top} > 0$, $S_{12} = Z_{12}^{\top} > 0 \in \mathbb{R}^{\bar{p}\times\bar{p}}$.

4) **Case 4** : LMI approach presented in [6].

TABLE I

Optimal values of $\sqrt{\mu}$ for different cases

	Case 1	**Case 2**	**Case 3**	**Case 4**
$\theta = 0.2$ and $\lambda = 0.3$	1.3731	2.1704	3.0151	1.3737
$\theta = 0.5$ and $\lambda = 0.1$	1.5282	2.0772	2.1778	1.5958
$\theta = 0.9$ and $\lambda = 0.25$	1.9875	6.3298	7.6210	2.0380
$\theta = 0.7$ and $\lambda = 1$	3.2500	8.3932	infeasible	10.6600
$\theta = 0.85$ and $\lambda = 1.5$	7.1515	10.1047	infeasible	33.8465

For all the above cases, LMIs are solved in the MATLAB toolbox, and the obtained optimal values of $\sqrt{\mu}$ are summarized in Table I. It highlights that LPV-based LMI (54) provides a better solution compared to other cases. In addition to this, the solution obtained from the LMI condition (48) is better than the one with matrices proposed in [11]. The matrix defined in [6] and [11] are particular forms of the matrices $\mathbb{Z}$ and $\mathbb{S}$, which are defined in (58). Thus, the solution provided in case 3 is the particular solution of the LMI (48). Hence, it is obvious that the derived LMI condition (48) is more generalized and provides larger sets of solutions. This is due to the number of additional decision variables. Furthermore, the LPV-based LMI (54) is less conservative than LMI (48) because of relationship between two Young's inequality (5) and (6). Therefore, the combination of the LPV approach with the proposed matrices helps to relax the existing LMI conditions in terms of feasibility and noise attenuation.

VI. Conclusion

In this paper, two new LMI techniques for the design of a nonlinear observer are presented. The key element of the proposed methods is the use of a generalized matrix multiplier, which leads to an additional number of decision variables. Such extra variables lead to additional degrees of freedom, thus improving the LMI feasibility. Further, the proposed novel matrix multiplier technique is combined with the LPV-based approach, which enhances the feasibility of the previous LMI conditions. Numerical comparisons are provided to show the validity and superiority of the proposed design methods compared to existing results in the literature.

As future work, we aim to extend the techniques proposed in this paper to the problem of observer-based control design and reference trajectory tracking. Indeed, in such a more general situation, coupling between the observer gains and the controller gains lead to bilinear terms which are difficult to linearize. Then, the objective consists in using our matrix multiplier based approach to convert such bilinear inequalities into non conservative LMI conditions. We also aim to exploit the proposed techniques for real-world applications, namely connected and autonomous vehicles, where there are several unknown input variables to be estimated together with the system states.

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