

Digital control of negative imaginary systems: a discrete-time hybrid integrator-gain system approach

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Abstract—A hybrid integrator-gain system (HIGS) is a control element that switches between an integrator and a gain, which overcomes some inherent limitations of linear controllers. In this paper, we consider using discrete-time HIGS controllers for the digital control of negative imaginary (NI) systems. We show that the discrete-time HIGS themselves are step-advanced negative imaginary systems. For a minimal linear NI system, there always exists a HIGS controller that can asymptotically stabilize it.

Index Terms—negative imaginary system, hybrid integrator-gain system, discrete-time system, digital control, feedback stability, switched system.

I. INTRODUCTION

Hybrid integrator-gain systems (HIGS) are hybrid control elements introduced in [1] to overcome fundamental limitations of linear time-invariant (LTI) control systems [2], [3]. A HIGS switches between an integrator mode and a gain mode so that a certain sector constraint is satisfied. To be specific, a HIGS is primarily designed to operate as an integrator, and it switches to the gain mode when its integrator dynamics tend to violate the sector constraint. The describing function of a HIGS has a phase lag of only 38.15 degrees, which is much smaller than the 90 degrees phase lag of an integrator. Reset elements including the Clegg integrators [4] and first-order reset elements [5], [6] also have such advantages. However, they generate discontinuous control signals which may cause chattering and degrade the system performance [7], while HIGS generate continuous control signals. HIGS controllers have attracted attention since it was introduced (e.g., see [8]–[13]) and have found application on wafer scanners [14] and atomic force microscopy [15], where the latter work was motivated by the negative imaginary property of HIGS.

Negative imaginary (NI) systems theory was introduced by Lanzon and Petersen in [16] and [17], and has attracted attention from many control theorists [18]–[22]. A typical example of NI systems is a mechanical system with colocated force actuators and position sensors. Motivated by the robust control of flexible structures [23]–[25], which have highly resonant dynamics, NI systems theory uses positive position feedback control. Roughly speaking, a square real-rational proper transfer matrix $F(s)$ is said to be NI if it has no poles on the open right half-plane and $j(F(j\omega) - F(j\omega)^*) \geq 0$

for all $\omega \geq 0$. The Nyquist plot of a single-input single-output (SISO) NI system is contained in the lower half of the complex plane. Under mild assumptions, an NI system $F(s)$ can be asymptotically stabilized using a strictly negative imaginary (SNI) system $F_s(s)$ in positive feedback if and only if the DC loop gain has all its eigenvalues less than unity; i.e., $\lambda_{max}(F(0)F_s(0)) < 1$. Compared with passivity theory which can deal with systems having relative degree of zero and one [26], NI systems theory can deal with systems having relative degree of zero, one and two [27]. NI systems theory has been applied in many fields including nano-positioning control [28]–[31], the control of lightly damped structures [32]–[34], and the control of power systems [35], etc.

NI systems theory was extended to nonlinear systems in [36]–[38]. Roughly speaking, a system is said to be nonlinear NI if it has a positive semidefinite storage function $V(x)$ such that $\dot{V}(x) \leq u^T y$, where x , u and y are the state, input and output of the system, respectively. Under some assumptions, a nonlinear NI system can be stabilized using another nonlinear NI system with a certain strictness property; e.g., output strictly negative imaginary systems [38], or weakly strictly negative imaginary systems [36]. It is shown in [15] that a HIGS controller is a nonlinear NI system. Also, for any minimal SISO linear NI system, there exists a HIGS controller such that their closed-loop interconnection is asymptotically stable. Motivated by the effectiveness of HIGS in the control of NI systems, the paper [39] showed the nonlinear NI property of two variants of HIGS including the multi-HIGS which was introduced in [40], and the cascade of two HIGS. It was also proved in [39] that these two variants of HIGS controllers can be used in stabilizing linear NI systems. This stability result was then applied on a MEMS nanopositioner [39].

However, although the use of HIGS as NI controllers follows from the stability analysis in continuous time, the control of physical systems often requires construction of digital controllers. For the purpose of digital control, a discrete-time HIGS was introduced in [41], which has a similar working mechanism as the continuous-time HIGS. Meanwhile, a novel discrete-time NI systems definition was introduced in [42], which characterizes the dissipativity property for a ZOH sampled continuous-time NI system. Note that the discrete-time NI systems definition in [42] is different from the previously introduced definition in [43], which was mapped from the continuous-time NI systems definition using a bilinear transform. Since the definition of discrete-time NI systems in [42] is obtained using ZOH

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sampling, it is guaranteed to be satisfied by any ZOH sampled physical plant with the NI property. It is shown in [42] that the closed-loop interconnection of a discrete-time NI system and a so-called step-advanced negative imaginary (SANI) system is asymptotically stable, given that either of the systems has some strictness property.

In this paper, we use discrete-time HIGS as controllers for NI systems. We show that a discrete-time HIGS is an SANI system. Furthermore, we establish the following stability result: for any discrete-time NI system, there exists a discrete-time HIGS controller that ensures closed-loop asymptotic stability. This paper contributes in providing a specific digital control framework for physical systems with the NI property. The implementation process of a HIGS controller only involves the selection of parameters in order to satisfy a simple condition. In comparison to the continuous-time design approach where a continuous-time controller is constructed based on the continuous-time model of the plant and subsequently discretized [44], the advantages of the framework in the present paper are two-fold: (a) the design and implementation processes are simpler; (b) closed-loop stability is more rigorously guaranteed.

The rest of the paper is organized as follows. Section II provides preliminary definitions and lemmas for discrete-time NI systems that are introduced in [42]. Section II also provides the state-space model of a discrete-time HIGS. Section III contains the main results of this paper. We show in Section III the NI property of a discrete-time HIGS. We also show that given a linear discrete-time NI plant, there always exists a HIGS controller that can stabilize the NI plant. Section IV concludes the paper and discusses potential future work. An illustrative example is provided in the full archive version of the paper (see [45]).

Notation: The notation in this paper is standard. \mathbb{R} denotes the field of real numbers. \mathbb{N} denotes the set of nonnegative integers. $\mathbb{R}^{m \times n}$ denotes the space of real matrices of dimension $m \times n$. A^T denotes the transpose of a matrix A . A^{-T} denotes the transpose of the inverse of A ; that is, $A^{-T} = (A^{-1})^T = (A^T)^{-1}$. $\lambda_{max}(A)$ denotes the largest eigenvalue of a matrix A with real spectrum. $\|\cdot\|$ denotes the standard Euclidean norm. For a real symmetric or complex Hermitian matrix P , $P > 0$ ($P \geq 0$) denotes the positive (semi-)definiteness of a matrix P and $P < 0$ ($P \leq 0$) denotes the negative (semi-)definiteness of a matrix P . A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$.

II. PRELIMINARIES

A. Discrete-time NI systems

Consider the system

$$x_{k+1} = f(x_k, u_k), \quad (1a)$$

$$y_k = h(x_k), \quad (1b)$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$. Here $u_k, y_k \in \mathbb{R}^p$ and $x_k \in \mathbb{R}^n$ are the input, output and state of the system at time step $k \in \mathbb{N}$, respectively.

Definition 1: [42] The system (1) is said to be a discrete-time negative imaginary (NI) system if there exists a positive definite function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for arbitrary x_k and u_k ,

$$V(x_{k+1}) - V(x_k) \leq u_k^T (y_{k+1} - y_k), \quad (2)$$

for all k .

We provide the necessary and sufficient linear matrix inequalities (LMI) conditions under which Definition 1 is satisfied by a linear system of the form

$$\Sigma : x_{k+1} = Ax_k + Bu_k, \quad (3a)$$

$$y_k = Cx_k, \quad (3b)$$

where $x_k \in \mathbb{R}^n$, $u_k, y_k \in \mathbb{R}^p$ are the system state, input and output, respectively.

Lemma 1: [42] Suppose the linear system (3) satisfies $\det(I - A) \neq 0$. Then the system (3) is NI with a positive definite quadratic storage function satisfying (2) if and only if there exists a real matrix $P = P^T > 0$ such that

$$A^T P A - P \leq 0 \quad \text{and} \quad C = B^T (I - A)^{-T} P.$$

We present in the following, the definition of SANI systems. Consider the system

$$\tilde{x}_{k+1} = \tilde{f}(\tilde{x}_k, \tilde{u}_k), \quad (4a)$$

$$\tilde{y}_k = \tilde{h}(\tilde{x}_k, \tilde{u}_k), \quad (4b)$$

where $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ and $\tilde{h} : \mathbb{R}^n \rightarrow \mathbb{R}^p$. Here $\tilde{u}, \tilde{y} \in \mathbb{R}^p$ and $\tilde{x} \in \mathbb{R}^n$ are the input, output and state of the system at time step $k \in \mathbb{N}$, respectively.

Definition 2: [42] The system (4) is said to be a step-advanced negative imaginary (SANI) system if there exists a function $\hat{h}(x_k)$ such that:

- 1) $\tilde{h}(\tilde{x}_k, \tilde{u}_k) = \hat{h}(\tilde{f}(\tilde{x}_k, \tilde{u}_k))$;
- 2) there exists a positive definite function $\tilde{V} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for arbitrary state \tilde{x}_k and input \tilde{u}_k ,

$$\tilde{V}(\tilde{x}_{k+1}) - \tilde{V}(\tilde{x}_k) \leq \tilde{u}_k^T (\hat{h}(\tilde{x}_{k+1}) - \hat{h}(\tilde{x}_k))$$

for all k .

Remark 1: Definition 2 can be regarded as a variant of Definition 1 in a way such that the system output takes one step advance. To be specific, suppose the system (1) is NI as per Definition 1. Then a system with the same state equation (1a) and an output equation $\tilde{y}_k = h(x_{k+1}) = h(f(x_k, u_k))$ is an SANI system. Note that this does not affect the causality of the system because $h(f(x_k, u_k))$ is a function of the state x_k and input u_k of the current step k .

B. Discrete-time hybrid integrator-gain systems

Discrete-time HIGS were introduced in [41]. We adapt the model in [41] to fit the system model (1) in the following.

$$\mathcal{H} : \begin{cases} x_h(k+1) = x_h(k) + \omega_h e(k), & \text{if } (x_h(k), e(k)) \in \mathcal{F} \\ x_h(k+1) = k_h e(k), & \text{if } (x_h(k), e(k)) \notin \mathcal{F} \\ y_h(k) = x_h(k+1). \end{cases} \quad (5)$$

Here, $e(k), x_h(k), y_h(k) \in \mathbb{R}$ are the system input, state and output, respectively. The constant parameters $\omega_h \geq 0$ and

$k_h > 0$ are called the integrator frequency and the gain value, respectively. The HIGS is designed to operate under the sector constraint $(x_h(k), e(k)) \in \mathcal{F}$, where \mathcal{F} is given by

$$\mathcal{F} = \{(x_h(k), e(k)) \in \mathbb{R}^2 \mid (x_h(k) + \omega_h e(k))e(k) \geq \frac{1}{k_h}(x_h(k) + \omega_h e(k))^2\}. \quad (6)$$

At time step k , if $(e(k), y_h(k)) \in \mathcal{F}$, then $(e(k), y_h(k))$ is contained in the sector $[0, k_h]$. The HIGS is designed to operate primarily in the integrator mode if the input $e(k)$ leads to an output $y_h(k)$ within the sector $[0, k_h]$ under the integrator mode dynamics. Otherwise, the system operates in the gain mode so that $y_h(k) = k_h e(k)$, which automatically satisfies the sector constraint $[0, k_h]$. According to (5), regardless of the initial condition $x_h(0)$, the discrete-time HIGS will remain in the sector given in \mathcal{F} from the time step $k = 1$. In what follows, we denote $e(k)$, $x_h(k)$ and $y_h(k)$ by e_k , \tilde{x}_k and \tilde{y}_k respectively for convenience. Note that the parameter ω_h in the present paper corresponds to the product $\omega_h T_s$ in [41], where ω_h is the integrator frequency of the corresponding continuous-time integrator and T_s is the sampling period. Since we only consider the discrete-time case in the present paper, we regard ω_h as the discrete-time integrator frequency.

III. MAIN RESULTS

A. SANI property of the HIGS

We show in the following that the HIGS given in (5) is an SANI system.

Theorem 1: The system given in (5) is an SANI system with the storage function

$$\tilde{V}(\tilde{x}_k) = \frac{1}{2k_h} \tilde{x}_k^2 \quad (7)$$

satisfying

$$\tilde{V}(\tilde{x}_{k+1}) - \tilde{V}(\tilde{x}_k) \leq e_k(\tilde{x}_{k+1} - \tilde{x}_k), \quad (8)$$

for any input e_k and state \tilde{x}_k .

Proof: According to Definition 2 and Remark 1, the HIGS is an SANI system if it is NI from the input e_k to the state \tilde{x}_k . Hence, we prove in the following that (8) is satisfied in both integrator mode and gain mode. Substituting (7) into (8) yields

$$\frac{1}{2k_h} \tilde{x}_{k+1}^2 - \frac{1}{2k_h} \tilde{x}_k^2 \leq e_k(\tilde{x}_{k+1} - \tilde{x}_k), \quad (9)$$

which is required to be satisfied in both modes.

Case 1. In the integrator mode, we have the state equation $\tilde{x}_{k+1} = \tilde{x}_k + \omega_h e_k$ and also $(\tilde{x}_k, e_k) \in \mathcal{F}$. In this case, (9) becomes

$$2\tilde{x}_k e_k \leq (2k_h - \omega_h) e_k^2, \quad (10)$$

which is always satisfied when $e_k = 0$. When $e_k \neq 0$, (10) can be rewritten as

$$2\frac{\tilde{x}_k}{e_k} \leq 2k_h - \omega_h. \quad (11)$$

The condition $(\tilde{x}_k, e_k) \in \mathcal{F}$ implies

$$\tilde{x}_k^2 + (2\omega_h - k_h)\tilde{x}_k e_k + (\omega_h - k_h\omega) e_k^2 \leq 0.$$

This implies that for $e_k \neq 0$,

$$\left(\frac{\tilde{x}_k}{e_k}\right)^2 + (2\omega_h - k_h)\frac{\tilde{x}_k}{e_k} + (\omega_h^2 - k_h\omega_h) \leq 0. \quad (12)$$

By solving (12), we have that operating in the integrator mode requires the HIGS input e_k and state \tilde{x}_k to satisfy

$$-\omega_h \leq \frac{\tilde{x}_k}{e_k} \leq k_h - \omega_h.$$

Such a pair of \tilde{x}_k and e_k always satisfies (11).

Case 2. In the gain mode, we have that $\tilde{x}_{k+1} = k_h e_k$ and $(\tilde{x}_k, e_k) \notin \mathcal{F}$. In this case, (9) becomes

$$\tilde{x}_k^2 - 2k_h \tilde{x}_k e_k + k_h^2 e_k^2 \geq 0,$$

which always holds because

$$\tilde{x}_k^2 - 2k_h \tilde{x}_k e_k + k_h^2 e_k^2 = (\tilde{x}_k - k_h e_k)^2 \geq 0.$$

Since condition (8) is satisfied in both modes, then the system (5) is an SANI system. \blacksquare

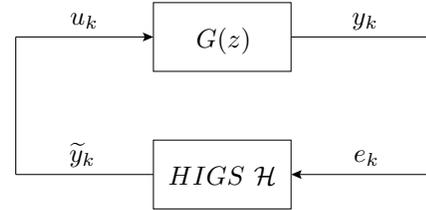


Fig. 1. Closed-loop interconnection of the system (3) with the transfer matrix $G(z)$ and the HIGS \mathcal{H} given in (5).

B. Stability for the interconnection of a linear NI system and a HIGS

Motivated by the SANI property of the HIGS, we investigate whether a HIGS controller can be applied in the control of a minimal SISO linear NI system. Consider a SISO system of the form (3) with $u_k, y_k \in \mathbb{R}$, which has a transfer function matrix $G(z)$. We show in the following that if the system Σ in (3) is NI, then there exists a HIGS controller \mathcal{H} such that the positive feedback interconnection of Σ and \mathcal{H} shown in Fig. 1 is asymptotically stable. The setting of the interconnection can be described as follows:

$$\begin{aligned} e_k &= y_k; \\ u_k &= \tilde{y}_k. \end{aligned}$$

This means the HIGS \mathcal{H} takes the output of the system Σ as its input and feeds back its output to the system Σ as its input.

Theorem 2: Suppose the SISO minimal system (3) with transfer function matrix $G(z)$ is NI and satisfies $\det(I-A) \neq 0$. Suppose the HIGS \mathcal{H} of the form (5) satisfies $0 < \omega_h \leq$

$k_h < \frac{1}{G(1)}$. Then the closed-loop interconnection of $G(z)$ and \mathcal{H} as shown in Fig. 1 is asymptotically stable.

Proof: According to Lemma 1, the minimal system (3) is NI if and only if there exists a matrix $P = P^T > 0$ such that

$$A^T P A - P \leq 0, \quad \text{and} \quad C = B^T (I - A)^{-T} P.$$

We construct the following Lyapunov function for the closed-loop interconnection:

$$\begin{aligned} W(x_k, \tilde{x}_k) &= V(x_k) + \tilde{V}(\tilde{x}_k) - C x_k \tilde{x}_k \\ &= \frac{1}{2} x_k^T P x_k + \frac{1}{2k_h} x_k^2 - C x_k \tilde{x}_k. \end{aligned}$$

Rewriting this as a quadratic form, we have that

$$W(x_k, \tilde{x}_k) = \frac{1}{2} \begin{bmatrix} x_k^T & \tilde{x}_k \end{bmatrix} \begin{bmatrix} P & -C^T \\ -C & \frac{1}{k_h} \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix}.$$

Using the Schur complement theorem, to ensure that $W(x_k, \tilde{x}_k)$ is positive definite, we need

$$\frac{1}{k_h} - C P^{-1} C^T > 0. \quad (13)$$

Since $C = B^T (I - A)^{-T} P$, then (13) can be rewritten as

$$\frac{1}{k_h} - C(I - A)^{-1} B > 0,$$

which is satisfied because $G(1) = C(I - A)^{-1} B$ and

$$k_h G(1) < 1. \quad (14)$$

Note that $G(1) \neq 0$ according to the positive definiteness of P and the fact that C is not a zero row vector, which is guaranteed by the minimality of the system. We use Lyapunov's direct method [46] in the following. Taking the difference between $W(x_{k+1}, \tilde{x}_{k+1})$ and $W(x_k, \tilde{x}_k)$, we have

$$\begin{aligned} &W(x_{k+1}, \tilde{x}_{k+1}) - W(x_k, \tilde{x}_k) \\ &= V(x_{k+1}) + \tilde{V}(\tilde{x}_{k+1}) - C x_{k+1} \tilde{x}_{k+1} - V(x_k) - \tilde{V}(\tilde{x}_k) \\ &\quad + C x_k \tilde{x}_k \\ &\leq u_k (y_{k+1} - y_k) + e_k (\tilde{x}_{k+1} - \tilde{x}_k) - C x_{k+1} \tilde{x}_{k+1} + C x_k \tilde{x}_k \\ &= \tilde{x}_{k+1} (e_{k+1} - e_k) + e_k (\tilde{x}_{k+1} - \tilde{x}_k) - e_{k+1} \tilde{x}_{k+1} + e_k \tilde{x}_k \\ &= 0. \end{aligned} \quad (15)$$

which implies that the system is Lyapunov stable. Furthermore, $W(x_{k+1}, \tilde{x}_{k+1}) - W(x_k, \tilde{x}_k) = 0$ only if the inequality in (15) is an equality. That is

$$V(x_{k+1}) - V(x_k) = u_k (y_{k+1} - y_k); \quad (16)$$

$$\tilde{V}(\tilde{x}_{k+1}) - \tilde{V}(\tilde{x}_k) = e_k (\tilde{x}_{k+1} - \tilde{x}_k). \quad (17)$$

We prove in the following that (16) and (17) cannot hold together at all time indices k unless $(x_k, \tilde{x}_k) = (0, 0)$. We consider the case that (16) and (17) hold for some index k and all future indices $k + 1, k + 2, \dots$. When (17) holds, we have that

$$\frac{1}{2k_h} \tilde{x}_{k+1}^2 - \frac{1}{2k_h} \tilde{x}_k^2 = e_k (\tilde{x}_{k+1} - \tilde{x}_k). \quad (18)$$

We consider the following two cases, where the HIGS is assumed to work in the integrator mode and the gain mode, respectively.

Case 1. Integrator mode. In this case, $(\tilde{x}_k, e_k) \in \mathcal{F}$ and

$$\tilde{x}_{k+1} = \tilde{x}_k + \omega_h e_k. \quad (19)$$

Substituting (19) in (18) yields

$$(\omega_h - 2k_h) e_k^2 + 2\tilde{x}_k e_k = 0. \quad (20)$$

Case 1a. Suppose $e_k \neq 0$. Then we have $\tilde{x}_k = (k_h - \frac{\omega_h}{2}) e_k$, which can be substituted in the inequality in (6) and yields

$$(k_h + \frac{\omega_h}{2}) e_k^2 \geq \frac{1}{k_h} (k_h + \frac{\omega_h}{2})^2 e_k^2.$$

This, after simplification, becomes

$$\omega_h^2 + 2k_h \omega_h \leq 0.$$

Considering the fact that $k_h > 0$ and $\omega > 0$, the above condition can never be satisfied. Hence, *Case 1a* can never happen.

Case 1b. Suppose $e_k = 0$. Then (20) is always satisfied. In this case, $(\tilde{x}_k, e_k) \in \mathcal{F}$ implies that $\tilde{x}_k = 0$. According to (19), we have that $\tilde{x}_{k+1} = 0$ as well. The condition for $(\tilde{x}_{k+1}, e_{k+1}) \in \mathcal{F}$ can be simplified to be

$$(k_h - \omega_h) e_{k+1}^2 \geq 0.$$

The fact that $k_h - \omega_h \geq 0$ guarantees that the next active mode is the integrator mode. Note that this condition is irrelevant to the HIGS input or state. Indeed, since *Case 1a* can never happen, then the system will fall in *Case 1b* for all future time indices $k + 1, k + 2, \dots$. Following from a similar analysis, we have that

$$0 = e_k = e_{k+1} = e_{k+2} = \dots, \quad (21)$$

and also

$$0 = \tilde{x}_k = \tilde{x}_{k+1} = \tilde{x}_{k+2} = \dots. \quad (22)$$

Since $u_k = \tilde{y}_k = \tilde{x}_{k+1}$, then according to (22) and (3a), we have that

$$x_{k+1} = A x_k, \quad x_{k+2} = A x_{k+1} = A^2 x_k, \quad \dots \quad (23)$$

Since $e_k = y_k = C x_k$, then according to (21) and (23), we have that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_k = 0.$$

This implies that $x_k = 0$ due to the observability of $G(z)$. In this case, $(x_k, \tilde{x}_k) = (0, 0)$. The closed-loop system is already in its equilibrium.

Case 2. Gain mode. In this case, $(\tilde{x}_k, e_k) \notin \mathcal{F}$, and we have that

$$\tilde{x}_{k+1} = k_h e_k. \quad (24)$$

Substituting (24) in (18), we have that

$$(\tilde{x}_k - k_h e_k)^2 = 0.$$

That is

$$\tilde{x}_k = k_h e_k = \tilde{x}_{k+1}. \quad (25)$$

The condition $(\tilde{x}_k, e_k) \notin \mathcal{F}$ implies that

$$(k_h + \omega_h) e_k^2 > 0.$$

This implies that $e_k \neq 0$. We only need consider the case that the HIGS operates in the gain mode for all future indices. This is because that under the constraints (17), if it enters the integrator mode, it will never exit the integrator mode, according to the analysis in *Case 1b*. Then it falls into *Case 1*. In the case that the system keeps operating in the gain mode, following from the same derivation of (25), we have that

$$\tilde{x}_{k+1} = k_h e_{k+1} = \tilde{x}_{k+2}. \quad (26)$$

Comparing (25), (26) and similar equations for future time indices, we have that

$$\tilde{x}_k = k_h e_k = \tilde{x}_{k+1} = k_h e_{k+1} = \tilde{x}_{k+2} = k_h e_{k+2} = \dots$$

That is

$$e_k = e_{k+1} = e_{k+2} = \dots$$

This implies that

$$y_k = y_{k+1} = y_{k+2} = \dots \quad (27)$$

In this case, we have that

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k = Ax_k + B\tilde{y}_k = Ax_k + B\tilde{x}_{k+1} \\ &= Ax_k + Bk_h e_k = Ax_k + k_h BCx_k \\ &= (A + k_h BC)x_k. \end{aligned}$$

Similarly, we have

$$\begin{aligned} x_{k+2} &= (A + k_h BC)x_{k+1} = (A + k_h BC)^2 x_k, \\ &\vdots \\ x_{k+n-1} &= (A + k_h BC)^{n-1} x_k. \end{aligned}$$

According to (27), we have that

$$\begin{bmatrix} y_{k+1} - y_k \\ y_{k+2} - y_{k+1} \\ \vdots \\ y_{k+n} - y_{k+n-1} \end{bmatrix} = 0,$$

which implies

$$\begin{bmatrix} C \\ C(A + k_h BC) \\ \vdots \\ C(A + k_h BC)^{n-1} \end{bmatrix} (x_{k+1} - x_k) = 0. \quad (28)$$

We use eigenvector test to prove that observability of (A, C) implies that of $(A + k_h BC, C)$. Suppose $\eta \neq 0$ is a vector in the kernel of C ; i.e., $C\eta = 0$. Then it is not an eigenvector of A ; i.e., $A\eta \neq \lambda\eta$ for all scalars λ . Then η is not an eigenvector of $A + k_h BC$ as well because $(A + k_h BC)\eta = A\eta + k_h BC\eta = A\eta \neq \lambda\eta$ for all λ , considering $C\eta = 0$. Hence, $(A + k_h BC, C)$ is observable and (28) implies that

$x_{k+1} = x_k$. That is, x_k is an eigenvector of $A + k_h BC$ with an eigenvalue $\lambda = 1$. This implies that

$$x_k = x_{k+1} = x_{k+2} = \dots$$

In this case, we also have that

$$\begin{aligned} x_k &= x_{k+1} = Ax_k + Bu_k = Ax_k + B\tilde{y}_k = Ax_k + B\tilde{x}_{k+1} \\ &= Ax_k + Bk_h e_k. \end{aligned}$$

This implies that

$$x_k = k_h (I - A)^{-1} B e_k.$$

Also, we have that

$$e_k = Cx_k = k_h C(I - A)^{-1} B e_k. \quad (29)$$

Since we have $e_k \neq 0$ in *Case 2*, then (29) implies that

$$k_h C(I - A)^{-1} B = 1,$$

which is

$$k_h G(1) = 1.$$

This contradicts (14). To conclude, we have shown that if (16) and (17) hold together for all future time indices, then the HIGS cannot stay in the gain mode according to the analysis in *Case 2*. It will eventually switch to the integrator mode. Then, according to the analysis in *Case 1*, the HIGS will stay in the integrator mode. However, we have shown in *Case 1b* that this is only possible if the system is already at the equilibrium. In other words, if the system is not at the equilibrium, then (16) and (17) cannot hold together for all future indices, and $W(x_k, \tilde{x}_k)$ will eventually decrease again until the system reaches its equilibrium. This means that the system is asymptotically stable. ■

IV. CONCLUSION AND FUTURE WORK

We proposed a control framework for the digital control of linear NI systems using HIGS controllers. Discrete-time HIGS are shown to be SANI systems. For any linear discrete-time NI systems obtained via ZOH sampling, there exists a HIGS controller such that their closed-loop interconnection is asymptotically stable.

REFERENCES

- [1] D. A. Deenen, M. F. Heertjes, W. Heemels, and H. Nijmeijer, "Hybrid integrator design for enhanced tracking in motion control," in *2017 American Control Conference (ACC)*. IEEE, 2017, pp. 2863–2868.
- [2] R. H. Middleton, "Trade-offs in linear control system design," *Automatica*, vol. 27, no. 2, pp. 281–292, 1991.
- [3] J. Freudenberg, R. Middleton, and A. Stefanopoulou, "A survey of inherent design limitations," in *Proceedings of the 2000 American Control Conference. ACC (IEEE Cat. No. 00CH36334)*, vol. 5. IEEE, 2000, pp. 2987–3001.
- [4] J. C. Clegg, "A nonlinear integrator for servomechanisms," *Transactions of the American Institute of Electrical Engineers, Part II: Applications and Industry*, vol. 77, no. 1, pp. 41–42, 1958.
- [5] I. Horowitz and P. Rosenbaum, "Non-linear design for cost of feedback reduction in systems with large parameter uncertainty," *International Journal of Control*, vol. 21, no. 6, pp. 977–1001, 1975.
- [6] Y. Chait and C. Hollot, "On Horowitz's contributions to reset control," *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal*, vol. 12, no. 4, pp. 335–355, 2002.

- [7] G. Bartolini, "Chattering phenomena in discontinuous control systems," *International journal of systems science*, vol. 20, no. 12, pp. 2471–2481, 1989.
- [8] S. Van den Eijnden, M. F. Heertjes, W. Heemels, and H. Nijmeijer, "Hybrid integrator-gain systems: A remedy for overshoot limitations in linear control?" *IEEE Control Systems Letters*, vol. 4, no. 4, pp. 1042–1047, 2020.
- [9] D. A. Deenen, B. Sharif, S. van den Eijnden, H. Nijmeijer, M. Heemels, and M. Heertjes, "Projection-based integrators for improved motion control: Formalization, well-posedness and stability of hybrid integrator-gain systems," *Automatica*, vol. 133, p. 109830, 2021.
- [10] S. J. Van Den Eijnden, M. F. Heertjes, W. M. Heemels, and H. Nijmeijer, "Frequency-domain tools for stability analysis of hybrid integrator-gain systems," in *2021 European Control Conference (ECC)*. IEEE, 2021, pp. 1895–1900.
- [11] D. Van Dintther, B. Sharif, S. Van den Eijnden, H. Nijmeijer, M. F. Heertjes, and W. Heemels, "Overcoming performance limitations of linear control with hybrid integrator-gain systems," *IFAC-PapersOnLine*, vol. 54, no. 5, pp. 289–294, 2021.
- [12] S. van den Eijnden, M. Heertjes, H. Nijmeijer, and W. Heemels, "A small-gain approach to incremental input-to-state stability analysis of hybrid integrator-gain systems," *IEEE Control Systems Letters*, 2023.
- [13] W. Heemels and A. Tanwani, "Existence and completeness of solutions to extended projected dynamical systems and sector-bounded projection-based controllers," *IEEE Control Systems Letters*, 2023.
- [14] M. Heertjes, S. van Den Eijnden, and B. Sharif, "An overview on hybrid integrator-gain systems with applications to wafer scanners," in *2023 IEEE International Conference on Mechatronics (ICM)*. IEEE, 2023, pp. 1–8.
- [15] K. Shi, N. Nikooienejad, I. R. Petersen, and S. R. Moheimani, "A negative imaginary approach to hybrid integrator-gain system control," in *2022 IEEE 61st Conference on Decision and Control (CDC)*. IEEE, 2022, pp. 1968–1973.
- [16] A. Lanzon and I. R. Petersen, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1042–1046, 2008.
- [17] I. R. Petersen and A. Lanzon, "Feedback control of negative-imaginary systems," *IEEE Control Systems Magazine*, vol. 30, no. 5, pp. 54–72, 2010.
- [18] J. Xiong, I. R. Petersen, and A. Lanzon, "A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2342–2347, 2010.
- [19] Z. Song, A. Lanzon, S. Patra, and I. R. Petersen, "A negative-imaginary lemma without minimality assumptions and robust state-feedback synthesis for uncertain negative-imaginary systems," *Systems & Control Letters*, vol. 61, no. 12, pp. 1269–1276, 2012.
- [20] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Generalizing negative imaginary systems theory to include free body dynamics: Control of highly resonant structures with free body motion," *IEEE Transactions on Automatic Control*, vol. 59, no. 10, pp. 2692–2707, 2014.
- [21] J. Wang, A. Lanzon, and I. R. Petersen, "Robust cooperative control of multiple heterogeneous negative-imaginary systems," *Automatica*, vol. 61, pp. 64–72, 2015.
- [22] P. Bhowmick and S. Patra, "On LTI output strictly negative-imaginary systems," *Systems & Control Letters*, vol. 100, pp. 32–42, 2017.
- [23] A. Preumont, *Vibration control of active structures: an introduction*. Springer, 2018, vol. 246.
- [24] D. Halim and S. R. Moheimani, "Spatial resonant control of flexible structures-application to a piezoelectric laminate beam," *IEEE Transactions on Control Systems Technology*, vol. 9, no. 1, pp. 37–53, 2001.
- [25] H. Pota, S. R. Moheimani, and M. Smith, "Resonant controllers for smart structures," *Smart Materials and Structures*, vol. 11, no. 1, p. 1, 2002.
- [26] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative systems analysis and control: theory and applications*. Springer, London, 2007, vol. 2.
- [27] K. Shi, I. R. Petersen, and I. G. Vladimirov, "Necessary and sufficient conditions for state feedback equivalence to negative imaginary systems," *IEEE Transactions on Automatic Control (Early Access)*, 2024.
- [28] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Spectral conditions for negative imaginary systems with applications to nanopositioning," *IEEE/ASME Transactions on Mechatronics*, vol. 19, no. 3, pp. 895–903, 2013.
- [29] S. K. Das, H. R. Pota, and I. R. Petersen, "A MIMO double resonant controller design for nanopositioners," *IEEE Transactions on Nanotechnology*, vol. 14, no. 2, pp. 224–237, 2014.
- [30] —, "Resonant controller design for a piezoelectric tube scanner: A mixed negative-imaginary and small-gain approach," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 5, pp. 1899–1906, 2014.
- [31] —, "Multivariable negative-imaginary controller design for damping and cross coupling reduction of nanopositioners: a reference model matching approach," *IEEE/ASME Transactions on Mechatronics*, vol. 20, no. 6, pp. 3123–3134, 2015.
- [32] C. Cai and G. Hagen, "Stability analysis for a string of coupled stable subsystems with negative imaginary frequency response," *IEEE Transactions on Automatic Control*, vol. 55, no. 8, pp. 1958–1963, 2010.
- [33] M. A. Rahman, A. Al Mamun, K. Yao, and S. K. Das, "Design and implementation of feedback resonance compensator in hard disk drive servo system: A mixed passivity, negative-imaginary and small-gain approach in discrete time," *Journal of Control, Automation and Electrical Systems*, vol. 26, no. 4, pp. 390–402, 2015.
- [34] B. Bhikkaji, S. R. Moheimani, and I. R. Petersen, "A negative imaginary approach to modeling and control of a collocated structure," *IEEE/ASME Transactions on Mechatronics*, vol. 17, no. 4, pp. 717–727, 2011.
- [35] Y. Chen, K. Shi, I. R. Petersen, and E. L. Ratnam, "A nonlinear negative imaginary systems framework with actuator saturation for control of electrical power systems," *To appear in 2024 European Control Conference, also available as arXiv preprint:2311.06820*, 2023.
- [36] A. G. Ghallab, M. A. Mabrok, and I. R. Petersen, "Extending negative imaginary systems theory to nonlinear systems," in *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 2018, pp. 2348–2353.
- [37] K. Shi, I. G. Vladimirov, and I. R. Petersen, "Robust output feedback consensus for networked identical nonlinear negative-imaginary systems," *IFAC-PapersOnLine*, vol. 54, no. 9, pp. 239–244, 2021.
- [38] K. Shi, I. R. Petersen, and I. G. Vladimirov, "Output feedback consensus for networked heterogeneous nonlinear negative-imaginary systems with free-body motion," *IEEE Transactions on Automatic Control*, vol. 68, no. 9, pp. 5536–5543, 2023.
- [39] K. Shi, N. Nikooienejad, I. R. Petersen, and S. R. Moheimani, "Negative imaginary control using hybrid integrator-gain systems: Application to MEMS nanopositioner," *IEEE Transactions on Control Systems Technology (Early Access)*, 2023.
- [40] A. S. P., "HIGS-based skyhook damping design of a multivariable vibration isolation system," Master's thesis, Eindhoven University of Technology, 2020.
- [41] B. Sharif, D. W. Alferink, M. F. Heertjes, H. Nijmeijer, and W. Heemels, "A discrete-time approach to analysis of sampled-data hybrid integrator-gain systems," in *2022 IEEE 61st Conference on Decision and Control (CDC)*. IEEE, 2022, pp. 7612–7617.
- [42] K. Shi, I. R. Petersen, and I. G. Vladimirov, "Discrete-time negative imaginary systems from zoh sampling," *Submitted to the 26th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2024), also available as arXiv preprint:2312.05419v2*, 2023.
- [43] A. Ferrante, A. Lanzon, and L. Ntogramatzidis, "Discrete-time negative imaginary systems," *Automatica*, vol. 79, pp. 1–10, 2017.
- [44] D. Nešić, A. R. Teel, and P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations," *Systems & Control Letters*, vol. 38, no. 4-5, pp. 259–270, 1999.
- [45] K. Shi and I. R. Petersen, "Digital control of negative imaginary systems: a discrete-time hybrid integrator-gain system approach," *arXiv preprint:2403.16046*, 2024.
- [46] R. E. Kalman and J. E. Bertram, "Control system analysis and design via the "second method" of Lyapunov: II-discrete-time systems," *Journal of Basic Engineering*, vol. 82, pp. 394–400, 1960.