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Static Output Feedback for a Certain Class of Systems of Order Four

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***Abstract*— An outstanding, albeit important, problem in control theory is the static output feedback problem. It deals with the case when one can neither measure nor actuate all state variables and exerts control without the use of a state observer. In this paper, we consider systems of order 4, where at least 2 states are measured and also directly actuated, and present means that provide a definite answer to the question whether a stabilising static output feedback exists or not. We characterise cases, where such a control law cannot exist, and show that, for all other cases, either the answer can be provided by means of using semidefinite and linear programming or that, for surprisingly many cases, a stabilising static output feedback always exists. Finally, we show that, for many cases, the stabilising feedback can be obtained by solving a semidefinite or linear programme, outperforming off-the-shelf solutions.**

I. INTRODUCTION

Solving the static output feedback problem is central to control theory, since often one cannot measure all state variables. An early survey on the topic from 1994 calls the output feedback problem “probably the most important open question in control engineering” [1], where early approaches had all in common that they searched for solutions of matrix equations or inequalities. A more recent survey from 2016 still considers the problem theoretically challenging, of great importance in practice, and worthy of the great attention that it receives from the control community [2]. Particularly, the survey concludes by highlighting that, still, neither an exact solution to this prominent design problem exists nor ways to guarantee the existence or non-existence of such a feedback and by classifying all main methods to solve the problem. On the one hand, notable recent approaches that provide solutions to the problem, albeit without guarantee, are presented in [3], [4], and [5], where the latter introduces the HIFOO toolbox. On the other hand, the authors of [2] believe that purely convex results are of much importance, even if they are only for systems with a specific structure. It is in this area that the contribution of this paper lies.

In the following, we present different convex results from the literature, of which not many exist to our knowledge. Traditionally, for a linear dynamical system given by

$$\begin{aligned}&\dot{x} = Ax + Bu,\ y = Cx,\\ &x \in \mathbb{R}^n,\ u \in \mathbb{R}^p,\ y \in \mathbb{R}^q,\\ &A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times p},\ C \in \mathbb{R}^{q\times n},\end{aligned} \tag{1}$$

where x is the system state with n state variables, u the input, A the system matrix representing the system dynamics, B the input matrix, y the output, and C the output matrix, if either matrix B or matrix C has rank n then the static output feedback problem can be solved efficiently. This is also true if there exists a matrix $\hat{P}$ such that $PB = B\hat{P}$ or if there exists a matrix $\hat{Q}$ such that $\hat{Q}C = CQ$, where P and Q are positive definite matrices, which we denote by $P \succ 0$ and $Q \succ 0$, respectively [6], [7]. The authors of [8] extend the latter case to include static linear parameter-varying output feedback controllers.

Nonnegative dynamical system models are derived from mass and energy balance considerations that involve dynamic states whose values are nonnegative. A known result for nonnegative linear dynamical systems of the form $\dot{x} = Ax$ is that they are asymptotically stable if and only if there exists a positive vector p such that $p^{\mathrm{T}}Ax > 0$ for all nonnegative x, $x \neq 0$ [9]. In [10], the authors provide an approach that is based on linear programming for the design of an L_1-optimal controller for nonnegative linear dynamical systems with single input or single output. In [11], the authors provide necessary and sufficient conditions for the existence of a static output feedback for plants of minimum phase if CB has full row-rank. In a recent paper [12], for a certain class of linear dynamical systems, we present sufficient conditions for the existence of a stabilising control gain matrix. The class of plants considered is of those ones, for which we can measure at least half of all state variables and where those measurements affect at least half of all state variables. Note that this class is different from the one considered in [11].

In this paper, for systems of order 4, where at least 2 states are measured and also directly actuated, we present efficient means to provide a definite answer to the question whether a stabilising static output feedback exists or not. For most cases, also the controller can be synthesised efficiently. The outline of the paper is the following. Section I-A formulates the problem considered. Section II briefly presents previous work relevant to this paper. Section III-A provides necessary and sufficient conditions for a stabilising matrix K to exist for linear dynamical systems of order 4, where at least 2 states are measured and also directly actuated. Section III-B shows how the conditions can be checked using semidefinite programming and linear programming. We perform numerical experiments that highlight the significance of our results in Section IV-A, and also apply them to an aircraft model from the literature (Section IV-B). We conclude the paper in Section V and deal with special cases in the Appendix.

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A. Problem Statement

Static output feedback seeks control gain matrix K that stabilises system (1) by closing the loop through the feedback law given by $u = Ky = KCx$, $K \in \mathbb{R}^{p\times q}$. In this paper, if the real part of all eigenvalues is negative then we say that the matrix is stable. To understand why the output feedback problem is difficult, let

$$\dot{x} = (A + BKC)\, x = A_F x. \tag{2}$$

It follows from Lyapunov stability theory [13] that the origin of closed loop system (2) is globally asymptotically stable if and only if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$A_F P + P A_F^{\mathrm{T}} \prec 0. \tag{3}$$

See also Theorem 2 in [14]. However, finding matrices K and P, where $P \succ 0$, such that the inequality given by (3) holds, requires to solve a so-called bilinear matrix inequality, which is known to be NP-hard to solve [15], [16], [17]. Thus, finding a matrix K such that the system given by (2) is globally asymptotically stable is difficult.

In [12], for the certain class of systems under consideration, we also show that we can determine the stabilising control gain matrix from the solution of a linear matrix inequality and that for random matrices A of dimension n (`A=randn(n)` in MATLAB [18]), $C = \begin{bmatrix} I_{\frac{n}{2}} & 0 \end{bmatrix}$, where $I_{\frac{n}{2}}$ denotes identity matrix of dimension $\frac{n}{2}$, and $B = C^{\mathrm{T}}$, we obtain a stabilising feedback matrix $K \in \mathbb{R}^{\frac{n}{2}\times\frac{n}{2}}$ in many cases (see Table I). Notably, for a random matrix A such a matrix K does not necessarily exist. Thus, we conclude the paper by asking, under which circumstances are the sufficient conditions stated in Theorem 1 (see Subsection II) also necessary?

While [12] provides only sufficient conditions, in this paper, if $n = 4$ and there exist real 4-dimensional transformation matrices T, T_1, and T_2 such that (4) holds and $\bar{K} \in \mathbb{R}^{2\times 2}$ has full rank, then we provide necessary and sufficient conditions for the existence of a feedback matrix K that stabilises the system given by (2). That is, we provide a certificate that guarantees that either a feedback matrix K that stabilises (2) exists or that it does not. We use linear programming and semidefinite programming to obtain the certificate and, in many cases, also matrix K; otherwise, we use `fmincon` to obtain K (through the YALMIP toolbox [19]).

TABLE I

THE NUMBER OF TIMES THAT A SOLUTION WAS FOUND SOLVING PROBLEM (10), WHERE n IS THE DIMENSION OF RANDOM MATRIX A.

n	N	# of solutions using (10)
4	100	82
6	100	97
20	30	30
50	10	10
100	10	9

II. Previous Work

In this section, we summarise those parts of our previous work [12] that are relevant to this paper. If n is even, $p = q = \frac{n}{2}$, and real n-dimensional transformation matrices T, T_1, and T_2 exist such that

$$\bar{A}_F = T_2 T T_1 A_F T_1^{-1} T^{-1} T_2^{-1} = \begin{bmatrix} A_{11} + \bar{K} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{4}$$

where A_{11}, A_{12}, A_{21}, A_{22}, and $\bar{K}$ are $\in \mathbb{R}^{\frac{n}{2}\times\frac{n}{2}}$, then it follows from Theorem 3.2 in [20] that matrix $\bar{A}_F$ is stable if and only if $\bar{K} + A_{11} + A_{12}R$ and $A_{22} - RA_{12}$ are stable, where

$$R\bar{K} + RA_{11} + RA_{12}R = A_{21} + A_{22}R,\ R \in \mathbb{R}^{\frac{n}{2}\times\frac{n}{2}}. \tag{5}$$

Note that free variable matrix $\bar{K}$ has full rank. It follows that if R is nonsingular then $\bar{K} + A_{11} + A_{12}R$ is stable if and only if

$$R\left(R^{-1}A_{21} + R^{-1}A_{22}R\right)R^{-1} = A_{21}R^{-1} + A_{22} \tag{6}$$

is stable. Additionally, if matrix A_{21} is nonsingular then we pose the following theorem [12].

Theorem 1: If matrix A_{21} is nonsingular and there exist positive definite matrices $Q_3 \in \mathbb{R}^{\frac{n}{2}\times\frac{n}{2}}$ and $Q_4 \in \mathbb{R}^{\frac{n}{2}\times\frac{n}{2}}$ such that

$$Q_3 A_{12} + Q_4 A_{22} + A_{12}^{\mathrm{T}} Q_3 + A_{22}^{\mathrm{T}} Q_4 \prec 0, \tag{7}$$

$$Q_3 A_{21}^{-1} A_{22} A_{21} - Q_4 A_{21} + (\cdots)^{\mathrm{T}} \prec 0, \tag{8}$$

and $R = -Q_4^{-1} Q_3$ then matrix $\bar{A}_F$ in (4) is stable for

$$\bar{K} = R^{-1}A_{21} + R^{-1}A_{22}R - A_{12}R - A_{11}. \tag{9}$$

Note that this result can be equivalently applied to matrices A^T, C^T, and B^T instead of matrices A, B, and C, respectively, for example, if A_{21} is singular but A_{12} is nonsingular. Moreover, the conditions in Theorem 1 can be implemented as the following semidefinite programme:

$$\begin{aligned} \text{given} \quad & A_{12},\ A_{21},\ A_{22}, \\ \text{find} \quad & Q_3,\ Q_4, \\ \text{sub. to} \quad & Q_3 \succ 0,\ Q_4 \succ 0,\ (7)\ \&\ (8). \end{aligned} \tag{10}$$

Problems such as the one presented in (10) can be efficiently solved using the MATLAB toolbox YALMIP and the SeDuMi solver [21].

III. Main Results

In this section, we present the main results of this paper. Note that for the reminder of the paper, $n = 4$.

A. Sufficient and Necessary Condition for $n = 4$

First, we assume that B and C have, both, rank 2. Thus, by Lemma 9 in [12], without loss of generality, we can assume that $p = q = 2$. Furthermore, we assume that the system under consideration can be transformed such that, with slight abuse of notation, A_F is given by (4), where $\bar{K}$ has full rank.

For clarity of presentation, in the following, we assume that matrix A_{21} is nonsingular. (In Appendix B, we discuss the case when A_{21} is singular.) If A_{21} is nonsingular then, again, with slight abuse of notation, let $A_F =$

$$SA_FS^{-1} = \begin{bmatrix} A_{11}+\bar{K} & A_{12} \\ I_2 & A_{22} \end{bmatrix}, \ S = \begin{bmatrix} A_{21} & 0 \\ 0 & I_2 \end{bmatrix}. \tag{11}$$

Then, A_F is stable if and only if matrices $A_{22}+R^{-1}$ and $A_{22}-RA_{12}$ are stable, where $A_{21}=I_2$ in (5).

Now, let $A_{22} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $A_{12} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, and $R = \begin{bmatrix} r_1 & r_3 \\ r_2 & r_4 \end{bmatrix}$, where a_i, b_i, and r_i are $\in \mathbb{R}$ and $i = 1,2,3,4$. Then, the characteristic polynomials of matrices $A_{22}+R^{-1}$ and $A_{22}-RA_{12}$ are given by

$$\lambda_{1,2}^2 + \frac{f_1}{c}\lambda_{1,2} + \frac{f_2}{c} \tag{12}$$

and

$$\lambda_{3,4}^2 + f_3\lambda_{3,4} + f_4, \tag{13}$$

respectively, where $\lambda_{1,2}$ and $\lambda_{3,4}$ denote the eigenvalues of matrix $A_{22}+R^{-1}$ and matrix $A_{22}-RA_{12}$, respectively,

$$\begin{aligned} f_1 &= -(r_1+r_4+a_1c+a_4c), \\ f_2 &= a_1r_1+a_2r_2+a_3r_3+a_4r_4+a_1a_4c-a_2a_3c+1, \\ f_3 &= b_1r_1+b_2r_2+b_3r_3+b_4r_4-a_1-a_4, \\ f_4 &= (a_3b_2-a_4b_1)r_1+(a_2b_1-a_1b_2)r_2 \\ &\quad +(a_3b_4-a_4b_3)r_3+(a_2b_3-a_1b_4)r_4 \\ &\quad +(b_1b_4-b_2b_3)c+a_1a_4-a_2a_3, \end{aligned} \tag{14}$$

and

$$c = r_1r_4 - r_2r_3. \tag{15}$$

It follows from the Routh-Hurwitz criterion [13] that matrices $A_{22}+R^{-1}$ and $A_{22}-RA_{12}$ are stable if and only if either

$$f_1>0, \ f_2>0, \ f_3>0, \ f_4>0, \ c>0 \tag{16}$$

or

$$f_1<0, \ f_2<0, \ f_3>0, \ f_4>0, \ c<0. \tag{17}$$

Remark 1: Note that one can use MATLAB's general purpose optimiser `fmincon` [18] to solve (14), (15), and either (16) or (17); however, this approach is not guaranteed to provide a solution even if it exists.

Remark 2: Note that the functions in (14) are affine in r. In the following, this allows us to rewrite them as in (18). For even $n>4$, they become nonlinear in r and cannot be rewritten as in (18). Furthermore, note that the case $n=2$ is (relatively) trivial.

B. Certificate for Stabilisability

In Theorem 2, we provide efficient means to determine whether a solution to (15), under the constraints given by (14) and either (16) or (17), exists. First, we use matrix and vector notation to rewrite (14) as

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_4 \end{bmatrix} = Fr + vc + d, \ r = \begin{bmatrix} r_1 \\ \vdots \\ r_4 \end{bmatrix}, \tag{18}$$

where $F =$

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_3b_2-a_4b_1 & a_2b_1-a_1b_2 & a_3b_4-a_4b_3 & a_2b_3-a_1b_4 \end{bmatrix},$$

$$v = \begin{bmatrix} -a_1-a_4 \\ a_1a_4-a_2a_3 \\ 0 \\ b_1b_4-b_2b_3 \end{bmatrix}, \ d = \begin{bmatrix} 0 \\ 1 \\ -a_1-a_4 \\ a_1a_4-a_2a_3 \end{bmatrix}. \tag{19}$$

If matrix F is nonsingular then we rewrite condition (15) as

$$\begin{aligned} 0 &= c - r_1r_4 + r_2r_3 = z_5 - r_1r_4 + r_2r_3 \\ &= z^{\mathrm{T}}Ez + z^{\mathrm{T}}G^{\mathrm{T}} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} Gz \\ &= z^{\mathrm{T}}Hz, \\ r &= F^{-1}(f-vc-d) = Gz, \\ G &= F^{-1}\begin{bmatrix} I_4 & -v & -d \end{bmatrix}, \ z = \begin{bmatrix} f \\ c \\ 1 \end{bmatrix}, \end{aligned} \tag{20}$$

where matrix $E \in \mathbb{R}^{6\times 6}$ is empty but for $E_{56}=1$. We also rewrite condition (16) as $z>0$, which means that $z_i>0$ for all i, $i=1,\ldots,6$, and we let $z_6=1$. Thus, if we require (14) to (16) then we rewrite these conditions as

$$z^{\mathrm{T}}Hz = 0, \ z>0. \tag{21}$$

If we require (14), (15), and (17) then we rewrite these as

$$z^{\mathrm{T}}\bar{H}z = 0, \ z>0, \bar{H} = DHD, \ D = \mathrm{diag}\begin{bmatrix} -1 \\ -1 \\ +1 \\ +1 \\ -1 \\ +1 \end{bmatrix}. \tag{22}$$

Before stating the theorem, we need the following definition.

Definition 1: A matrix $A \in \mathbb{R}^{n\times n}$ is strictly co-positive (co-negative) if $y^{\mathrm{T}}Ay>0$ ($y^{\mathrm{T}}Ay<0$) for all positive y, that is, $y_i>0$ for all i, $i=1,\ldots,n$.

Theorem 2: Condition (21) (or (22)) holds if and only if H ($\bar{H}$) is neither strictly co-positive nor strictly co-negative.

Proof: The property stated in the theorem follows directly from Definition 1 and continuity of $z^{\mathrm{T}}Hz$ ($z^{\mathrm{T}}\bar{H}z$).

In Appendix A, we discuss the case when F is singular. Now, using Theorem 2 we prove the following.

Theorem 3: If matrix F is nonsingular and either A_{22} or A_{12} has real nonzero eigenvalues then a matrix K that stabilises A_F always exists.

Proof: First, we assume that A_{22} has real nonzero eigenvalues and apply a similarity transformation to matrix A_F such that matrix F is given by

$$F = \begin{bmatrix} -1 & 0 & 0 & -1 \\ a_1 & 0 & 0 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ -a_4b_1 & -a_1b_2 & -a_4b_3 & -a_1b_4 \end{bmatrix},$$

where a_1 and a_4 are the real nonzero eigenvalues of A_{22}. Now, evaluating matrix H reveals that

$$\left[\begin{array}{cc} H_{33} & H_{34} \\ H_{43} & H_{44} \end{array}\right] = \left[\begin{array}{cc} 2a_1a_4 & a_1+a_4 \\ a_1+a_4 & 2 \end{array}\right]\alpha, \tag{23}$$

where $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Since F is nonsingular, $a_1 \neq a_4$, and, thus, the matrix given by (23) has a negative and a positive eigenvalue. Therefore, matrix H is neither strictly co-positive nor strictly co-negative and, by Theorem 2, a matrix K that stabilises A_F exists. If A_{12} has real nonzero eigenvalues then the proof is analogous and, thus, omitted.

If A_{22} and A_{12} have, both, complex eigenvalues then A_F might not be stabilisable. For instance, consider

$$A_{12} = \left[\begin{array}{cc} 1.886 & 0.333 \\ -0.333 & 1.886 \end{array}\right],\ A_{22} = \left[\begin{array}{cc} 0.157 & -1.454 \\ 1.266 & 1.030 \end{array}\right].$$

In general, to provide a definite answer to whether condition (21) (or (22)) holds or not and, thus, to whether a static output feedback that stabilises the system given by (1) exists or not, we use Theorem 2 and the following two theorems from [22] and [23], where we note that $z^{\mathrm{T}}Hz = \frac{1}{2}z^{\mathrm{T}}\left(H+H^{\mathrm{T}}\right)z$ and that $H+H^{\mathrm{T}}$ is symmetric.

Theorem 4: A symmetric matrix A of order $n \leq 4$ is strictly co-positive if and only if it is decomposable as the sum $A = A_1 + A_2$ of a positive definite symmetric matrix A_1 and a nonnegative symmetric matrix A_2 (that is, $A_2 \in \overline{\mathbb{R}}^n_+$). Thus, matrix $A \in \mathbb{R}^4$ is strictly co-positive if and only if the following semidefinite programme has a solution:

$$\begin{aligned} \text{given} \quad & A, \\ \text{find} \quad & A_1,\ A_2, \\ \text{sub. to} \quad & A_1 \succ 0,\ A_2 \in \overline{\mathbb{R}}^4_+,\ A_2 = A_2^{\mathrm{T}},\ A = A_1 + A_2. \end{aligned} \tag{24}$$

Theorem 5: Let A be a symmetric matrix of order n such that each principal sub-matrix of order $n-1$ is strictly co-positive. Then A is strictly co-positive if and only if the minimum of the following linear programme is positive:

$$\begin{aligned} \text{minimise} \quad & t_1 - t_2, \\ \text{sub. to} \quad & y_1 + \cdots + y_n = 1, \\ & a_i^{\mathrm{T}}y - t_1 + t_2 + s_i = 0,\ i = 1,\ldots,n, \\ & y \in \mathbb{R}^n,\ s \in \mathbb{R}^n,\ y_i > 0\ \forall i,\ s_i > 0\ \forall i, \\ & t_1 > 0,\ t_2 > 0, \end{aligned} \tag{25}$$

where a_i denotes the ith column of A.
Importantly, efficient solvers exist for solving (24) and (25).

IV. Numerical Experiments and Applications

A. Numerical Experiments

We created $N = 1000$ random matrices $A \in \mathbb{R}^{4\times4}$ and applied different approaches to search for a matrix K that stabilises (1) for $C = \left[\begin{array}{cc} I_{\frac{n}{2}} & 0 \end{array}\right]$ and $B = C^{\mathrm{T}}$ (see Table II). When solving (10) or using the gradient-based HIFOO toolbox [5] to find a stabilising matrix K, where we allowed up to ten initialisations for the latter, we obtained a solution in 854 or 956 of the cases, respectively.

When using MATLAB's general purpose optimiser `fmincon` to solve for (14), (15), and either (16) or (17), we obtained 993 solutions. Note that we used the algorithms `interior-point`, `sqp`, as well as `active-set` when employing `fmincon` in the search for a solution but, otherwise, have not changed the function settings. For the remaining 7 cases, we checked whether H ($\bar{H}$) is either strictly co-positive or strictly co-negative. Specifically, we searched for a feasible solution to programme (24) for all 4×4 principal sub-matrices of H ($\bar{H}$) as well as for a feasible solution to programme (25) for all 5×5 principal sub-matrices of H ($\bar{H}$) and for H ($\bar{H}$) itself. We obtained a certificate that neither (21) nor (22) hold only for 4 of these cases, which means that the associated system matrices, A, cannot be stabilised by means of static output feedback.

For the remaining 3 cases, we used `fmincon` to solve the following problem to obtain K:

$$\begin{aligned} \text{find} \quad & w, \\ \text{sub. to} \quad & w\tilde{H}w > 0 \quad \text{or} \quad -w\tilde{H}w > 0, \\ & w_i > 0\ \forall i, \end{aligned} \tag{26}$$

where $\tilde{H}$ is the non-strictly-co-positive or non-strictly-co-negative principal sub-matrix of H ($\bar{H}$), for which either programme (24) or programme (25) failed to find a solution. (Note that `fmincon` failed to solve the minimisation problem given by (26) for $\tilde{H} = H$.) Finally, we repeated the numerical experiments for diagonal matrices A_{12} and A_{22} ($N = 100$). Significantly, the linear programme given by (38) was not only guaranteed to find a solution if it existed, it also outperformed other approaches (see Table II).

B. Lateral-directional Aircraft Dynamics

We consider the lateral-directional dynamics of a passenger aircraft in cruise configuration given by $\dot{x} = Ax + Bu$ from [24]. The system states are roll angle ϕ and sideslip angle β and the respective angular rates p and r. State vector x is given by $x^{\mathrm{T}} = \left[\begin{array}{cccc} p & r & \phi & \beta \end{array}\right]$, where angles are in radians and angular rates are in rad/s. The input consists of aileron and rudder deflections, which are in radians, and is given by $u^{\mathrm{T}} = \left[\begin{array}{cc} \delta_a & \delta_r \end{array}\right]$. System matrix A and input

TABLE II

For N runs, we provide the number of times that a solution was found solving (10), using the approach from [5], solving (14) – (15) and either (16) or (17) using `fmincon`, or solving (24) – (26); for A_{12} and A_{22} being both diagonal, we compare the performance when solving (10), using the approach from [5], solving (14) – (15) and either (16) or (17), or solving (38).

Solving $N = 1000$	(10)	[5]	(14) – (16) / (17)	(24) – (26)
# of solutions	854	956	993	996
Solving $N = 100$	(10)	[5]	(14) – (16) / (17)	(38)
# of solutions	52	73	55	75

matrix B are given by

$$A = \begin{bmatrix} -1.699 & 0.1717 & 0 & -4.546 \\ -0.0654 & -0.0893 & 0 & 3.382 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0.0487 & -0.0829 \end{bmatrix}, \quad (27)$$

$$B = \begin{bmatrix} 27.276 & 0.5758 \\ 0.3952 & -1.362 \\ 0 & 0 \\ 0 & 0.0116 \end{bmatrix}. \quad (28)$$

For $y^{\mathrm{T}} = \begin{bmatrix} p & r \end{bmatrix}$, we solve (10) such that $u = Ky$ stabilises the system, where $K = \begin{bmatrix} -0.0538 & 0.1084 \\ -0.8569 & 3.3839 \end{bmatrix}$.

V. DISCUSSION AND CONCLUSIONS

In this paper, for a specific class of plants, we answer the question presented in [2], whether there are ways to provide guarantees either that a static output feedback exists or that it does not exist. Specifically, to the best of our knowledge, for the first time, for the class of four-dimensional systems under consideration, we provide a certificate that guarantees that either a feedback matrix K that stabilises the system given by (2) exists or that it does not. Significantly, the certificate can be obtained efficiently by means of semidefinite and linear programming, which lets our approach outperform other approaches. Interestingly, in this paper as well as in our previous one [12], we observe that the special systems under consideration are almost always stabilisable. In the future, we will investigate the implication of this observation. Finally, we note that we provide the necessary feasible initial condition for methods that seek to solve bilinear matrix inequalities for optimal and robust control such as the one presented in [25].

APPENDIX

A. Matrix F is Singular

Let us transform matrix A_F as in the following. With a slight abuse of notation, we let

$$A_F = \tilde{V} A_F \tilde{V}^{-1},\ \tilde{V} = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix},\ V \in \mathbb{R}^{2\times 2}, \quad (29)$$

such that either

$$A_{12} = \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \end{bmatrix}, \quad (30)$$

where b_1 and b_4 are the real eigenvalues of matrix A_{12}, or

$$A_{12} = \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{bmatrix}, \quad (31)$$

where $b_1 \pm ib_2$ are the complex eigenvalues of matrix A_{12}. It follows that if (30) holds then

$$F = \begin{bmatrix} -1 & 0 & 0 & -1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & 0 & 0 & b_4 \\ -a_4 b_1 & a_2 b_1 & a_3 b_4 & -a_1 b_4 \end{bmatrix} \quad (32)$$

and if (31) holds then

$$F = \begin{bmatrix} -1 & 0 & 0 & -1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & -b_2 & b_1 \\ a_3 b_2 - a_4 b_1 & a_2 b_1 - a_1 b_2 & a_3 b_1 + a_4 b_2 & -a_2 b_2 - a_1 b_1 \end{bmatrix}. \quad (33)$$

1) Matrix F has Rank 3: Before showing that if matrix F has rank 3 then a matrix K that stabilises A_F can be efficiently determined, we need the following two propositions.

Proposition 1: If matrix F has rank 3 then it is given by (32) and either $a_2 = 0$ or $a_3 = 0$.

Proof: First, if, for example, $a_2 = 0$ and $a_3 = 1$ or $a_2 = 1$, $a_3 = 0$, $a_1 = a_4 = b_1 = 1$, and $b_4 = 4$ then matrix F, given by (32), has rank 3. Second, if neither $a_2 = 0$ nor $a_3 = 0$ and F, given by (32), is singular then there must exist a real nonzero scalar α such that $\alpha a_2 + a_2 b_1 = \alpha a_3 + a_3 b_4 = 0$, which is only possible, for $a_2 \neq 0$ and $a_3 \neq 0$, if $\alpha = -b_1$ and $\alpha = -b_4$, which implies that $b_4 = b_1$. Then,

$$F = \begin{bmatrix} -1 & 0 & 0 & -1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & 0 & 0 & b_1 \\ -a_4 b_1 & a_2 b_1 & a_3 b_1 & -a_1 b_1 \end{bmatrix} \quad (34)$$

and $n^{\mathrm{T}} F = \tilde{n}^{\mathrm{T}} F = 0$, where

$$n = \begin{bmatrix} -(a_1 b_1 + a_4 b_1) \\ -b_1 \\ 0 \\ 1 \end{bmatrix},\ \tilde{n} = \begin{bmatrix} 0 \\ -b_1 \\ \frac{a_1 b_1 + a_4 b_1}{b_1} \\ 1 \end{bmatrix}, \quad (35)$$

which implies that (34) is of rank < 3. Third, if matrix F is given by (33) and $b_2 \neq 0$ then, from looking at the first and third row of matrix F, it is singular only if either $Fn = 0$ or $F\tilde{n} = 0$, where

$$n = \begin{bmatrix} -1 \\ 0 \\ 0 \\ +1 \end{bmatrix},\ \tilde{n} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}. \quad (36)$$

Then, $a_4 = a_1$ and $a_3 = -a_2$ must also hold and, thus,

$$F = \begin{bmatrix} -1 & 0 & 0 & -1 \\ a_1 & a_2 & -a_2 & a_1 \\ b_1 & b_2 & -b_2 & b_1 \\ -a_2 b_2 - a_1 b_1 & a_2 b_1 - a_1 b_2 & a_1 b_2 - a_2 b_1 & -a_2 b_2 - a_1 b_1 \end{bmatrix} \quad (37)$$

has rank < 3. Finally, if $b_2 = 0$ then matrix F is given by (34) and, thus, is of rank < 3.

Proposition 2: If matrix F has an empty column then (14), (15), and either (16) or (17) can be solved through a linear programme.

Proof: Consider the following linear programme:

$$\begin{aligned} &\text{find} && f,\ c,\ r, \\ &\text{sub. to} && f = Fr + vc + d \ \ \&\ \ (16) \text{ or } (17). \end{aligned} \quad (38)$$

If the i-th column of matrix F is empty then (38) is independent of r_i. If (38) is feasible then we choose r_i such that, together with the solution of (38), (15) is solved.

Theorem 6: If matrix F has rank 3 and a matrix K that stabilises A_F exists then K can be obtained by solving (38).

Proof: If matrix F has rank 3 then, by Proposition 1, it is given by (32) and either $a_2 = 0$ or $a_3 = 0$. Thus, either the second or the third column of matrix F is empty. By Proposition 2, if a matrix K that stabilises A_F exists then it can be obtained by solving linear programme (38).

2) Matrix F has Rank 2 or 1: Let the rank of matrix F be 2 and matrices A_{12} and A_{22} be, both, diagonal. Note that (34) can be linearly transformed to have this form through a transformation similar to the one given by (29). Then,

$$F = \begin{bmatrix} -1 & 0 & 0 & -1 \\ a_1 & 0 & 0 & a_4 \\ b_1 & 0 & 0 & b_4 \\ -a_4 b_1 & 0 & 0 & -a_1 b_4 \end{bmatrix}. \tag{39}$$

By proposition 2, if a stabilising matrix K exists then such a matrix can be obtained by solving linear programme (38).

Now, let matrix F be given by (37). Then,

$$\begin{aligned} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} &= F_1 \begin{bmatrix} \tilde{r} \\ \bar{r} \end{bmatrix} + \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} c + \begin{bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \end{bmatrix}, \\ \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} &= F_2 \begin{bmatrix} \tilde{r} \\ \bar{r} \end{bmatrix} + \begin{bmatrix} \tilde{v}_3 \\ \tilde{v}_4 \end{bmatrix} c + \begin{bmatrix} \tilde{d}_3 \\ \tilde{d}_4 \end{bmatrix}, \end{aligned} \tag{40}$$

where

$$\begin{aligned} F_1 &= \begin{bmatrix} -1 & 0 \\ a_1 & a_2 \end{bmatrix}, \\ F_2 &= \begin{bmatrix} b_1 & b_2 \\ -a_2 b_2 - a_1 b_1 & a_2 b_1 - a_1 b_2 \end{bmatrix}, \\ \tilde{v} &= \begin{bmatrix} -2a_1 \\ a_1^2 + a_2^2 \\ 0 \\ b_1^2 + b_2^2 \end{bmatrix}, \ \tilde{d} = \begin{bmatrix} 0 \\ 1 \\ -2a_1 \\ a_1^2 + a_2^2 \end{bmatrix}, \end{aligned}$$

$\tilde{r} = r_1 + r_4$, and $\bar{r} = r_2 - r_3$. It follows that

$$\begin{aligned} \begin{bmatrix} \tilde{r} \\ \bar{r} \end{bmatrix} &= F_1^{-1} \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} c - \begin{bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \end{bmatrix} \right) \\ &= F_2^{-1} \left(\begin{bmatrix} f_3 \\ f_4 \end{bmatrix} - \begin{bmatrix} \tilde{v}_3 \\ \tilde{v}_4 \end{bmatrix} c - \begin{bmatrix} \tilde{d}_3 \\ \tilde{d}_4 \end{bmatrix} \right) \end{aligned} \tag{41}$$

and that

$$\begin{aligned} 0 &= c - \tilde{r} r_4 + r_4^2 + \bar{r} r_3 + r_3^2 \\ &= \begin{bmatrix} r_3 & r_4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{\bar{r}}{2} \\ 0 & 1 & -\frac{\tilde{r}}{2} \\ \frac{\bar{r}}{2} & -\frac{\tilde{r}}{2} & c \end{bmatrix} \begin{bmatrix} r_3 \\ r_4 \\ 1 \end{bmatrix}, \end{aligned} \tag{42}$$

which has a real solution if either $c < 0$ or $c > 0$ and $h = \bar{r}^2 + \tilde{r}^2 - 4c \geq 0$. We formulate the first condition as the following linear programme:

$$\begin{aligned} &\text{find} && c, \\ &\text{sub. to} && \text{(41) and (17).} \end{aligned} \tag{43}$$

We formulate the second condition as the following semidefinite programme:

$$\begin{aligned} &\text{find} && c, \\ &\text{sub. to} && \bar{r}^2 + \tilde{r}^2 \leq 4c, \text{ (41), and (16).} \end{aligned} \tag{44}$$

Thus, a stabilising matrix K exists if and only if either (43) or (44) is feasible. Finally, the rank of matrix F is 1 if and only if $A_{12} = \alpha I_2$ and $A_{22} = \beta I_2$, where $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq 0$. If $\beta \geq 0$ then matrix $A_{22} + R^{-1}$ and matrix $A_{22} - RA_{12}$ are, both, stable if and only if $\alpha < -\beta$. If $\beta < 0$ then they are stable if we choose a stable matrix R, $R \approx 0$.

B. Matrix A_{21} is Singular

If matrix A_{21} is singular then we can apply a similarity transformation to the system such that matrix A_{21} of transformed matrix A_F is given by either $A_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $A_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, the characteristic polynomial of matrix $A_{22} + A_{21}R^{-1}$ is given by $\lambda_{1,2}^2 + \frac{f_1}{c}\lambda_{1,2} + \frac{f_2}{c}$, where

$$\begin{aligned} f_1 &= -(sr_i + a_1 c + a_4 c), \\ f_2 &= sa_3 r_j + sa_4 r_i + a_1 a_4 c - a_2 a_3 c, \end{aligned} \tag{45}$$

and either $i = 4$, $j = 3$, and $s = 1$ or $i = 2$, $j = 1$, and $s = -1$, depending on A_{21}. It follows that either

$$F = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_3 b_2 - a_4 b_1 & a_2 b_1 - a_1 b_2 & a_3 b_4 - a_4 b_3 & a_2 b_3 - a_1 b_4 \end{bmatrix} \tag{46}$$

or

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_3 & -a_4 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 \\ a_3 b_2 - a_4 b_1 & a_2 b_1 - a_1 b_2 & a_3 b_4 - a_4 b_3 & a_2 b_3 - a_1 b_4 \end{bmatrix}, \tag{47}$$

while vectors v and d are as before in (19), but for $d_2 = 0$. Note that both cases can be handled similarly. Thus, for brevity, we continue with (46) only. First, the following theorem shows that if matrix F is nonsingular then a matrix K that stabilises A_F always exists.

Theorem 7: If matrix A_{21} is singular and matrix F is nonsingular then matrix K that stabilises A_F always exists.

Proof: Evaluating matrix H reveals that

$$\begin{bmatrix} H_{22} & H_{24} \\ H_{42} & H_{44} \end{bmatrix} = \begin{bmatrix} 2(b_2 b_3 - b_1 b_4) & b_1 \\ b_1 & 0 \end{bmatrix} \alpha, \tag{48}$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Since F is nonsingular, if $b_1 = 0$ then $b_2 \neq 0$ and vice versa. Thus, matrix (48) has a negative and a positive eigenvalue, which, by Theorem 2, implies that matrix H is neither strictly co-positive nor strictly co-negative and that a matrix K that stabilises matrix A_F exists.

Next, note that if matrix F is singular and $a_3 \neq 0$ then either $a_1 = a_4 + a_2 \frac{b_1}{b_2} - a_3 \frac{b_2}{b_1}$, $b_1 = b_2 = 0$, or $a_2 = b_2 = 0$. The following theorem shows that if matrix F is singular, $a_3 \neq 0$, and $a_1 = a_4 + a_2 \frac{b_1}{b_2} - a_3 \frac{b_2}{b_1}$ then matrix K that stabilises A_F always exists.

Theorem 8: If A_{21} and F are singular, $a_3 \neq 0$, and $a_1 = a_4 + a_2 \frac{b_1}{b_2} - a_3 \frac{b_2}{b_1}$ then K that stabilises A_F always exists.

Proof: If matrix F is singular, $a_3 \neq 0$, and $a_1 = a_4 + a_2 \frac{b_1}{b_2} - a_3 \frac{b_2}{b_1}$ then, for $w = \begin{bmatrix} a_3 b_2 - a_4 b_1 \\ -b_1 \end{bmatrix}$ and $F_3 = \begin{bmatrix} b_3 & b_4 \\ a_3 b_4 - a_4 b_3 & a_2 b_3 - a_1 b_4 \end{bmatrix}$, the following holds:

$$\begin{bmatrix} r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ a_3 & a_4 \end{bmatrix}^{-1} \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} c \right) \tag{49}$$

and

$$0 = w^{\mathrm{T}} \begin{bmatrix} b_1 & b_2 \\ a_3 b_2 - a_4 b_1 & a_2 b_1 - a_1 b_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$
$$= w^{\mathrm{T}} \left(\begin{bmatrix} f_3 \\ f_4 \end{bmatrix} - F_3 \begin{bmatrix} r_3 \\ r_4 \end{bmatrix} - \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} c - \begin{bmatrix} d_3 \\ d_4 \end{bmatrix} \right). \tag{50}$$

Note that, for either (16) or (17), (50) is always solvable.

If matrix F is singular, $a_3 \neq 0$, and either $b_1 = b_2 = 0$ or $a_2 = b_2 = 0$ then at least one column of F is empty. By Proposition 2, if a stabilising matrix K exists then it can be obtained by solving linear programme (38). Finally, if $a_3 = 0$ and a matrix K that stabilises A_F exists then it can be also obtained by solving (38), as the next theorem shows. Note that such a matrix K cannot exist if $a_4 \geq 0$.

Theorem 9: If $a_3 = 0$ and a matrix K that stabilises A_F exists then such a matrix K can be obtained by solving (38).

Proof: If $a_3 = 0$ then $a_4 < 0$ must hold for stability. If a matrix K that makes matrix A_F stable exists then it does it independently of the values of b_2, b_4, and a_2, which we, therefore, set to 0. Then, the second column of matrix F becomes empty and it follows from Proposition 2 that if a stabilising matrix K exists then such a matrix K can be obtained by solving linear programme (38).

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