2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Accelerating Distributed Nash Equilibrium Seeking

Tatiana Tatarenko[1] and Angelia Nedić[2]

Abstract— **This work proposes a novel distributed approach for computing a Nash equilibrium in convex games with restricted strongly monotone pseudo-gradients. By leveraging the idea of the centralized operator extrapolation method presented in [4] to solve variational inequalities, we develop the algorithm converging to Nash equilibria in games, where players have no access to the full information but are able to communicate with neighbors over some communication graph. The convergence rate is demonstrated to be geometric and improves the rates obtained by the previously presented procedures seeking Nash equilibria in the class of games under consideration.**

I. INTRODUCTION

Game theory deals with a specific class of optimization problems arising in multiagent systems, in which each agent, also called player, aims to minimize its local cost function coupled through decision variables (actions) of all agents (players) in a system. The applications of game-theoretic optimization can be found, for example, in electricity markets, communication networks, autonomous driving systems and the smart grids [1], [7], [11], [12]. Solutions to such optimization problems are Nash equilibria which characterize stable joint actions in games. To find these solutions in a so called convex game, one can use their equivalent characterization as the solutions to the variational inequality defined for the game's pseudo-gradient over the joint action set [10]. Moreover, it is known that, given a strongly monotone and Lipschitz continuous mapping, the projection algorithm converges geometrically fast to the unique solution of the variational inequality and, thus, to the unique Nash equilibrium of the game. The convergence rate, in terms of the kth iterate's distance to the solution, is in the order of $O\left(\exp\left\{-\frac{k}{\gamma^2}\right\}\right)$(see [8]), where $\gamma = L/\mu \geq 1$ with L and μ being the Lipschitz continuity and strong monotonicity constants of the mapping, respectively. This rate has been improved in [8] to the rate of $O\left(\exp\left\{-\frac{k}{\gamma}\right\}\right)$ by a more sophisticated algorithm that requires, at each iteration, two operator evaluations and two projections. To relax these requirements, the paper [4] presents the so called operator extrapolation method achieving the same rate $O\left(\exp\left\{-\frac{k}{\gamma}\right\}\right)$ with one operator evaluation and one projection per iteration. Moreover, geometrically fast convergence of the operator extrapolation method takes place under a weaker condition of restricted strong monotonicity (see **Notations**). However, these fast algorithms require full information in the sense that each player observes actions of all other players at every iteration.

Since in the modern large-scale systems each agent has access only to some partial information about joint actions, *fast distributed communication-based* optimization procedures in games have gained a lot of attention over the recent years (see [18] for an extensive review and bibliography). In particular, the work [2] presents a proximal-point algorithm for converging to the Nash equilibrium with a geometric rate. However, this algorithm requires the evaluation of a proximal operator, at each iteration, and cannot be rewritten as iterations that give the next state in terms of the current one. On the other hand, the papers [15] and [3] propose the distributed procedures based on the gradient algorithm and demonstrate their geometric convergence rate in the order $O\left(\exp\left\{-\frac{k}{\gamma^4}\right\}\right)$ for strongly monotone games with player communications over time-invariant and time-varying graphs, respectively. The works [16], [17] focus on a reformulation of a Nash equilibrium in distributed setting in terms of a so called augmented variational inequality, which takes into account the communication network that players are using. The main goal of such reformulations has been to adjust the fast centralized procedure from [8] to the distributed settings and accelerate learning Nash equilibria under such settings. However, the acceleration (to the rate $O\left(\exp\left\{-\frac{k}{\gamma^3}\right\}\right)$) has been guaranteed only for a restrictive subclass of games with strongly monotone and Lipschitz continuous pseudo-gradients. This restriction is due to the fact that the mapping defining the augmented variational inequality is generally not strongly monotone but it can be made restricted strongly monotone. Moreover, the augmented variational inequality requires introduction of an extra parameter which has to be properly set up to guarantee convergence of the proposed algorithms.

This article presents a novel fast distributed discrete-time algorithm for seeking Nash equilibria in games with restricted strongly monotone pseudo-gradients. To avoid issues related to the augmented variational inequality arising in the distributed settings and still to be able to accelerate the previously known rates, this algorithm leverages the idea of the operator extrapolation method from [4] instead of the Nesterov's acceleration approach presented in [8]. We

*This work was not supported by any organization

[1]Tatiana Tatarenko is with the Department of Control Theory and Intelligent Systems, TU Darmstadt, German tatiana.tatarenko@tu-darmstadt.de

[2]Angelia Nedić is with School of Electrical, Computer and Energy Engineering, Arizona State University, USA. Angelia.Nedich@asu.edu. Te work has been done under support of the Office of Naval Research award N00014-21-1-2242.

develop a procedure converging to the Nash equilibrium with the rate $O\left(\exp\left\{-\frac{k}{\gamma^2}\right\}\right)$ and requiring one projection and gradient calculation per iteration.

Notations. The set $\{1,\dots,n\}$ is denoted by $[n]$. For any function $f: K\to\mathbb{R}$, $K\subseteq\mathbb{R}^n$, $\nabla_i f(x)=\frac{\partial f(x)}{\partial x_i}$ is the partial derivative taken in respect to the ith coordinate of the vector variable $x\in\mathbb{R}^n$. For any real vector space $\tilde{E}$ its dual space is denoted by $\tilde{E}^*$ and the inner product is denoted by $\langle u,v\rangle$, $u\in\tilde{E}^*$, $v\in\tilde{E}$. A mapping $g:\tilde{E}\to\tilde{E}^*$ is said to be *strongly monotone with the constant* $\mu>0$ *on the set* $Q\subseteq\tilde{E}$, if $\langle g(u)-g(v),u-v\rangle\geq\mu\|u-v\|^2$ for all $u,v\in Q$. It is said to be *restricted strongly monotone with respect to* $u^*\in Q$, if $\langle g(u)-g(u^*),u-u^*\rangle\geq\mu\|u-u^*\|^2$ for all $u\in Q$. A mapping $g:\tilde{E}\to\tilde{E}^*$ is said to be Lipschitz continuous on the set $Q\subseteq\tilde{E}$ with the constant L, if $\|g(u)-g(v)\|\leq L\|u-v\|$ for all $u,v\in Q$. We consider real vector space E, which is either space of real vectors $E=E^*=\mathbb{R}^n$ or the space of real matrices $E=E^*=\mathbb{R}^{n\times n}$. In the case $E=\mathbb{R}^n$ we use $\|\cdot\|$ to denote the Euclidean norm induced by the standard dot product in $\mathbb{R}^n$. In the case $E=\mathbb{R}^{n\times n}$, the inner product $\langle u,v\rangle\triangleq\sqrt{\mathrm{trace}(u^Tv)}$ is the Frobenius inner product on $\mathbb{R}^{n\times n}$ and $\|\cdot\|$ denotes the Frobenius norm induced by the Frobenius inner product, i.e., $\|v\|\triangleq\sqrt{\mathrm{trace}(v^Tv)}$. We use $\mathcal{P}_\Omega\{v\}$ to denote the projection of $v\in E$ on a set $\Omega\subseteq E$. For any matrix A, the vector of diagonal entries of the matrix A is denoted by $\mathrm{diag}(A)$.

II. Distributed Learning in Convex Games

We consider a non-cooperative game between n players. Let J_i and $\Omega_i\subseteq\mathbb{R}$ denote[1] respectively the cost function and the feasible action set of the player i. We denote the joint action set by $\Omega=\Omega_1\times\cdots\times\Omega_n$. Each function $J_i(x_i,x_{-i})$, $i\in[n]$, depends on x_i and x_{-i}, where $x_i\in\Omega_i$ is the action of the player i and $x_{-i}\in\Omega_{-i}=\Omega_1\times\cdots\times\Omega_{i-1}\times\Omega_{i+1}\times\cdots\times\Omega_n$ denotes the joint action of all players except for the player i. We assume that the players can interact over an undirected communication graph $\mathcal{G}([n],\mathcal{A})$. The set of nodes is the set $[n]$ of players, and the set $\mathcal{A}$ of undirected arcs is such that $\{i,j\}\in\mathcal{A}$ whenever there is an undirected communication link between i to j and, thus, some information (message) can be passed between the players i and j. For each player i, the set $\mathcal{N}_i$ is the set of neighbors in the graph $\mathcal{G}([n],\mathcal{A})$, i.e., $\mathcal{N}_i\triangleq\{j\in[n]:\{i,j\}\in\mathcal{A}\}$. We denote this game by $\Gamma(n,\{J_i\},\{\Omega_i\},\mathcal{G})$, and we make the following assumptions regarding the game.

Assumption 1. *[Convex Game] For all* $i\in[n]$, *the set* Ω_i *is convex and closed, while the function* $J_i(x_i,x_{-i})$ *is convex and continuously differentiable in* x_i *for each fixed* x_{-i}.

When the cost functions $J_i(\cdot,x_{-i})$ are differentiable, we can define the pseudo-gradient.

[1]All results below are applicable for games with different dimensions $\{d_i\}$ of the action sets $\{\Omega_i\}$. The one-dimensional case is considered for the sake of notation simplicity.

Definition 1. *The* pseudo-gradient $F(x):\Omega\to\mathbb{R}^n$ *of the game* $\Gamma(n,\{J_i\},\{\Omega_i\},\mathcal{G})$ *is defined as follows:* $F(x)\triangleq[\nabla_1 J_1(x_1,x_{-1}),\dots,\nabla_n J_n(x_n,x_{-n})]^T\in\mathbb{R}^n$, *where* ∇_i *denotes the partial derivative with respect to* x_i *(see* **Notations).**

A solution to a game is a Nash equilibrium, defined below.

Definition 2. *A vector* $x^*=[x_1^*,x_2^*,\cdots,x_n^*]^T\in\Omega$ *is a* Nash equilibrium *if for all* $i\in[n]$ *and all* $x_i\in\Omega_i$

$$J_i(x_i^*,x_{-i}^*)\leq J_i(x_i,x_{-i}^*).$$

By Assumption 1 and the connection between Nash equilibria and solutions of variational inequalities [10], the point $x^*\in\Omega$ is a Nash equilibrium of the game $\Gamma(n,\{J_i\},\{\Omega_i\},\mathcal{G})$ if and only if the following variational inequality holds

$$\langle F(x^*),x-x^*\rangle\geq 0\quad\text{for all } x\in\Omega. \tag{1}$$

We make further assumptions regarding the players cost functions, as follows.

Assumption 2. *The game* $\Gamma(n,\{J_i\},\{\Omega_i\},\mathcal{G})$ *has a Nash equilibrium* x^*, *and the pseudo-gradient mapping* $F(x)$ *is defined on the whole space* $\mathbb{R}^n$ *and is* restricted strongly monotone with respect to x^* *on* $\mathbb{R}^n$ *with a constant* $\mu>0$.

The existence of a Nash equilibrium is guaranteed if, for example, Assumption 1 holds and the action sets Ω_i, $i\in[n]$, are bounded [10].

Assumption 3. *For every* $i\in[n]$ *the function* $\nabla_i J_i(x_i,x_{-i})$ *is Lipschitz continuous in* x_i *on* Ω_i *for every fixed* $x_{-i}\in\mathbb{R}^{n-1}$, *that is, there exist a constant* $L_i\geq 0$ *such that for all* $x_{-i}\in\mathbb{R}^{n-1}$ *we have for all* $x_i,y_i\in\Omega_i$,

$$|\nabla_i J_i(x_i,x_{-i})-\nabla_i J_i(y_i,x_{-i})|\leq L_i|x_i-y_i|.$$

Moreover, for every $i\in[n]$ *the function* $\nabla_i J_i(x_i,x_{-i})$ *is Lipschitz continuous in* x_{-i} *on* $\mathbb{R}^{n-1}$, *for every fixed* $x_i\in\Omega_i$, *that is, there is a constant* $L_{-i}\geq 0$ *such that for all* $x_i\in\Omega_i$ *we have for all* $x_{-i},y_{-i}\in\mathbb{R}^{n-1}$,

$$|\nabla_i J_i(x_i,x_{-i})-\nabla_i J_i(x_i,y_{-i})|\leq L_{-i}\|x_{-i}-y_{-i}\|.$$

The players' communications are restricted to the underlying connectivity graph $\mathcal{G}([n],\mathcal{A})$, with which we associate a nonnegative symmetric mixing matrix W, i.e., a symmetric matrix with nonnegative entries and with positive entries w_{ij} only when $\{i,j\}\in\mathcal{A}$. To ensure sufficient information "mixing" in the network, we assume that the graph is connected. These assumptions are formalized, as follows.

Assumption 4. *The underlying undirected communication graph* $\mathcal{G}([n],\mathcal{A})$ *is connected. The associated non-negative symmetric mixing matrix* $W=[w_{ij}]\in\mathbb{R}^{n\times n}$ *defines the weights on the undirected arcs such that* $w_{ij}>0$ *if and only if* $\{i,j\}\in\mathcal{A}$ *and* $\sum_{j=1}^n w_{ij}=1$ *for all* $i\in[n]$.

Remark 1. *There are some simple strategies for generating symmetric mixing matrices over undirected graphs for which Assumption 4 holds (see Section 2.4 in [13] for a summary of such strategies).*

Assumption 4 implies that the second largest singular value σ of W is such that $\sigma \in (0,1)$ and for any $x \in \mathbb{R}^n$ the following average property holds (see [9]):

$$\|Wx - \mathbf{1}\bar{x}\| \leq \sigma\|x - \mathbf{1}\bar{x}\|, \quad (2)$$

where $\bar{x} = \frac{1}{n}\langle \mathbf{1}, x\rangle$ is the average of the elements of x.

In this work, we are interested in *distributed seeking of the Nash equilibrium* in a game $\Gamma(n, \{J_i\}, \{\Omega_i\}, \mathcal{G})$ for which Assumptions 1–4 hold. We note that under Assumptions 1–2, the game has a unique Nash equilibrium (see [16]).

III. Algorithm Development

A. Direct Acceleration

Throughout the paper, we let player i hold a *local copy* of the global decision variable[2] x, which is denoted by

$$x_{(i)} = [\tilde{x}_{(i)1}; \dots; \tilde{x}_{(i)i-1}; x_i; \tilde{x}_{(i)i+1}; \dots; \tilde{x}_{(i)n}] \in \mathbb{R}^n.$$

Here $\tilde{x}_{(i)j}$ can be viewed as a temporary estimate of x_j by player i. In this notation, we always have $\tilde{x}_{(i)i} = x_i$. Also, we compactly denote the temporary estimates that player i has for all decisions of the other players as

$$\tilde{x}_{-i} = [\tilde{x}_{(i)1}; \dots; \tilde{x}_{(i)i-1}; \tilde{x}_{(i)i+1}; \dots; \tilde{x}_{(i)n}] \in \mathbb{R}^{n-1}.$$

We introduce the following estimation matrix:

$$\mathbf{x} \triangleq \begin{pmatrix} x_{(1)}^{\mathrm{T}} \\ x_{(2)}^{\mathrm{T}} \\ \vdots \\ x_{(n)}^{\mathrm{T}} \end{pmatrix} \in \mathbb{R}^{n\times n},$$

where x^{T} denotes the transpose of a column-vector x. We let Ω_a to denote an augmented action set, consisting of the vectors on the diagonals of the estimation matrices, i.e., $\Omega_a = \{\mathbf{x} \in \mathbb{R}^{n\times n} \,|\, \mathrm{diag}(\mathbf{x}) \in \Omega\}$. The pseudo-gradient estimation of the game is defined as $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^{n\times n}$:

$$\mathbf{F}(\mathbf{x}) \triangleq \begin{pmatrix} \nabla_1 J_1(x_{(1)}) & 0 & \cdots & 0 \\ 0 & \nabla_2 J_2(x_{(2)}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nabla_n J_n(x_{(n)}) \end{pmatrix}. \quad (3)$$

The algorithm starts with an arbitrary initial $\mathbf{x}^0 \in \Omega_a$, that is, each player i holds an arbitrary point $x_{(i)}^0 \in \mathbb{R}^{i-1} \times \Omega_i \times \mathbb{R}^{n-i}$. All the subsequent estimation matrices $\mathbf{x}_1, \mathbf{x}_2, \dots$ are obtained through the updates described by Algorithm 1.

[2]Note that global decision variable x is a fictitious variable which never exists in the designed decentralized computing system.

Remark 2. *Algorithm 1 is inspired by the operator extrapolation approach presented in [4] for solving variational inequalities in a centralized setting. The extrapolation here corresponds to the expression* $\nabla_i J_i(\hat{x}_{(i)}^k) + \lambda[\nabla_i J_i(x_{(i)}^k) - \nabla_i J_i(\hat{x}_{(i)}^{k-1})]$ *in the update of the individual actions* x_i^{k+1}, $i \in [n]$. *It is inspired by the connection between the Nesterov's acceleration and the gradients' extrapolation (see [6]). We call our proposed distributed algorithm* Accelerated Direct Method *to emphasize that, in contrast to the work [16], the communication step* $\hat{x}_{(i)}^k = \sum_{j\in\mathcal{N}_i} w_{ij}x_{(j)}^k$ *is directly implemented in the corresponding centralized procedure without any augmented game mapping.*

Algorithm 1: Accelerated Direct Method

Set mixing matrix W;
Choose step size $\alpha > 0$ and parameter $\lambda > 0$;
Pick arbitrary $x_{(i)}^0 \in \mathbb{R}^{i-1} \times \Omega_i \times \mathbb{R}^{n-i}$, $i = 1, \dots, n$;
Set $\hat{x}_{(i)}^0 = \sum_{j\in\mathcal{N}_i} w_{ij}x_{(j)}^0$ and $x_{(i)}^1 = \hat{x}_{(i)}^0$,
for $k = 1, 2, \dots$, all players $i = 1, \dots, n$ do
  $\hat{x}_{(i)}^k = \sum_{j\in\mathcal{N}_i} w_{ij}x_{(j)}^k$,
  $x_i^{k+1} = \mathcal{P}_{\Omega_i}\{\hat{x}_i^k - \alpha[\nabla_i J_i(\hat{x}_{(i)}^k) + \lambda[\nabla_i J_i(x_{(i)}^k) - \nabla_i J_i(\hat{x}_{(i)}^{k-1})]]\}$;
  for $\ell = \{1, \dots, i-1, i+1, \dots, n\}$
    $\tilde{x}_{(i)\ell}^{k+1} = \hat{x}_{(i)\ell}^k$;
  end for;
end for.

The compact expression of Algorithm 1 in terms of the pseudo-gradient estimation $\mathbf{F}(\cdot)$ (see (3)) and the estimation matrices $\{\mathbf{x}^k\}$ is as follows:

$$\hat{\mathbf{x}}^k = W\mathbf{x}^k,$$
$$\mathbf{x}^{k+1} = \mathcal{P}_{\Omega_a}\left\{\hat{\mathbf{x}}^k - \alpha(\mathbf{F}(\hat{\mathbf{x}}^k) + \lambda(\mathbf{F}(\mathbf{x}^k) - \mathbf{F}(\hat{\mathbf{x}}^{k-1})))\right\}. \quad (4)$$

In the further analysis of Algorithm 1, we will use its compact matrix form as given in (4).

B. Analysis

This subsection investigates convergence of Algorithm 1. To be able to prove its convergence to a unique solution and to state a linear rate, we first provide some important technical properties regarding the game pseudo-gradient $\mathbf{F}$ and the iteration of the procedure.

Lemma 1. *Under Assumption 3, the pseudo-gradient estimation* $\mathbf{F}(\cdot)$ *is Lipschitz continuous over* Ω_a *with the constant* $L = \max_{i\in[n]}\sqrt{L_i^2 + L_{-i}^2}$.

Proof: See Lemma 1 in [16].

Before formulating the next result, let us define the following decomposition for each matrix $\mathbf{x} \in \Omega_a$:

$$\mathbf{x} = \mathbf{x}_{||} + \mathbf{x}_{\perp}, \quad (5)$$

where $\mathbf{x}_{||} = \frac{1}{n}\mathbf{1}\mathbf{1}^T\mathbf{x}$ is the so called consensus matrix and $\mathbf{x}_\perp = \mathbf{x} - \mathbf{x}_{||}$ with the implied property $\langle \mathbf{x}_{||}, \mathbf{x}_\perp \rangle = 0$. This decomposition will be used in the proof of the following results as it allows for an estimation of the running weighted distance to the solution. In words, we aim to upper bound a sum $\sum_{t=1}^k \omega_{1,t} \left\| \mathbf{x}_{||}^{t+1} - \mathbf{x}^* \right\|^2 + \omega_{2,t} \|\mathbf{x}_\perp^{t+1}\|^2$ by a sum $\sum_{t=1}^k \nu_{1,t} \left\| \mathbf{x}_{||}^{t+1} - \mathbf{x}^* \right\|^2 + \nu_{2,t} \|\mathbf{x}_\perp^{t+1}\|^2$ for some weights $\omega_{1,t}$, $\omega_{2,t}$, $\nu_{1,t}$, and $\nu_{2,t}$ in such a way that the final conclusion on linear convergence $\left\| \mathbf{x}_{||}^{k+1} - \mathbf{x}^* \right\|^2 + \|\mathbf{x}_\perp^{k+1}\|^2 = \left\| \mathbf{x}^{k+1} - \mathbf{x}^* \right\|^2 \leq \rho^k \|\mathbf{x}^1 - \mathbf{x}^*\|$, $\rho \in (0,1)$ is implied.

Proposition 1. *Let Assumptions 1-4 hold. Let $\mathbf{x}^*$ be the consensus matrix with each row equal to the unique Nash equilibrium x^* in the game Γ. Moreover, let a scalar $\eta \in (0,1)$ and a positive sequence $\{\theta_t\}$ be such that $\theta_{t-1}\eta(1-\eta) \geq \theta_t L^2\alpha^2\lambda^2$ for all $t \geq 1$ and $\theta_{t+1}\lambda = \theta_t$ for all $t \geq 0$, where $\lambda > 0$ is the stepsize from Algorithm 1. Then, we have*

$$\sum_{t=1}^k [\theta_t a_1 \left\| \mathbf{x}_{||}^{t+1} - \mathbf{x}^* \right\|^2 + \theta_t a_2 \|\mathbf{x}_\perp^{t+1}\|^2] - \frac{\theta_k L^2\alpha^2}{2(1-\eta)} \|\mathbf{x}^{k+1} - \mathbf{x}^*\|^2 \leq \sum_{t=1}^k [\theta_t b_1 \left\| \mathbf{x}_{||}^t - \mathbf{x}^* \right\|^2 + \theta_t b_2 \|\mathbf{x}_\perp^t\|^2], \quad (6)$$

where $a_1 = \frac{1-\eta}{2} + \frac{\mu}{2n}\alpha$, $a_2 = \frac{1-\eta}{2} - \left(L + \frac{2nL^2}{\mu}\right)\alpha$, $b_1 = \frac{1+\eta\|W-I\|^2}{2}$, *and* $b_2 = \frac{(1-\eta)\sigma^2 + \eta(1+\|W-I\|^2)}{2}$.

The proof of this technical result is based on the so called three-points lemma presented in Lemma 3.1 of the book [5] which is applied to the matrix form of the algorithm in (4) and implies that

$$\alpha \left\langle \mathbf{F}(\hat{\mathbf{x}}^t) + \lambda(\mathbf{F}(\mathbf{x}^t) - \mathbf{F}(\hat{\mathbf{x}}^{t-1})), \mathbf{x}^{t+1} - \mathbf{x}^* \right\rangle + \frac{1}{2}\|\mathbf{x}^{t+1} - \hat{\mathbf{x}}^t\|^2 \leq \frac{1}{2}\|\hat{\mathbf{x}}^t - \mathbf{x}^*\|^2 - \frac{1}{2}\|\mathbf{x}^{t+1} - \mathbf{x}^*\|^2$$

for all $t \geq 1$ and x^* being the Nash equilibrium from Assumption 2. Next, the both sides of the inequality above are multiplied by some time-dependent positive constant θ_t and summed up over $t = 1, \dots, k$. After that the each side of the resulting inequality are analyzed separately to set up such parameters θ_t that the relation (6) holds. As it has been mentioned above, we use the equation (5) to get the decomposed result on the squared distance to the solution $\mathbf{x}^*$. Moreover, in the reasoning we leverage Assumption 1 to apply (1) and, thus, $\langle \mathbf{F}(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle = \langle F(x^*), x - x^* \rangle \geq 0$. Assumption 2 is required to get the estimation $\langle \mathbf{F}(\mathbf{x}_{||}^{t+1}) - \mathbf{F}(\mathbf{x}^*), \mathbf{x}_{||}^{t+1} - \mathbf{x}^* \rangle = \langle F(x_{||}^{t+1}) - F(x^*), x_{||}^{t+1} - x^* \rangle \geq \mu\|x_{||}^{t+1} - x^*\|^2 = \frac{\mu}{n}\|\mathbf{x}_{||}^{t+1} - \mathbf{x}^*\|^2$. We require Assumptions 3 and 4 to apply Lemma 1 and the relation $\|W(\mathbf{x}_{||}^t + \mathbf{x}_\perp^t) - \mathbf{x}^*\|^2 = \|\mathbf{x}_{||}^t - \mathbf{x}^*\|^2 + \|W\mathbf{x}_\perp^t\|^2 \overset{(2)}{\leq} \|\mathbf{x}_{||}^t - \mathbf{x}^*\|^2 + \sigma^2\|\mathbf{x}_\perp^t\|^2$ respectively. All further details can be found in the full version of this paper [14].

Next we formulate our main result.

Theorem 1. *Let the parameters in the Algorithm 1 be chosen as follows:*

$$\alpha \leq \min\{g_1, g_2, g_3, g_4\}, \quad \lambda = \frac{1}{1+\epsilon(\alpha)}, \quad (7)$$

where

$$g_1 = \frac{n\mu(1-\sigma^2)}{4(\mu + 2nL)^2(1 + \|I - W\|^2)},$$
$$g_2 = \frac{n(1 + \|I - W\|^2)}{2\mu},$$
$$g_3 = \frac{\mu n(1 + \|I - W\|^2)}{\mu^2 + L^2(1 + \|I - W\|^2)^2 n^2},$$
$$g_4 = \frac{\mu}{\sqrt{4L^2\mu^2 + 16(L\mu + 2nL^2)^2}},$$
$$\epsilon(\alpha) = \frac{2\mu\alpha/n - (1 + \|I - W\|^2)(1 - \sqrt{1 - 4L^2\alpha^2})}{2 + \|I - W\|^2(1 - \sqrt{1 - 4L^2\alpha^2})}.$$

Then $\epsilon(\alpha) > 0$ and

$$\|\mathbf{x}^{k+1} - \mathbf{x}^*\|^2 \leq \frac{8 + 4\|I - W\|^2 - 4\|I - W\|^2\sqrt{1 - 4L^2\alpha^2}}{(1 + \epsilon(\alpha))^{k-1}} \|\mathbf{x}^1 - \mathbf{x}^*\|^2.$$

Proof: Let $\theta_t = c^t$, where

$$c = \min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\right\}, \quad (8)$$

and a_1, a_2, b_1, and b_2 are defined in Proposition 1, namely,

$$a_1 = \frac{1-\eta}{2} + \frac{\mu}{2n}\alpha, a_2 = \frac{1-\eta}{2} - \left(L + \frac{2nL^2}{\mu}\right)\alpha,$$
$$b_1 = \frac{1 + \eta\|W - I\|^2}{2},$$
$$b_2 = \frac{(1-\eta)\sigma^2 + \eta(1 + \|W - I\|^2)}{2}.$$

Moreover, let us choose $\lambda = \frac{\theta_t}{\theta_{t+1}}$ and $\eta = \frac{1-\sqrt{1-4L^2\alpha^2}}{2}$. Next, we demonstrate that under the condition (7), $c > 1$. For this purpose we check that in this case both $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are larger than 1. Indeed,

$$\frac{a_1}{b_1} = \frac{1 - \eta + \mu\alpha/n}{1 + \eta\|W - I\|^2},$$
$$\frac{a_2}{b_2} = \frac{1 - \eta - 2\left(L + \frac{2nL^2}{\mu}\right)\alpha}{(1-\eta)\sigma^2 + \eta(1 + \|W - I\|^2)}.$$

First, we notice that under the condition $\alpha \leq g_1$, $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$ (see Appendix). Next, we have that

$$c = \frac{a_1}{b_1} > 1,$$

if and only if

$$\eta < \frac{\mu\alpha}{n(1+\|W-I\|^2)}. \quad (9)$$

Given the definition of η, we conclude that under the condition $\alpha < \min\{g_2, g_3\}$,

$$\eta = \frac{1-\sqrt{1-4L^2\alpha^2}}{2} < \frac{\mu\alpha}{n(1+\|W-I\|^2)},$$

and, thus, (9) holds.

Since $c > 1$, it follows that

$$L^2\alpha^2 = \eta(1-\eta) \le c\eta(1-\eta),$$

which implies $\theta_{t-1}\eta(1-\eta) \ge \theta_t L^2\alpha^2\lambda^2$. Hence, the conditions of Proposition 1 hold and we conclude that

$$\theta_k[a_1\|\mathbf{x}_{||}^{k+1} - \mathbf{x}^*\|^2 + a_2\|\mathbf{x}_\perp^{t+1}\|^2] - \frac{\theta_k L^2\alpha^2}{2(1-\eta)}\|\mathbf{x}^{k+1} - \mathbf{x}^*\|^2$$
$$\le \theta_1[b_1\|\mathbf{x}_{||}^1 - \mathbf{x}^*\|^2 + b_2\|\mathbf{x}_\perp^1\|^2],$$

where we used definition of c in (8) implying $\theta_t b_1 \le \theta_{t-1}a_1$ and $\theta_t b_2 \le \theta_{t-1}a_2$. We also use the fact that $\mathbf{x}_\perp^* = 0$. Thus, $\theta_k \min\{a_1, a_2\}\|\mathbf{x}^{k+1} - \mathbf{x}^*\|^2 - \frac{\theta_k L^2\alpha^2}{2(1-\eta)}\|\mathbf{x}^{k+1} - \mathbf{x}^*\|^2 \le \theta_1 \max\{b_1, b_2\}\|\mathbf{x}^1 - \mathbf{x}^*\|^2$. As $\min\{a_1, a_2\} = a_2$ and $\max\{b_1, b_2\} = b_1$, we conclude that

$$\theta_k\left[a_2 - \frac{L^2\alpha^2}{2(1-\eta)}\right]\|\mathbf{x}^{k+1} - \mathbf{x}^*\|^2 \le \theta_1 b_1\|\mathbf{x}^1 - \mathbf{x}^*\|^2. \quad (10)$$

Next, we notice that $a_2 = \frac{1-\eta}{2} - \left(L + \frac{2nL^2}{\mu}\right)\alpha$ is larger or equal to $1/8$, if the conditions $\alpha \le \frac{\mu}{4(L\mu+2nL^2)}$ and $\alpha \le \frac{\sqrt{3}}{4L}$ hold (see Appendix). On the other hand, given the condition $\alpha \le \frac{\sqrt{7}}{8L}$, we have $\frac{L^2\alpha^2}{2(1-\eta)} \le \frac{1}{16}$. Taking this inequality together with $a_2 \ge \frac{1}{8}$ into account, we obtain from (10) that, given $\alpha \le \min\{g_4, g_5\}$,

$$\frac{\theta_k}{16}\|\mathbf{x}_{k+1} - \mathbf{x}^*\|^2 \le \theta_1 b_1\|\mathbf{x}_1 - \mathbf{x}^*\|^2. \quad (11)$$

Finally, we use that $\theta_k = c^k = (1+\epsilon(\alpha))^k$, where $\epsilon(\alpha) = \frac{a_1}{b_1} - 1 = \frac{2\mu\alpha/n - (1+\|I-W\|^2)(1-\sqrt{1-4L^2\alpha^2})}{2+\|I-W\|^2(1-\sqrt{1-4L^2\alpha^2})}$ to conclude the result from (11).

Corollary 1. *Taking into account the conditions of the theorem above, one can choose the step size* $\alpha = O\left(\frac{\mu}{L^2 n}\right)$ *to obtain the convergence rate* $O\left(\exp\left\{-\frac{k}{\gamma^2 n^2}\right\}\right)$ *which is faster than the rates of previously proposed methods for distributed learning of Nash equilibria in restricted strongly monotone games [2], [15], [16]. Indeed, the GRANE algorithm from [16] is proven to converge to the Nash equilibrium with the rate* $O\left(\exp\left\{-\frac{k}{\gamma^6 n^6}\right\}\right)$*, whereas the direct distributed procedure in [2], [15] improves this rate to* $O\left(\exp\left\{-\frac{k}{\gamma^4 n^3}\right\}\right)$.

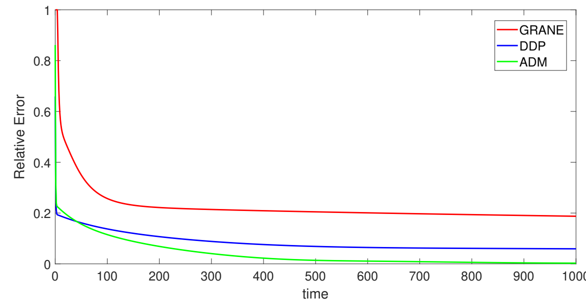

Fig. 1. Comparison of the presented accelerated direct method (ADM) with GRANE and DDP

IV. Simulations

Let us consider a class of games with strongly monotone game mappings. Specifically, we have players $\{1, 2, \dots, 20\}$ and each player i's objective is to minimize the cost function $J_i(x_i, x_{-i}) = f_i(x_i) + l_i(x_{-i})x_i$, where $f_i(x_i) = 0.5a_i x_i^2 + b_i x_i$ and $l_i(x_{-i}) = \sum_{j\ne i} c_{ij}x_j$. The local cost function is dependent on actions of all players, but the underlying communication graph is a randomly generated tree graph. We randomly select $a_i > 0$, b_i, and c_{ij} for all possible i and j to guarantee strong monotonicity of the pseudo-gradient.

We simulate the proposed gradient play algorithm and compare its implementation with the implementations of the algorithm GRANE presented in [16] and direct distributed procedure (DDP) from [2], [15]. Figure 1 demonstrates the simulation results which support theoretic ones stated in Corollary 1.

V. Conclusion

This work extends centralized operator extrapolation method presented in [4] to distributed settings in restricted strongly monotone games where players can exchange their information only with local neighbors via some communication graph. The proposed procedure is proven to possess a geometric rate and to outperform the previously developed algorithms calculating Nash equilibria in games under the same assumptions. Future research directions include consideration of a more general communication topology and study of lower bounds for convergence rates of distributed methods in such class of games.

References

[1] T. Alpcan and T. Başar. Distributed Algorithms for Nash Equilibria of Flow Control Games. In *Advances in Dynamic Games*, pages 473–498. Springer, 2005.

[2] M. Bianchi, G. Belgioioso, and S. Grammatico. A fully-distributed proximal-point algorithm for nash equilibrium seeking with linear convergence rate. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 2303–2308, 2020.

[3] M. Bianchi and S. Grammatico. Fully distributed nash equilibrium seeking over time-varying communication networks with linear convergence rate. *IEEE Control Systems Letters*, 5(2):499–504, 2021.

[4] G. Kotsalis, G. Lan, and T. Li. Simple and optimal methods for stochastic variational inequalities, i: Operator extrapolation. *SIAM Journal on Optimization*, 32(3):2041–2073, 2022.
[5] G. Lan. First-order and stochastic optimization methods for machine learning. Springer, 2020.
[6] Guanghui Lan and Yi Zhou. Random gradient extrapolation for distributed and stochastic optimization. *SIAM Journal on Optimization*, 28(4):2753–2782, 2018.
[7] N. Li, Y. Yao, I. Kolmanovsky, E. Atkins, and A. R. Girard. Game-theoretic modeling of multi-vehicle interactions at uncontrolled intersections. *IEEE Transactions on Intelligent Transportation Systems*, 23(2):1428–1442, 2022.
[8] Yu. Nesterov and L. Scrimali. Solving strongly monotone variational and quasi-variational inequalities. *Discrete and Continuous Dynamical Systems - A*, 31(4):1383–1396, 2011.
[9] A. Olshevsky and J. Tsitsiklis. Convergence speed in distributed consensus and averaging. *SIAM Journal on Control and Optimization*, 48(1):33–55, 2009.
[10] J.-S. Pang and F. Facchinei. *Finite-dimensional variational inequalities and complementarity problems : vol. 1*. Springer series in operations research. Springer, New York, Berlin, Heidelberg, 2003.
[11] W. Saad, H. Zhu, H. V. Poor, and T. Başar. Game-theoretic methods for the smart grid: An overview of microgrid systems, demand-side management, and smart grid communications. *IEEE Signal Processing Magazine*, 29(5):86–105, 2012.
[12] G. Scutari, S. Barbarossa, and D. P. Palomar. Potential games: A framework for vector power control problems with coupled constraints. In *2006 IEEE International Conference on Acoustics Speech and Signal Processing Proceedings*, volume 4, pages 241–244, May 2006.
[13] W. Shi, Q. Ling, G. Wu, and W. Yin. EXTRA: An Exact First-Order Algorithm for Decentralized Consensus Optimization. *SIAM Journal on Optimization*, 25(2):944–966, 2015.
[14] T. Tatarenko and A. Nedić. Accelerating distributed nash equilibrium seeking. *arXiv: 2310.15384*, 2023.
[15] T. Tatarenko and A. Nedić. Geometric convergence of distributed gradient play in games with unconstrained action sets. *IFAC-PapersOnLine*, 53(2):3367–3372, 2020. 21st IFAC World Congress.
[16] T. Tatarenko, W. Shi, and A. Nedić. Geometric convergence of gradient play algorithms for distributed nash equilibrium seeking. *IEEE Transactions on Automatic Control*, 66(11):5342–5353, 2021.
[17] T. Tatarenko, W. Shi, and A. Nedić. Accelerated gradient play algorithm for distributed nash equilibrium seeking. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 3561–3566, 2018.
[18] M. Ye, Q.-L. Han, L. Ding, and S. Xu. Distributed nash equilibrium seeking in games with partial decision information: A survey. *Proceedings of the IEEE*, 111(2):140–157, 2023.

APPENDIX

1. More details on the inequality $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$ under the condition $\alpha < g_1$. The condition $\alpha < g_1 = \frac{n\mu(1-\sigma^2)}{4(\mu+2nL)^2(1+\|I-W\|^2)}$ implies $\alpha\left(1+\frac{2nL}{\mu}\right)^2 \leq \frac{n(1-\sigma^2)}{4\mu(1+\|I-W\|^2)}$. Thus, since $\eta = \frac{1-\sqrt{1-4L^2\alpha^2}}{2} < \frac{1}{2}$, we conclude that $(1-\eta)^2 > \frac{1}{4}$, and, hence

$$\alpha\left(1+\frac{2nL}{\mu}\right)^2(1+\|I-W\|^2) \leq \frac{n}{\mu}(1-\sigma^2)(1-\eta)^2.$$

Next, we use $\eta < 1$ and $\sigma < 1$ to obtain

$$\left(1+\frac{2nL}{\mu}\right)^2(1+\|I-W\|^2)$$
$$\geq \left(1-\frac{2nL}{\mu}\right)^2(1+\eta\|I-W\|^2)-(1-\eta)(1-\sigma^2).$$

Combining two last inequalities, we get

$$\alpha[\left(1+\frac{2nL}{\mu}\right)^2(1+\eta\|I-W\|^2)-(1-\eta)(1-\sigma^2)]$$
$$\leq \frac{n}{\mu}(1-\sigma^2)(1-\eta)^2.$$

By multiplying both sides by $\frac{\mu}{n}$, we obtain

$$\alpha\left[\left(\frac{\mu}{n}+2L+\frac{4nL^2}{\mu}\right)(1+\eta\|I-W\|^2) -(1-\eta)(1-\sigma^2)\frac{\mu}{n}\right] \leq (1-\sigma^2)(1-\eta)^2$$
$$\Updownarrow$$
$$(1-\eta)(1+\eta\|I-W\|^2)+\alpha\frac{\mu}{n}(1+\eta\|I-W\|^2) - (1-\eta)^2(1-\sigma^2)-\frac{\mu}{n}\alpha(1-\eta)(1-\sigma^2)$$
$$\leq (1-\eta)(1+\eta\|I-W\|^2) - (1+\eta\|I-W\|^2)2\left(L+\frac{2nL^2}{\mu}\right)\alpha$$
$$\Updownarrow$$
$$(1+\eta\|I-W\|^2)\left(1-\eta+\alpha\frac{\mu}{n}\right) - (1-\eta)^2(1-\sigma^2)\left(1-\eta+\alpha\frac{\mu}{n}\right)$$
$$\leq (1+\eta\|I-W\|^2)\left(1-\eta-2\left(L+\frac{2nL^2}{\mu}\right)\alpha\right)$$
$$\Updownarrow$$
$$(1+\eta\|I-W\|^2-(1-\eta)^2(1-\sigma^2))\left(1-\eta+\alpha\frac{\mu}{n}\right)$$
$$\leq (1+\eta\|I-W\|^2)\left(1-\eta-2\left(L+\frac{2nL^2}{\mu}\right)\alpha\right)$$
$$\Updownarrow$$
$$\frac{1-\eta-2\left(L+\frac{2nL^2}{\mu}\right)\alpha}{(1-\eta)\sigma^2+\eta(1+\|W-I\|^2)} \leq \frac{1-\eta+\mu\alpha/n}{1+\eta\|W-I\|^2}.$$

2. More details on the inequality $a_2 = \frac{1-\eta}{2} - \left(L+\frac{2nL^2}{\mu}\right)\alpha \geq 1/8$ under the conditions $\alpha \leq \frac{\mu}{4(L\mu+2nL^2)}$ and $\alpha \leq \frac{\sqrt{3}}{4L}$.

As $\eta = \frac{1-\sqrt{1-4L^2\alpha^2}}{2}$, $\frac{1-\eta}{2} - \left(L+\frac{2nL^2}{\mu}\right)\alpha \geq 1/8$ if and only if

$$2+2\sqrt{1-4L^2\alpha^2}-8\left(L+\frac{2nL^2}{\mu}\right)\alpha \geq 1$$
$$\Updownarrow$$
$$2-8\left(L+\frac{2nL^2}{\mu}\right)\alpha \geq 1-2\sqrt{1-4L^2\alpha^2},$$

which holds, if $2-8\left(L+\frac{2nL^2}{\mu}\right)\alpha \geq 0$ and $1-2\sqrt{1-4L^2\alpha^2} \leq 0$. The first inequality is guaranteed by $\alpha \leq \frac{\mu}{4(L\mu+2nL^2)}$, whereas the second one is implied by $\alpha \leq \frac{\sqrt{3}}{4L}$.