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On the H_2 optimal control of uniformly damped mass-spring networks

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***Abstract*— In this paper we provide an analytical solution to an H_2 optimal control problem, that applies whenever the process corresponds to a uniformly damped network of masses and springs. The solution covers both stable and unstable systems, and illustrates analytically how damping affects the levels of achievable performance. Furthermore, the resulting optimal controllers can be synthesised using passive damped mass-spring networks, allowing for controller implementations without an energy source. We investigate the impact of both positive and negative damping through a small numerical example.**

I. INTRODUCTION

In this paper we study an H_2 optimal control problem for a process with dynamics modelled by the linear differential equations

$$M\ddot{q} + C\dot{q} + Kq = f,\ q(0) = \dot{q}(0) = 0, \tag{1}$$

under the restriction that the matrix M is positive definite, and K is positive semi-definite. This is a prototypical setup for the dynamics of a network of damped masses and springs, such as that illustrated in Figure 1, when linearised around an equilibrium point. The variable q is a vector of generalised coordinates describing the configuration of the system relative to equilibrium, and f is a vector of forces applied at those coordinates. The entries of M and K can typically be determined directly from the expressions for the kinetic and potential energy for the system, and our particular focus is on exploring the impact of the matrix C on the optimal control problem, and the corresponding optimal control law.

The motivation for studying this model class stems from the fact that networks of masses and springs are frequently used to model engineering processes, with applications ranging from electrical power systems, to vehicle suspension systems, to optimization algorithms, to structure stabilisation [1], [2], [3], [4]. Key to their importance is the balance they strike between simplicity and versatility. Moreover, models of even very large systems can be systematically built up through simple descriptions based on the underlying physics. Yet despite this structural simplicity, the resulting models can still describe a very rich range of behaviours, including resonances across a wide range of time and length scales, and even instability when allowing for negative damping.

In addition to their practical relevance, linear mass-spring networks also have a range of desirable theoretical properties.

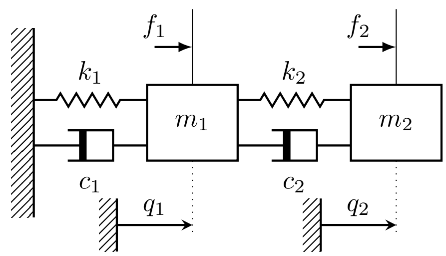

Fig. 1: Example of a mass-spring network, with damped springs. Masses m_1 and m_2 are connected with damped springs with spring constants k_1 and k_2 and damping constants c_1 and c_2. q_1 and q_2 are the positions of the masses, and f_1 and f_2 are external forces applied to the masses.

These are particularly striking in the lossless case (namely when $C = 0$), where for example central control theoretic results such as the Kalman-Yakubovich-Popov lemma simplify greatly, and the dynamics in (1) can be realised with highly structured state-space realisations [5]. These features can be exploited to simplify optimal control problems for lossless systems, which can result in optimal control laws that can both be described analytically and synthesised with simple passive networks [6], [7].

When damping is introduced, the underlying theoretical properties of (1) become significantly more complex (see for example [8], [9] and the references therein for a discussion of the quadratic eigenvalue problem). However if the damping is *uniform*, many of the desirable properties from the lossless case are preserved. Mathematically, uniform damping means that C is symmetric and satisfies $CM^{-1}K = KM^{-1}C$. This condition is equivalent to the existence of a congruence transformation that decouples (1) into a set of orthogonal modes [10] (we discuss this transformation further in section III). Although an assumption of convenience, uniform damping is surprisingly versatile, and will be satisfied by any symmetric C matrix on the form

$$C = Mg\left(M^{-1}K\right),$$

where g is a polynomial (or entire) function. Note in particular that such a C need not be positive semi-definite, and so uniformly damped networks do not need to be stable. Special cases include Rayleigh damping [11], where $C = \alpha_0 M + \alpha_1 K$ for $\alpha_0, \alpha_1 \in \mathbb{R}$. Given the difficulties in modelling damping phenomena from first principles, uniform models of damping are often adopted, at least as a first approximation, and methods for estimating uniform damping matrices from data are discussed in [12].

In this paper we study a natural H_2 optimal control

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problem for (1) under the uniform damping assumption. We start by developing analytical results for a highly structured optimal control problem. This is presented as Theorem 1 in section II. In section III we show how to exploit the uniform damping assumption to apply this result to solve optimal control problems for systems described by (1), and discuss and illustrate the structure in the resulting optimal controllers. A striking feature of the obtained optimal controller is that it has the same structure as the problem itself! For the damped mass-spring network in (1) this means that the optimal control law is itself a damped mass-spring network. The theorems give an analytical solution to problems with uniform damping, both positive and negative. Thus the optimal controller obtained works for both stable and unstable damped mass-spring networks. The H_2-gain from disturbances to performance outputs under optimal control is expressed in the M, C and K matrices of (1). The analytical nature of the result makes it to be well suited for large scale networks with damped mass-spring dynamics, like power system networks.

NOTATION

$\sqrt{E}$ denotes the unique positive semi-definite matrix square root of a positive semi-definite matrix E. The H_2 norm of a stable transfer function $G(s)$ is defined as

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\left(G(j\omega)^* G(j\omega)\right) d\omega\right)^{1/2}.$$

II. AN ANALYTICAL SOLUTION TO AN H_2 OPTIMAL CONTROL PROBLEM

In this section we study an H_2 optimal control problem for the feedback loop in Figure 2. This is a natural setup, in which the objective is to design a controller $K(s)$ to minimise the effects of process and sensor disturbances on the control effort and process output, as quantified by the H_2 norm. We will provide an analytical solution to this problem under a set of strict assumptions on the state-space realisation of the process transfer function $G(s)$. However in section III we will show how to use this result to obtain an analytical expression for the optimal control law when $G(s)$ describes the dynamics of a uniformly damped mass-spring network.

Problem 1: Let

$$\begin{aligned} \dot{x}_{\mathrm{G}} &= A_{\mathrm{G}} x_{\mathrm{G}} + B_{\mathrm{G}}\left(u + w_{\mathrm{u}}\right), \, x_{\mathrm{G}}(0) = 0, \\ z &= \begin{bmatrix} y_{\mathrm{G}} \\ u \end{bmatrix} = \begin{bmatrix} C_{\mathrm{G}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_{\mathrm{G}} \\ u \end{bmatrix}, \\ y &= C_{\mathrm{G}} x_{\mathrm{G}} + w_{\mathrm{y}}, \end{aligned} \tag{2}$$

where the matrices A_{G} B_{G} and C_{G} have the following structure

$$A_{\mathrm{G}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}, \, B_{\mathrm{G}} = \begin{bmatrix} Q \\ 0 \end{bmatrix}, \, C_{\mathrm{G}} = \begin{bmatrix} Q^{\mathsf{T}} & 0 \end{bmatrix}, \tag{3}$$

with the following properties of the sub-matrices:

- A_{11} is diagonal and square of size $n \times n$;
- A_{12} is $n \times m$, where $0 \leq m \leq n$ and only has entries on the main diagonal and these entries are non-zero;
- $A_{21} = -A_{12}^{\mathsf{T}}$;

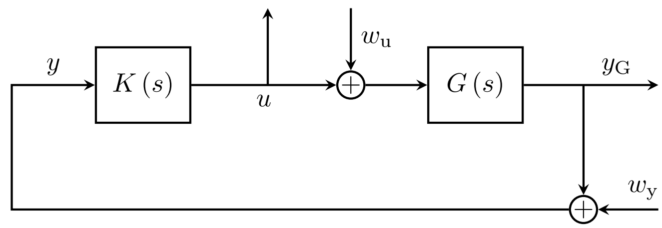

Fig. 2: Illustration of Problem 1, with external disturbances w_{u} and w_{y} acting on the system. The aim is to minimise the effects of the disturbances on the outputs y_{G} and u.

- Q has the property that $QQ^{\mathsf{T}} = I$ and is of size $n \times p$;
- the 0 in A_{G} is of size $m \times m$, the 0 in B_{G} is of size $m \times p$ and the 0 in C_{G} is of size $p \times m$, with $p \geq n$.

Suppose also that the controller $K(s)$ can be described by the state-space system

$$\begin{aligned} \dot{x}_{\mathrm{K}} &= A_{\mathrm{K}} x_{\mathrm{K}} + B_{\mathrm{K}} y, \, x_{\mathrm{K}}(0) = 0, \\ u &= C_{\mathrm{K}} x_{\mathrm{K}} + D_{\mathrm{K}} y. \end{aligned} \tag{4}$$

Define $T_{zw}(s)$ as the closed loop transfer function from $w = \begin{bmatrix} w_{\mathrm{u}}^{\mathsf{T}} & w_{\mathrm{y}}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}$ to z described by (2) and (4). Find

$$\gamma_{H_2}^* = \inf\left\{\gamma : \|T_{zw}(s)\|_{H_2} < \gamma\right\},$$

where the infimum is taken over A_{K}, B_{K}, C_{K}, and D_{K}. $\diamondsuit$

Remark 1: In (3) the state vector x_{G} has length $n + m$, and the length of the input vector u is p. The slightly unconventional naming of the size of the state vector will be explained by the type of problems this setup can describe in section III. $\diamondsuit$

Theorem 1: Under the conditions of Problem 1,

$$\gamma_{H_2}^* = \sqrt{\operatorname{tr}\left(Z^3\right) + \operatorname{tr}\left(Z\right)},$$

where $Z = A_{11} + \sqrt{A_{11}^2 + I}$, and an optimal controller is given by:

$$\begin{aligned} \dot{x}_{\mathrm{K}} &= \begin{bmatrix} A_{11} - 2Z & A_{12} \\ A_{21} & 0 \end{bmatrix} x_{\mathrm{K}} + \begin{bmatrix} ZQ \\ 0 \end{bmatrix} y, \, x_{\mathrm{K}}(0) = 0, \\ u &= \begin{bmatrix} -Q^{\mathsf{T}} Z & 0 \end{bmatrix} x_{\mathrm{K}}. \end{aligned} \tag{5}$$

Proof: Introduce the generalised plant

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A_{\mathrm{G}} & B_w & B_{\mathrm{G}} \\ C_z & 0 & D_{zu} \\ C_{\mathrm{G}} & D_{yw} & D_{yu} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix},$$

where

$$B_w = \begin{bmatrix} B_{\mathrm{G}} & 0 \end{bmatrix}, \, C_z = \begin{bmatrix} C_{\mathrm{G}} \\ 0 \end{bmatrix},$$

and

$$D_{zu} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \, D_{yw} = \begin{bmatrix} 0 & I \end{bmatrix}, \, D_{yu} = 0.$$

Under the conditions of Problem 1, the pair (A_{G}, B_w) is controllable and the pair (C_z, A_{G}) is observable. To see this, first note that since $QQ^{\mathsf{T}} = I$, thus $\operatorname{rank}(Q) = n$. Since the diagonal elements in A_{21} are all non-zero, $\operatorname{rank}(A_{21}) = m$. The two first sub-matrices of the controllability matrix are

$$\begin{bmatrix} B_\mathrm{G} & A_\mathrm{G} B_\mathrm{G} \end{bmatrix} = \begin{bmatrix} Q & A_{11}Q \\ 0 & A_{21}Q \end{bmatrix}.$$

Therefore the controllability matrix has rank $n+m$, which equals the state dimension. Thus $(A_\mathrm{G}, B_\mathrm{G})$ is controllable, which also implies that (A_G, B_w) controllable. The same argument shows that $(C_\mathrm{G}, A_\mathrm{G})$ and (C_z, A_G) are observable. Furthermore the matrices D_{zu} and D_{yw} are full rank. Therefore the H_2 solution to Problem 1 can be tackled within the Riccati equation framework of [13].

Let X denote the unique stabilising solution of

$$XA_\mathrm{G} + A_\mathrm{G}^\mathsf{T} X - XB_\mathrm{G}B_\mathrm{G}^\mathsf{T} X + C_z^\mathsf{T} C_z = 0, \tag{6}$$

and Y denote the unique stabilising solution of

$$YA_\mathrm{G}^\mathsf{T} + A_\mathrm{G} Y - YC_\mathrm{G}^\mathsf{T} C_\mathrm{G} Y + B_w B_w^\mathsf{T} = 0. \tag{7}$$

The solutions X and Y can be used to calculate $\gamma^*_{H_2}$ and define the optimal control laws. We will show under the conditions of Problem 1 that X and Y can be found analytically.

Ansatz: X is a diagonal matrix. Introduce

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

where the dimensions of the sub-matrices of X match those of A_G. Rewriting (6) in terms of the sub-matrices from Problem 1 (recall that $A_{22} = 0$), (6) is reduced to:

$$\begin{aligned} X_1 A_{11} + A_{11}^\mathsf{T} X_1 - X_1^2 + I = 0, \\ X_1 A_{12} + A_{21}^\mathsf{T} X_2 = 0, \\ X_2 A_{21} + A_{12}^\mathsf{T} X_1 = 0. \end{aligned} \tag{8}$$

Since A_{11}, X_1 and X_2 are all diagonal, the first equation reduces to n scalar quadratic equations, with unique positive definite solution $x_{1,i} = a_{11,i} + \sqrt{a_{11,i}^2 + 1}$, where $x_{1,i}$ denotes the ith diagonal element in X_1, and $a_{11,i}$ the ith diagonal element in A_{11}. Further, using that $A_{21} = -A_{12}^\mathsf{T}$, the final two equations in (8) reduce to the equations $x_{2,i} = x_{1,i}$ for $i = 1, \dots, m$. Inserting these solutions into one diagonal matrix gives:

$$X = \begin{bmatrix} Z & 0 \\ 0 & Z_m \end{bmatrix},$$

where $Z = A_{11} + \sqrt{A_{11}^2 + I}$, and Z_m is Z truncated to the first m rows and columns. Since (A_G, B_w) is stabilisable and (C_z, A_G) is detectable, the unique stabilising solution to (6) is equal to the unique positive semi-definite solution to (6) [14, Corollary 12.5]. Since the X we have found is positive definite, it is therefore the sought stabilising solution. Equation (7) can be solved in an analogous manner. Note that $B_\mathrm{G} B_\mathrm{G}^\mathsf{T} = C_\mathrm{G}^\mathsf{T} C_\mathrm{G}$ and $C_z^\mathsf{T} C_z = B_w B_w^\mathsf{T}$. This gives the same solution for Y, namely that $Y = X$.

By [13, Theorem 1],

$$\gamma^*_{H_2} = \sqrt{\|G_\mathrm{a}(s)\|_{H_2}^2 + \|G_\mathrm{b}(s)\|_{H_2}^2},$$

where

$$\begin{aligned} G_\mathrm{a}(s) &= B_\mathrm{G}^\mathsf{T} X \left(sI - A_\mathrm{G} + YC_\mathrm{G}^\mathsf{T} C_\mathrm{G}\right)^{-1} \left(B_w - YC_\mathrm{G}^\mathsf{T} D_{yw}\right) \\ &= \begin{bmatrix} Q^\mathsf{T} Z & 0 \end{bmatrix} \left(sI - \begin{bmatrix} A_{11} - Z & A_{12} \\ A_{21} & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} Q & -ZQ \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} G_\mathrm{b}(s) &= \left(C_z - D_{zu} B_\mathrm{G}^\mathsf{T} X\right) \left(sI - A_\mathrm{G} + B_\mathrm{G} B_\mathrm{G}^\mathsf{T} X\right)^{-1} B_w \\ &= \begin{bmatrix} Q^\mathsf{T} & 0 \\ -Q^\mathsf{T} Z & 0 \end{bmatrix} \left(sI - \begin{bmatrix} A_{11} - Z & A_{12} \\ A_{21} & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The H_2-norm of a state-space model can be calculated using either its controllability or its observability gramian. More specifically, if $G_\mathrm{g}(s) = C_\mathrm{g}(sI - A_\mathrm{g})^{-1} B_\mathrm{g}$ is stable, its H_2-norm is given by

$$\|G_\mathrm{g}(s)\|_{H_2}^2 = \operatorname{tr}\left(C_\mathrm{g} L_\mathrm{g}^\mathrm{c} C_\mathrm{g}^\mathsf{T}\right) = \operatorname{tr}\left(B_\mathrm{g}^\mathsf{T} L_\mathrm{g}^\mathrm{o} B_\mathrm{g}\right),$$

where L_g^o is the observability gramian, and L_g^c is the controllability gramian. These matrices are in turn given by the positive definite solutions to the two Lyapunov equations

$$A_\mathrm{g} L_g^\mathrm{c} + L_g^\mathrm{c} A_\mathrm{g}^\mathsf{T} + B_\mathrm{g} B_\mathrm{g}^\mathsf{T} = 0 \ \text{ and } \ A_\mathrm{g}^\mathsf{T} L_g^\mathrm{o} + A_\mathrm{g} L_g^\mathrm{o} + C_\mathrm{g}^\mathsf{T} C_\mathrm{g} = 0.$$

We will start by finding the controllability gramian for $G_\mathrm{a}(s)$, which we denote L_a^c.

Ansatz: L_a^c is diagonal. Introducing

$$L_\mathrm{a}^\mathrm{c} = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix},$$

with dimensions of the sub-matrices of L_a^c matching the sub-matrices of A_G, the Lyapunov equation is reduced to:

$$\begin{aligned} (A_{11} - Z)L_1 + L_1(A_{11} - Z)^\mathsf{T} + I + Z^2 = 0, \\ A_{12} L_2 + L_1 A_{21}^\mathsf{T} = 0, \\ A_{21} L_1 + L_2 A_{12}^\mathsf{T} = 0. \end{aligned} \tag{9}$$

Solving the above shows that $L_1 = Z$ and $L_2 = Z_m$.

Solving for L_b^o and making the ansatz that it is diagonal gives the exact same equations as in (9), and thus

$$L_\mathrm{b}^\mathrm{o} = L_\mathrm{a}^\mathrm{c} = \begin{bmatrix} Z & 0 \\ 0 & Z_m \end{bmatrix}.$$

Therefore the H_2 gains of $G_\mathrm{a}(s)$ and $G_\mathrm{b}(s)$ are

$$\|G_\mathrm{a}(s)\|_{H_2}^2 = \operatorname{tr}\left(\begin{bmatrix} Q^\mathsf{T} Z & 0 \end{bmatrix} L_\mathrm{a}^\mathrm{c} \begin{bmatrix} ZQ \\ 0 \end{bmatrix} \right) = \operatorname{tr}\left(Z^3\right),$$

and

$$\|G_\mathrm{b}(s)\|_{H_2}^2 = \operatorname{tr}\left(\begin{bmatrix} Q^\mathsf{T} & 0 \\ 0 & 0 \end{bmatrix} L_\mathrm{b}^\mathrm{o} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \right) = \operatorname{tr}(Z),$$

which implies that $\gamma^*_{H_2} = \sqrt{\operatorname{tr}(Z^3) + \operatorname{tr}(Z)}$ as required.

The realisation of the controller follows from [13, Theorem 1]. In particular

$$A_\mathrm{K} = A_\mathrm{G} - B_\mathrm{G} B_\mathrm{G}^\mathsf{T} X - YC_\mathrm{G}^\mathsf{T} C_\mathrm{G} = \begin{bmatrix} A_{11} - 2Z & A_{12} \\ A_{21} & 0 \end{bmatrix},$$

$$B_\mathrm{K} = YC_\mathrm{G}^\mathsf{T} = \begin{bmatrix} ZQ \\ 0 \end{bmatrix}, \text{ and } C_\mathrm{K} = -B_\mathrm{G}^\mathsf{T} X = \begin{bmatrix} -Q^\mathsf{T} Z & 0 \end{bmatrix},$$

and the proof is complete. ■

III. Applying Theorem 1 to uniformly damped mass-spring networks

In this section we will look at an optimal control problem for a damped mass-spring network. We will see that under the assumption of uniform damping, it is possible to convert this on to the form of Problem 1. We will further show how to write the optimal H_2-gain for this problem in terms of the mass and damper parameters of the network, and that the resulting H_2-controller is itself a passive, damped, mass-spring network, that inherits many of the structural properties of the system that is being controlled. A small numerical example is provided to illustrate the result.

A. The Optimal Control Problem

We first define the problem that is to be studied.

Problem 2: Consider a system described by

$$\begin{aligned} M\ddot{q} + C\dot{q} + Kq &= u + \sqrt{M}w_{\mathrm{u}},\ q\left(0\right) = \dot{q}\left(0\right) = 0, \\ y &= \dot{q} + \sqrt{M^{-1}}w_{\mathrm{y}}, \end{aligned} \tag{10}$$

where $M,\ C,\ K \in \mathbb{R}^{n\times n}$, satisfy the following conditions:

1) M is positive definite;
2) K is positive semi-definite;
3) C is any symmetric matrix that satisfies $CM^{-1}K = KM^{-1}C$.

Find a controller on the form of (5) that minimises the H_2 norm of the closed loop transfer function from disturbance w to performance output z, where

$$w = \begin{bmatrix} w_{\mathrm{u}} \\ w_{\mathrm{y}} \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} \sqrt{M}\dot{q} \\ \sqrt{M^{-1}}u \end{bmatrix}. \qquad \diamondsuit$$

In the context of damped mass-spring networks, the first equation in (10) is a statement of Newton's second law, where q is the generalised coordinates, M is the mass-matrix, C is the damper-matrix, and K is the stiffness-matrix. The input u is a force that can be applied to the network by a controller, and w_{u} is a disturbance acting on the network. The second equation in (10) describes the measurements taken, where it is assumed that the velocity of each generalised coordinate is measured subject to measurement noise w_{y}.

We now discuss the implications of the restrictions 1)–3) in Problem 2.

1) This condition makes M invertible, allowing for a simple conversion of (10) into state-space form. This condition can be relaxed at the expense of more complex derivations. For extensions of the concept of uniform damping to this setting, see [12, Theorem 1].
2) This condition is needed to ensure the skew-symmetric structure $A_{21} = -A_{12}^{\mathsf{T}}$ required in Problem 1 appears when converting Problem 2 into the form of Problem 1. In the damped mass-spring interpretation of Problem 2, this corresponds to that all springs have non-negative spring constants.
3) As discussed in the introduction, this is the uniform damping condition. As shown in [10], this condition is equivalent to the existence of an invertible matrix S such that $S^{\mathsf{T}}MS = I$, and both $S^{\mathsf{T}}CS$ and $S^{\mathsf{T}}KS$ are diagonal. Note in particular that this implies that $Q = S^{\mathsf{T}}\sqrt{M}$ satisfies $QQ^{\mathsf{T}} = I$. Since Q is square this further implies that $Q^{\mathsf{T}}Q = I$.

Remark 2: Any matrix C given by a Caughey series

$$C = M\sum_{j=0}^{n-1} \alpha_j (M^{-1}K)^j,$$

where $\alpha_j \in \mathbb{R}$, is uniformly damped. When $\alpha_j = 0$ for $j \geq 2$ this is typically called Rayleigh damping [11]. Conversely whenever K has distinct eigenvalues a uniformly damped C admits a Caughey series. Note that C need not be positive semi-definite (for example when all the α_j's are negative). When this is the case the dynamics in (10) are unstable. $\diamondsuit$

There are further implicit assumptions in Problem 1. Most significantly, the disturbances and performance outputs in Problem 2 are scaled. These scalings are required to transform Problem 2 into Problem 1. However this requirement can likely be significantly relaxed by generalising Problem 1 and Theorem 1. There are a number of ways this could be approached, but we keep these scalings here for simplicity.

From the application point of view, these scaling are not unreasonable as we now discuss. It is likely reasonable that w_{u} should be scaled by the size of the masses, since larger masses will likely be affected by larger disturbance. Looking at the performance output z, we see that the velocities of large masses incur larger penalties than those of small masses. This is again reasonable, since when larger masses move, they are harder to stop, so it is desirable to prevent this with the control. At the same time, the forces acting on large masses from the controller should be expected to be larger than those of the smaller masses, which is again reflected by the weight on u. The weight on w_{y} is harder to intuitively explain, but is needed for symmetry. Note also that the scalings in terms of the square root of M are not so unnatural, since the H_2-norm penalises the square of the signals in questions. In the case of the process output y, for example, it means that we are minimising the effect on $\int_0^\infty y\left(t\right)^{\mathsf{T}} My\left(t\right)\, \mathrm{d}t$.

B. The solution to Problem 2

In this subsection we describe and illustrate the solution to Problem 2. The derivation from Problem 1 and Theorem 1 will be given in the next subsection.

1) The optimal cost and control law: The optimal cost for Problem 2 is given by

$$\gamma_{H_2}^{*} = \sqrt{\mathrm{tr}\left(Z_C^3\right) + \mathrm{tr}\left(Z_C\right)},$$

where

$$Z_C = -\sqrt{M^{-1}}C\sqrt{M^{-1}} + \sqrt{\sqrt{M^{-1}}CM^{-1}C\sqrt{M^{-1}} + I}. \tag{11}$$

The optimal controller can either be expressed in state-space form, or as a second order differential equation. Taking the later option shows that the control law is given by

$$\begin{aligned} &M_{\mathrm{K}}\ddot{q}_{\mathrm{K}} + C_{\mathrm{K}}\dot{q}_{\mathrm{K}} + K_{\mathrm{K}}q_{\mathrm{K}} = y,\ q_{\mathrm{K}}\left(0\right) = \dot{q}_{\mathrm{K}}\left(0\right) = 0 \\ &u = -\dot{q}_{\mathrm{K}}, \end{aligned} \tag{12}$$

where the matrices are

- $T_{\mathrm{K}} = -C + \sqrt{\sqrt{M}CM^{-1}C\sqrt{M} + M^2}$,
- $M_{\mathrm{K}} = T_{\mathrm{K}}^{-1}MT_{\mathrm{K}}^{-1}$,
- $C_{\mathrm{K}} = T_{\mathrm{K}}^{-1}\left(-C + 2\sqrt{\sqrt{M}CM^{-1}C\sqrt{M} + M^2}\right)T_{\mathrm{K}}^{-1}$,
- $K_{\mathrm{K}} = T_{\mathrm{K}}^{-1}KT_{\mathrm{K}}^{-1}$.

It is interesting to note that the optimal controller in (12) is itself a damped mass-spring network, with parameters written in terms of the original network. A tedious but straightforward calculation in fact shows that C_{K} is positive definite (this follows from the fact that the north-west entry in the A_{K} in (5) is negative definite, and the negative of this entry eventually becomes C_{K} after a sequence of transformations that preserve sign-definiteness). This means that the controller can be implemented physically by building a suitable passive damped mass-spring network.

2) An illustrative case: If the damping matrix C is proportional to the mass matrix M, the controller expressions simplify considerably. If $C = \alpha_0 M$, then $T_{\mathrm{K}} = (\sqrt{\alpha_0^2+1} - \alpha_0)M$, and (12) becomes

$$\begin{aligned}&\frac{M^{-1}\ddot{q}_{\mathrm{K}}}{(\sqrt{\alpha_0^2+1}-\alpha_0)^2} + \frac{(2\sqrt{\alpha_0^2+1}-\alpha_0)M^{-1}\dot{q}_{\mathrm{K}}}{(\sqrt{\alpha_0^2+1}-\alpha_0)^2}\\ &+ \frac{M^{-1}KM^{-1}q_{\mathrm{K}}}{(\sqrt{\alpha_0^2+1}-\alpha_0)^2} = y,\\ &u = -\dot{q}_{\mathrm{K}}.\end{aligned}$$

The unstable case corresponds here to that $\alpha_0 < 0$. The more negative α_0 becomes, the more positive the term in front of $\dot{q}_{\mathrm{K}}$ becomes in relation to the terms in front of $\ddot{q}_{\mathrm{K}}$ and q_{K}. An interpretation of this is that for an unstable system the controller introduces more damping, and the more unstable the original system was, the greater the damping provided by the controller. It should also be noted that the denominator in all terms becomes larger with more negative α_0. This means that the more unstable the system is, the more important the measurement y becomes in the control dynamics.

3) A numerical example: Consider the network in Figure 1, where $m_1 = 1$ kg, $m_2 = 4$, $k_1 = 1$ N/m, and $k_2 = 2$ N/m. This gives the following mass and stiffness matrices

$$M = \begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix} \quad K = \begin{bmatrix}3 & -2\\ -2 & 2\end{bmatrix}.$$

Suppose that the damping is given as Rayleigh damping, then

$$C = \alpha_0 M + \alpha_1 K.$$

In Figure 3 the optimal H_2-gain from disturbances to outputs is plotted for different values of α_0 and α_1 on the interval from -5 to 5. Here it can clearly be seen that if both $\alpha_0 > 0$ and $\alpha_1 > 0$, which corresponds to positive damping, $\gamma^*_{H_2}$ is small. For strictly negative damping, corresponding to the third quadrant where both $\alpha_0 < 0$ and $\alpha_1 < 0$, the performance is much poorer, since the uncontrolled system is unstable. In the second and fourth quadrants of Figure 3 C is positive definite for some combinations of α_0 and α_1, and negative definite or indefinite for others. Depending on the definiteness of C, the performance metric $\gamma^*_{H_2}$ can either be larger or smaller than the case of no damping.

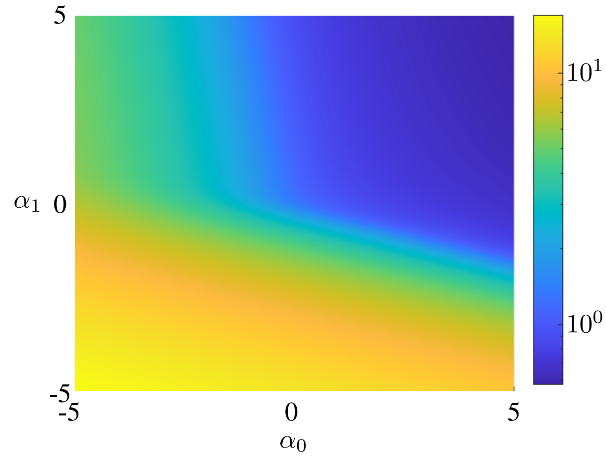

Fig. 3: Levels of H_2 performance achieved as Rayleigh damping parameters α_0 and α_1 vary between -5 and 5. Larger values of α_0 and α_1, which correspond to increased levels of damping in the original network, result in improved performance.

C. Solving Problem 2 with Theorem 1

1) Converting Problem 2 into Problem 1: We now describe the required transformations to convert Problem 2 into Problem 1. We start by introducing a new variable $p = \sqrt{M}q$, thus $q = \sqrt{M^{-1}}p$. Inserting this into (10) and multiplying both sides by $\sqrt{M^{-1}}$ from the left gives

$$\ddot{p} + \sqrt{M^{-1}}C\sqrt{M^{-1}}\dot{p} + \sqrt{M^{-1}}K\sqrt{M^{-1}}p = \sqrt{M^{-1}}u + w_u \tag{13}$$

Define $\tilde{u} = \sqrt{M^{-1}}u$. Since K is positive semi-definite and M is positive definite, $\sqrt{M^{-1}}K\sqrt{M^{-1}}$ is positive semi-definite. Under the uniform damping assumption, $\sqrt{M^{-1}}C\sqrt{M^{-1}}$ and $\sqrt{M^{-1}}K\sqrt{M^{-1}}$ commute. Since in addition $\sqrt{M^{-1}}C\sqrt{M^{-1}}$ and $\sqrt{M^{-1}}K\sqrt{M^{-1}}$ are symmetric, there exists a unitary transformation Q (that is $QQ^{\mathsf{T}} = Q^{\mathsf{T}}Q = I$) such that

$$\begin{aligned}\sqrt{M^{-1}}C\sqrt{M^{-1}} &= Q^{\mathsf{T}}\Lambda_C Q,\\ \sqrt{M^{-1}}K\sqrt{M^{-1}} &= Q^{\mathsf{T}}\begin{bmatrix}\Lambda_K & 0\\ 0 & 0\end{bmatrix}Q,\end{aligned}$$

where Λ_C and Λ_K are both diagonal and Λ_{K} is positive definite. Now define

$$L = \begin{bmatrix}\sqrt{\Lambda_K}\\ 0\end{bmatrix}.$$

This makes L of size $n \times m$, where n is the number of masses in the network, and m is the number of non-zero eigenvalues of K. Introduce the state variable

$$x = \begin{bmatrix}Q\dot{p}\\ L^{\mathsf{T}}Qp\end{bmatrix}.$$

Define $\tilde{y} = \sqrt{M}y$. Including the performance output z and measurement $\tilde{y}$, (13) admits the state-space realisation

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} -\Lambda_C & -L \\ L^\mathsf{T} & 0 \end{bmatrix} x + \begin{bmatrix} Q \\ 0 \end{bmatrix} (\tilde{u} + w_u), \\
z &= \begin{bmatrix} [Q^\mathsf{T} \quad 0] & 0 \\ [0 \quad 0] & I \end{bmatrix} \begin{bmatrix} x \\ \tilde{u} \end{bmatrix}, \\
\tilde{y} &= \begin{bmatrix} Q^\mathsf{T} & 0 \end{bmatrix} x + w_y.
\end{aligned} \tag{14}$$

Problem 2 has now been written on the form of Problem 1, with the sub-matrices fulfilling all conditions in the formulation of Problem 1.

2) Extracting the optimal solution to Problem 2: By Theorem 1, the optimal H_2-gain from disturbances to performance outputs is given by

$$\begin{aligned}
\gamma^*_{H_2} &= \sqrt{\operatorname{tr}(Z^3) + \operatorname{tr}(Z)}, \text{ where} \\
Z &= -\Lambda_C + \sqrt{\Lambda_C^2 + I}.
\end{aligned} \tag{15}$$

With $\Lambda_C = Q\sqrt{M^{-1}}C\sqrt{M^{-1}}Q^\mathsf{T}$, Z can be expressed in terms of the original system matrices according to

$$\begin{aligned}
Z = &-Q\sqrt{M^{-1}}C\sqrt{M^{-1}}Q^\mathsf{T} \\
&+ \sqrt{Q\sqrt{M^{-1}}CM^{-1}C\sqrt{M^{-1}}Q^\mathsf{T} + I}.
\end{aligned}$$

Using the cyclic property of the trace, and the fact that the matrices Q and Q^T can be pulled out of the square root, shows that

$$\operatorname{tr}\left(Z^3\right) = \operatorname{tr}\left(Z_C^3\right) \quad \text{and} \quad \operatorname{tr}(Z) = \operatorname{tr}(Z_C),$$

where Z_C is defined in (11).

From (5), the optimal controller in state-space form is given by

$$\begin{aligned}
\dot{x}_\mathrm{K} &= \begin{bmatrix} -\Lambda_C - 2Z & -L \\ L^\mathsf{T} & 0 \end{bmatrix} x_\mathrm{K} + \begin{bmatrix} ZQ \\ 0 \end{bmatrix} \tilde{y}, \; x_\mathrm{K}(0) = 0, \\
\tilde{u} &= \begin{bmatrix} -Q^\mathsf{T}Z & 0 \end{bmatrix} x_\mathrm{K},
\end{aligned} \tag{16}$$

where Z is defined as in (15). This structure is very similar to the structure of the problem in (14). Reversing the described transformations yields the controller expression in (12). This can be done by first introducing p_K through

$$x_\mathrm{K} = \begin{bmatrix} Q\dot{p}_\mathrm{K} \\ L^\mathsf{T}Qp_\mathrm{K} \end{bmatrix},$$

and then setting $q_\mathrm{K} = T_\mathrm{K}\sqrt{M^{-1}}p_\mathrm{K}$, where

$$T_\mathrm{K} = \sqrt{M}Q^\mathsf{T}ZQ\sqrt{M} = -C + \sqrt{\sqrt{M}CM^{-1}C\sqrt{M} + M^2},$$

and simplifying. T_K is non-singular since M is non-singular. This can most easily be seen in first expression of T_K above where Z is non-singular according to (15) and Q is non-singular due to it being unitary. Equation (16) can, with these transforms, be rewritten and simplified to

$$\begin{aligned}
&T_\mathrm{K}^{-1}MT_\mathrm{K}^{-1}\ddot{q}_\mathrm{K} + T_\mathrm{K}^{-1}KT_\mathrm{K}^{-1}q_\mathrm{K} \\
&+ T_\mathrm{K}^{-1}\left(-C + 2\sqrt{\sqrt{M}CM^{-1}C\sqrt{M} + M^2}\right)T_\mathrm{K}^{-1}\dot{q}_\mathrm{K} = y, \\
&u = -\dot{q}_\mathrm{K},
\end{aligned}$$

which is the result in (12).

IV. Conclusions

An analytical solution to a structured optimal control problem has been derived. This was used to analytically solve a corresponding problem for any system that can be modelled as a uniformly damped network of masses and springs. The results illustrate the impact of damping on system performance, and also that such systems can be optimally regulated by passive networks of damped masses and springs.

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