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Robust Stability Analysis for Continuous-Time Parameter-Varying Persidskii Systems

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Abstract—**The class of parameter-varying generalized Persidskii systems is introduced. For models within this class, characterized by nonlinearities satisfying the sector property, the conditions for (integral) input-to-state stability are proposed. These conditions are established using both, parameter-dependent and parameter-independent, Lyapunov functions. To formulate these conditions, parameterized matrix inequalities are used, which can be reduced into linear ones under additional assumptions concerning the model's dependence on scheduling variables. The efficiency of these stability conditions is illustrated through a numerical example.**

Index Terms—**Persidskii systems, LPV, Input-to-state stability, LMIs**

I. Introduction

The analysis of robust stability and the design of controllers or estimators represent fundamental problems in the field of automatic control theory [1], [2]. The input-to-state stability framework [3], [4] has emerged as one of the most widely used concepts for examining stability in the presence of various forms of uncertainty. In the context of general nonlinear dynamical systems, the synthesis of standard control or estimation algorithms can be challenging, often due to the complexity involved in constructing a Lyapunov function for stability assessment. A common way to overpass this issue consists in using the canonical models: linear parameter-varying (LPV) systems [5], [6], homogeneous dynamics [7], Lur'e models [8], [9]. A variation of the latter is given by Persidskii systems. This class of nonlinear models was first introduced for stability analysis in [10], where a linear combination of the integrals of the nonlinearities was used as a Lyapunov function. That result was extended in [11] by augmenting the Lyapunov function through a combination of the absolute values of the states. Furthermore, Persidskii systems were studied in the context of diagonal stability [12], [13], sliding mode control [14], [15], [16] and Lur'e systems [8], with applications to opinion dynamics [17], neural networks [18], [19], [20] and digital filters [21]. Following the foundational results [10], [11], one of the main advantages of Persidskii dynamics is the availability of a canonical form of Lyapunov function. Recently, several additional results have been proposed to enhance the existing theory, addressing various analysis problems [22], [23], [24], [25], [26], [27]. These developments have a common feature: they lead to the verification of linear matrix inequalities (LMIs), which is an interesting and useful feature of this class of nonlinear systems.

In many cases, it is not always possible to transform the model of a process into a known canonical form. In this way, the LPV framework is widely used, enabling the equivalent representation of a nonlinear system in a linear form with time-varying parameters [5], [6]. Subsequently, this opens the door to the application of the large spectrum of well-established methods and tools of linear system theory (including quadratic or non-quadratic Lyapunov functions [28]). Unfortunately, transformation of stabilizing nonlinearities to LPV form may introduce some conservatism due to a possible loss of stability property yielded by the nonlinear dynamics. In this work, an extension of the LPV framework to Persidskii systems is proposed, where the scheduling parameters can represent the additional time- or state-varying terms, parametric or signal uncertainty, etc., while the useful (passive or negative feedback) nonlinearities are kept in the model. Such a new development can help in analysis of a wide class of nonlinear systems being close to the Persidskii dynamics by using the related Lyapunov function, whose application usually results in LMI constructive stability conditions. It can also be linked with the absolute stability analysis of Lur'e systems considering a set of nonlinearities from a sector [29], [9], but here we will fix the nonlinearity introducing the scheduling parameters in the matrices.

Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ is used for the Euclidean norm on $\mathbb{R}^n$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ and $[t_0, t_1) \subset \mathbb{R}_+$ define the norm $\|d\|_{[t_0,t_1)} = \text{ess}\sup_{t\in[t_0,t_1)} \|d(t)\|$, then $\|d\|_\infty = \|d\|_{[0,+\infty)}$ and the set of d with the property $\|d\|_\infty < +\infty$ we further denote as $\mathcal{L}_\infty^m$ (the set of essentially bounded measurable functions).

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- A finite series of integers $1, 2, ..., n$ is denoted by $\overline{1, n}$, and $\{\overline{1, n}\} = \{1, 2, ..., n\}$.
- Denote the identity matrix of dimension $n \times n$ by I_n, the vector of dimension n or the matrix of dimension $n \times m$ with all elements equal 1 by $\mathtt{1}_n$ and $\mathtt{1}_{n\times m}$, respectively.
- $\mathrm{diag}\{g\} \in \mathbb{D}_+^n$ represents a diagonal matrix of dimension $n \times n$ with a vector $g \in \mathbb{R}_+^n$ on the main diagonal, where $\mathbb{D}_+^n \subset \mathbb{R}^{n\times n}$ is the set of nonnegative diagonal matrices.
- For a matrix $A \in \mathbb{R}^{n\times n}$, denote its i^{th} row and column by $A^{(i)}$ and $A^{[i]}$, respectively, for $i = \overline{1, n}$. The relation $P \prec 0$ ($P \preceq 0$) means that a symmetric matrix $P \in \mathbb{R}^{n\times n}$ is negative (semi-)definite.

II. Preliminaries

In this paper, it is conventionally assumed that if the upper limit of a summation or a sequence is smaller than the lower one, then the corresponding terms (conditions) have to be omitted.

A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is decreasing to zero for each fixed $s > 0$.

Lemma (Finsler's lemma). [30] *Let $x \in \mathbb{R}^n\backslash\{0\}$ and $P, R \in \mathbb{R}^{n\times n}$ are symmetric, then $x^\top Px \prec 0$ whenever $x^\top Rx = 0$ if and only if there exists $\rho \in \mathbb{R}$ such that $P - \rho R \prec 0$.*

A. Input-to-state stability

Consider a nonlinear system:

$$\dot{x}(t) = f(x(t), d(t)),\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathcal{L}_\infty^m$, and $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a locally Lipschitz continuous function, $f(0, 0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ and input $d \in \mathcal{L}_\infty^m$, define the corresponding solutions by $x(t, x_0, d)$ for any $t \geq 0$ for which the solution exists.

In this work we will be interested in the following stability properties [3], [4]:

Definition 1. The system (1) is called *input-to-state practically stable* (ISpS), if there are functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and a constant $c \geq 0$ such that

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma(||d||_{[0,t)}) + c \quad \forall t \geq 0$$

for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$. The function γ is called *nonlinear asymptotic gain*. The system is called *input-to-state stable* (ISS) if $c = 0$.

Definition 2. The system (1) is called *integral ISS* (iISS), if there are functions $\alpha \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$ the estimate holds:

$$\alpha(\|x(t, x_0, d)\|) \leq \beta(\|x_0\|, t) + \int_0^t \gamma(\|d(s)\|)\, ds \quad \forall t \geq 0.$$

These properties have the following characterizations in terms of existence of Lyapunov functions:

Definition 3. A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called *ISpS-Lyapunov function* for the system (1) if there are $r \geq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\eta \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$
$$DV(x)f(x, d) \leq r + \eta(\|d\|) - \alpha_3(\|x\|)$$

for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$. Such a function V is called *ISS-Lyapunov function* if $r = 0$, and it is *iISS-Lyapunov function* if additionally $\alpha_3 : \mathbb{R}_+ \to \mathbb{R}_+$ is just a positive definite function.

Note that an ISS-Lyapunov function can also satisfy the following equivalent condition for some $\chi \in \mathcal{K}$:

$$\|x\| > \chi(\|d\|) \Rightarrow DV(x)f(x, d) \leq -\alpha_3(\|x\|).$$

The relations between these Lyapunov characterizations and the robust stability properties are given below:

Theorem 1. *The system (1) is ISS (ISpS, iISS) if and only if it admits an ISS (ISpS, iISS)-Lyapunov function.*

A consequence of Theorem 1 and Definition 3 is that an ISS system (1) is also iISS.

B. Parameter-varying Persidskii systems

Consider the following class of systems [31], [22]:

$$\dot{x}(t) = A_0(\theta(t))x(t) + \sum_{j=1}^{M} A_j(\theta(t))f^j(H_jx(t)) + d(t), \tag{2}$$

where $x(t) = [x_1(t) \ldots x_n(t)]^\top \in \mathbb{R}^n$ is the state vector, $x(0) \in \mathbb{R}^n$; $d(t) \in \mathbb{R}^n$ is the external disturbance, $d \in \mathcal{L}_\infty^n$; $\theta(t) \in \mathbb{R}^q$ is the vector of time-varying parameters, $\theta \in \mathcal{L}_\infty^q$; $f^j : \mathbb{R}^{k_j} \to \mathbb{R}^{k_j}$ with diagonal structure $f^j(s) = [f_1^j(s_1) \ldots f_{k_j}^j(s_{k_j})]^\top$, $j = \overline{1, M}$ are continuous functions ensuring existence of solutions of the system (2) at least locally in the forward time; continuous matrix functions $A_g : \mathbb{R}^q \to \mathbb{R}^{n\times k_g}$ for $g = \overline{0, M}$ and matrices $H_j \in \mathbb{R}^{k_j\times n}$ for $j = \overline{1, M}$ are given. Further, for brevity and consistently with (2) we use the convention $k_0 = n$ and $H_0 = I_n$ with $f^0(x) = x$.

The model (2) belongs to the class of Persidskii system [11], [12] under the following sector condition imposed on the nonlinearities: for any $j = \overline{1, M}$ and $i = \overline{1, k_j}$,

$$sf_i^j(s) > 0 \quad \forall s \in \mathbb{R} \setminus \{0\}.$$

Consequently, all nonlinearities belong to a sector and may take zero values at zero only. If $\theta(t) = const$, $H_1 = I_n$ and $A_r = 0$ for all $r = \overline{2, M}$, then we recover the system studied by Persidskii in [11]. In the case of $M = 1$, (2) belongs also to the class of Lur'e systems widely investigated in the absolute stability theory [9].

After a proper re-indexing and decomposition of f^j, there exists $\nu \in \{\overline{0, M}\}$ such that for all $z = \overline{1, \nu}$ and $i = \overline{1, k_z}$:

$$\lim_{s \to \pm\infty} f_i^z(s) = \pm\infty;$$

and there exists $\mu \in \{\overline{\nu, M}\}$ such that for all $j = \overline{1, \mu}$ and $i = \overline{1, k_j}$:

$$\lim_{s \to \pm\infty} \int_0^s f_i^j(\sigma) d\sigma = +\infty.$$

Thus, some of the nonlinearities are radially unbounded, and $\nu = 0$ corresponds to the case when all nonlinearities are bounded (at least for negative or positive argument).

III. Problem statement

Consider a parameter-varying Persidskii (PVP) system (2). The following assumptions are first introduced to formulate/state the problem addressed in this paper.

Assumption 1. *Let $\theta(t) \in \Theta$, where $\Theta \subset \mathbb{R}^q$ is a known compact set.*

Assumption 2. *Let $\dot{\theta} \in \mathcal{L}_\infty^q$ and $\|\dot{\theta}\|_\infty \leq \dot{\theta}_{\max}$ for a known constant $\dot{\theta}_{\max} > 0$.*

Assumptions 1 and 2 formulate standard hypotheses for stability analysis of LPV systems: the first just restricts the set of admissible values for the vector of scheduling parameters, while the latter allows us to introduce in Lyapunov functions the dependence on $\theta(t)$ [5]. In general, the dependence on $\theta(t)$ should make the stability conditions less restrictive. However, this comes at the cost of requiring a more intricate numerical procedure for verification. In order to make the conditions more constructive, we will consider the case with linear dependence of the matrix functions in the vector of scheduling parameters:

Assumption 3. *Let $A_g(\theta) = \sum_{k=1}^q \theta_k A_{gk}$ for $g = \overline{0, M}$, where $A_{gk} \in \mathbb{R}^{n \times k_g}$ are known matrices and $\theta_k \in [0, 1]$ for $k = \overline{1, q}$, $\sum_{k=1}^q \theta_k = 1$.*

The objective of this work is to propose conditions of IS(p)S and iISS for the system (2), using a parameter-independent Lyapunov function under Assumption 1, or parameter-dependent one introducing assumptions 1–3 (i.e., in both cases the information about the set Θ and the velocity bound $\dot{\theta}_{\max}$ should be used, but not the properties of a particular trajectory $\theta(t)$). The conventional LPV framework transforms all nonlinearities, which may be difficult and restrictive, while the Persidskii or Lur'e system frameworks do not consider uncertainty presented in the matrices describing the dynamics. In this work we are going to fill these gaps.

IV. Parameter-independent Lyapunov functions

Recalling [22], [23], [25], consider the following structure of a candidate Lyapunov function for (2):

$$V(x) = x^\top P x + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \Lambda_i^j \int_0^{H_j^{(i)} x} f_i^j(s) ds, \qquad (3)$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric matrix and $\Lambda^j = \mathrm{diag}[\Lambda_1^j \dots \Lambda_{k_j}^j] \in \mathbb{D}_+^{k_j}$ are parameters to be tuned in order that (3) verifies the properties stated in Definition 3.

Our main result is stated in the following theorem:

Theorem 2. *Let Assumption 1 be satisfied. Assume there exist matrices $P^\top = P \in \mathbb{R}^{n \times n}$, $\Lambda^j \in \mathbb{D}_+^{k_j}$ for $j = \overline{1, M}$, $\Phi^\top = \Phi \in \mathbb{R}^{n \times n}$, $\Psi \in \mathbb{R}^{n \times n}$ and matrix functions $\Omega_g : \mathbb{R}^q \to \mathbb{R}^{n \times k_g}$, $\Upsilon_{g,g} : \mathbb{R}^q \to \mathbb{D}_+^{k_g}$ ($\tilde{\Upsilon}_{g,g}(\theta) = H_g^\top \Upsilon_{g,g}(\theta) H_g$) for $g = \overline{0, M}$, $\Upsilon_{0,j} : \mathbb{R}^q \to \mathbb{D}_+^{k_j}$ ($\tilde{\Upsilon}_{0,j}(\theta) = H_j^\top \Upsilon_{0,j}(\theta)$) for $j = \overline{1, M}$, $\Upsilon_{z,s} : \mathbb{R}^q \to \mathbb{D}_+^n$ ($\tilde{\Upsilon}_{z,s}(\theta) = H_z \Upsilon_{z,s}(\theta) H_s^\top$) for $z = \overline{1, M-1}$ and $s = \overline{z+1, M}$ such that the following matrix inequalities are verified for all $\theta \in \Theta$:*

$$P \succeq 0, \ \Phi \succ 0, \ Q(\theta) \preceq 0,$$

$$P + \rho_1 \sum_{j=1}^{\mu} H_j^\top \Lambda^j H_j \succ 0,$$

$$\sum_{r=0}^{\nu} \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{\nu-1} \sum_{s=z+1}^{\nu} H_z^\top \tilde{\Upsilon}_{z,s}(\theta) H_s$$
$$+ \rho_3 \sum_{j=1}^{\nu} H_j^\top \Upsilon_{0,j}(\theta) H_j \succ 0$$

for some $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$, where

$$Q(\theta) = \begin{bmatrix} -\Psi^\top - \Psi & P + \Psi^\top A_0 - \Omega_0 & H_1^\top \Lambda^1 + \Psi^\top A_1 - \Omega_1 & \cdots & H_M^\top \Lambda^M + \Psi^\top A_M - \Omega_M & \Psi^\top \\ P + A_0^\top \Psi - \Omega_0^\top & \Omega_0^\top A_0 + A_0^\top \Omega_0 + \Upsilon_{0,0} & \Omega_0^\top A_1 + A_0^\top \Omega_1 + \tilde{\Upsilon}_{0,1} & \cdots & \Omega_0^\top A_M + A_0^\top \Omega_M + \tilde{\Upsilon}_{0,M} & \Omega_0^\top \\ \Lambda^1 H_1 + A_1^\top \Psi - \Omega_1^\top & A_1^\top \Omega_0 + \Omega_1^\top A_0 + \tilde{\Upsilon}_{0,1}^\top & \Omega_1^\top A_1 + A_1^\top \Omega_1 + \Upsilon_{1,1} & \cdots & \Omega_1^\top A_M + A_1^\top \Omega_M + \tilde{\Upsilon}_{1,M} & \Omega_1^\top \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda^M H_M + A_M^\top \Psi - \Omega_M^\top & A_M^\top \Omega_0 + \Omega_M^\top A_0 + \tilde{\Upsilon}_{0,M}^\top & A_M^\top \Omega_1 + \Omega_M^\top A_1 + \tilde{\Upsilon}_{1,M} & \cdots & \Omega_M^\top A_M + A_M^\top \Omega_M + \Upsilon_{M,M} & \Omega_M^\top \\ \Psi & \Omega_0 & \Omega_1 & \cdots & \Omega_M & -\Phi \end{bmatrix},$$

then the system (2) *is ISS. If the last inequality is replaced with*

$$\sum_{r=0}^{M}\tilde{\Upsilon}_{r,r}(\theta)+\rho_2\sum_{z=1}^{M-1}\sum_{s=z+1}^{M}H_z^\top\tilde{\Upsilon}_{z,s}(\theta)H_s$$
$$+\rho_3\sum_{j=1}^{M}H_j^\top\Upsilon_{0,j}(\theta)H_j\succ 0$$

satisfying for all $\theta\in\Theta$, then the system (2) *is iISS.*

All proofs are omitted due to space limitations.

Remark 1. A necessary condition for feasibility of the matrix inequality $Q(\theta)\preceq 0$ required in the formulation of Theorem 2 is that all matrices on the main diagonal of Q are nonnegative definite, i.e., $\Omega_g^\top(\theta)A_g(\theta)+A_g^\top(\theta)\Omega_g(\theta)\preceq 0$ for $g=\overline{0,M}$, which can be easily checked separately. It may also provide a hint on the choice of the structure of the functions $\Omega_g(\theta)$ for known shape of $A_g(\theta)$.

A drawback of the conditions formulated in Theorem 2 is that their verification for all $\theta\in\Theta$ may be computationally costly. Imposing affinity of the system matrix functions $A_g(\theta)$ in θ and restricting the matrix functions $\Omega_g(\theta)$ to be constant for all $g=\overline{0,M}$, these stability conditions can be reduced to LMIs:

Corollary 1. *Let assumptions 1 and 3 be satisfied. Assume there exist matrices $P^\top=P\in\mathbb{R}^{n\times n}$, $\Lambda^j\in\mathbb{D}_+^{k_j}$ for $j=\overline{1,M}$, $\Phi^\top=\Phi\in\mathbb{R}^{n\times n}$, $\Psi\in\mathbb{R}^{n\times n}$, $\Omega_g\in\mathbb{R}^{n\times k_g}$, $\Upsilon_{g,g}\in\mathbb{D}_+^{k_g}$ ($\tilde{\Upsilon}_{g,g}=H_g^\top\Upsilon_{g,g}H_g$) for $g=\overline{0,M}$, $\Upsilon_{0,j}\in\mathbb{D}_+^{k_j}$ ($\tilde{\Upsilon}_{0,j}=H_j^\top\Upsilon_{0,j}$) for $j=\overline{1,M}$, $\Upsilon_{z,s}\in\mathbb{D}_+^{n}$ ($\tilde{\Upsilon}_{z,s}=H_z\Upsilon_{z,s}H_s^\top$) for $z=\overline{1,M-1}$ and $s=\overline{z+1,M}$ such that the following LMIs are verified:*

$$P\succeq 0,\ \Phi\succ 0;\ Q_k\preceq 0,\ \forall k=\overline{1,q};$$
$$P+\rho_1\sum_{j=1}^{\mu}\mathcal{H}_j^\top\Lambda^j\mathcal{H}_j\succ 0,$$
$$\sum_{r=0}^{\nu}\tilde{\Upsilon}_{r,r}+\rho_2\sum_{z=1}^{\nu-1}\sum_{s=z+1}^{\nu}H_z^\top\tilde{\Upsilon}_{z,s}H_s+\rho_3\sum_{j=1}^{\nu}H_j^\top\Upsilon_{0,j}H_j\succ 0$$

for some $\rho_1,\rho_2,\rho_3\in\mathbb{R}$, where Q_k are given in (4)*, then the system* (2) *is ISS. If the last LMI is replaced with*

$$\sum_{r=0}^{M}\tilde{\Upsilon}_{r,r}+\rho_2\sum_{z=1}^{M-1}\sum_{s=z+1}^{M}H_z^\top\tilde{\Upsilon}_{z,s}H_s$$
$$+\rho_3\sum_{j=1}^{M}H_j^\top\Upsilon_{0,j}H_j\succ 0,$$

then the system (2) *is iISS.*

V. Parameter-dependent Lyapunov functions

Consider for (2) the following modification of (3):

$$V(x,\theta)=x^\top P(\theta)x+2\sum_{j=1}^{M}\sum_{i=1}^{k_j}\Lambda_i^j(\theta)\int_0^{H_j^{(i)}x}f_i^j(s)ds, \tag{5}$$

where $P:\mathbb{R}^q\to\mathbb{R}^{n\times n}$ is a symmetric matrix function and denote by $\Lambda^j(\theta)=\mathrm{diag}[\Lambda_1^j(\theta)\ldots\Lambda_{k_j}^j(\theta)]\in\mathbb{D}_+^{k_j}$ a nonnegative diagonal matrix function. We assume that V is differentiable with respect to θ. In such a case, since the coefficients Λ_i^j are assumed to be dependent on θ, the derivative of (5) contains the integrals of the nonlinearities, which requires the introduction of additional hypotheses on their relations with nonlinearities of the system:

Assumption 4. *Let assume that, for any $j=\overline{1,M}$ and $i=\overline{1,k_j}$, there exist $W_{g,g}^{i,j}\in\mathbb{D}_+^{k_g}$ ($\tilde{W}_{g,g}^{i,j}=H_g^\top W_{g,g}^{i,j}H_g$) for $g=\overline{0,M}$, $W_{0,z}^{i,j}\in\mathbb{D}_+^{k_j}$ ($\tilde{W}_{0,z}^{i,j}=H_z^\top W_{0,z}^{i,j}$) for $z=\overline{1,M}$, $W_{k,s}^{i,j}\in\mathbb{D}_+^{n}$ ($\tilde{W}_{k,s}^{i,j}=H_kW_{k,s}^{i,j}H_s^\top$) for $k=\overline{1,M-1}$ and $s=\overline{k+1,M}$ such that*

$$\int_0^{H_j^{(i)}x}f_i^j(s)ds\le\sum_{g=0}^{M}f^g(H_gx)^\top W_{g,g}^{i,j}f^g(H_gx)$$
$$+2\sum_{k=0}^{M-1}\sum_{s=k+1}^{M}f^k(H_kx)^\top\tilde{W}_{k,s}^{i,j}f^s(H_sx)$$

for all $x\in\mathbb{R}^n$.

Assumption 4 is naturally satisfied for polynomial functions, for example: if $f_i^j(s)=s^a$ with $a>0$, then $\int_0^{H_j^{(i)}x}f_i^j(s)ds=\frac{1}{1+a}\left(H_j^{(i)}x\right)^{1+a}\le\left(H_j^{(i)}f^0(H_0x)\right)\left(W_{0,j}^{i,j}\right)_{i,i}f_i^j(H_j^{(i)}x)$ for any $\left(W_{0,j}^{i,j}\right)_{i,i}\ge\frac{1}{1+a}$ all other elements in the matrices $W_{k,s}^{i,j}$ can be selected to be zero.

Theorem 3. *Let assumptions 1, 2 and 4 be satisfied. Assume there exist matrices $\Phi^\top=\Phi\in\mathbb{R}^{n\times n}$, $\Psi\in\mathbb{R}^{n\times n}$, symmetric continuously differentiable matrix functions $P:\mathbb{R}^q\to\mathbb{R}^{n\times n}$, $\Lambda^j:\mathbb{R}^q\to\mathbb{D}_+^{k_j}$ for $j=\overline{1,M}$, and matrix functions $\Omega_g:\mathbb{R}^q\to\mathbb{R}^{n\times k_g}$, $\Upsilon_{g,g}:\mathbb{R}^q\to\mathbb{D}_+^{k_g}$ ($\tilde{\Upsilon}_{g,g}(\theta)=H_g^\top\Upsilon_{g,g}(\theta)H_g$) for $g=\overline{0,M}$, $\Upsilon_{0,j}:\mathbb{R}^q\to\mathbb{D}_+^{k_j}$ ($\tilde{\Upsilon}_{0,j}(\theta)=H_j^\top\Upsilon_{0,j}(\theta)$) for $j=\overline{1,M}$, $\Upsilon_{z,s}:\mathbb{R}^q\to\mathbb{D}_+^{n}$ ($\tilde{\Upsilon}_{z,s}(\theta)=H_z\Upsilon_{z,s}(\theta)H_s^\top$) for $z=\overline{1,M-1}$ and $s=\overline{z+1,M}$ such that the following matrix inequalities are verified for all $\theta\in\Theta$ and $\dot{\theta}\in[-\dot{\theta}_{\max},\dot{\theta}_{\max}]^q$:*

$$P(\theta)\succeq 0,\ \Phi\succ 0,\ Q(\theta,\dot{\theta})\preceq 0,$$
$$P(\theta)+\rho_1\sum_{j=1}^{\mu}H_j^\top\Lambda^j(\theta)H_j\succ 0,$$
$$\sum_{r=0}^{\nu}\tilde{\Upsilon}_{r,r}(\theta)+\rho_2\sum_{z=1}^{\nu-1}\sum_{s=z+1}^{\nu}H_z^\top\tilde{\Upsilon}_{z,s}(\theta)H_s$$
$$+\rho_3\sum_{j=1}^{\nu}H_j^\top\Upsilon_{0,j}(\theta)H_j\succ 0$$

for some $\rho_1,\rho_2,\rho_3\in\mathbb{R}$, where the matrices are defined in (6)*, then the system* (2) *is ISS. If the last inequality is replaced*

$$
Q_k = \begin{bmatrix} -\Psi^\top - \Psi & P + \Psi^\top A_{0k} - \Omega_0 & H_1^\top \Lambda^1 + \Psi^\top A_{1k} - \Omega_1 & \cdots & H_M^\top \Lambda^M + \Psi^\top A_{Mk} - \Omega_M & \Psi^\top \\ P + A_{0k}^\top \Psi - \Omega_0^\top & \Omega_0^\top A_{0k} + A_{0k}^\top \Omega_0 + \Upsilon_{0,0} & \Omega_0^\top A_{1k} + A_{0k}^\top \Omega_1 + \tilde{\Upsilon}_{0,1} & \cdots & \Omega_0^\top A_{Mk} + A_{0k}^\top \Omega_M + \tilde{\Upsilon}_{0,M} & \Omega_0^\top \\ \Lambda^1 H_1 + A_{1k}^\top \Psi - \Omega_1^\top & A_{1k}^\top \Omega_0 + \Omega_1^\top A_{0k} + \tilde{\Upsilon}_{0,1}^\top & \Omega_1^\top A_{1k} + A_{1k}^\top \Omega_1 + \Upsilon_{1,1} & \cdots & \Omega_1^\top A_{Mk} + A_{1k}^\top \Omega_M + \tilde{\Upsilon}_{1,M} & \Omega_1^\top \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda^M H_M + A_{Mk}^\top \Psi - \Omega_M^\top & A_{Mk}^\top \Omega_0 + \Omega_M^\top A_{0k} + \tilde{\Upsilon}_{0,M}^\top & A_{Mk}^\top \Omega_1 + \Omega_M^\top A_{1k} + \tilde{\Upsilon}_{1,M} & \cdots & \Omega_M^\top A_{Mk} + A_{Mk}^\top \Omega_M + \Upsilon_{M,M} & \Omega_M^\top \\ \Psi & \Omega_0 & \Omega_1 & \cdots & \Omega_M & -\Phi \end{bmatrix} \tag{4}
$$

$$
Q = \begin{bmatrix} -\Psi^\top - \Psi & P + \Psi^\top A_0 - \Omega_0 & H_1^\top \Lambda^1 + \Psi^\top A_1 - \Omega_1 & \cdots & H_M^\top \Lambda^M + \Psi^\top A_M - \Omega_M & \Psi^\top \\ P + A_0^\top \Psi - \Omega_0^\top & \Omega_0^\top A_0 + A_0^\top \Omega_0 + \Gamma_{0,0} & \Omega_0^\top A_1 + A_0^\top \Omega_1 + \Gamma_{0,1} & \cdots & \Omega_0^\top A_M + A_0^\top \Omega_M + \Gamma_{0,M} & \Omega_0^\top \\ \Lambda^1 H_1 + A_1^\top \Psi - \Omega_1^\top & A_1^\top \Omega_0 + \Omega_1^\top A_0 + \Gamma_{0,1}^\top & \Omega_1^\top A_1 + A_1^\top \Omega_1 + \Gamma_{1,1} & \cdots & \Omega_1^\top A_M + A_1^\top \Omega_M + \Gamma_{1,M} & \Omega_1^\top \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda^M H_M + A_M^\top \Psi - \Omega_M^\top & A_M^\top \Omega_0 + \Omega_M^\top A_0 + \Gamma_{0,M}^\top & A_M^\top \Omega_1 + \Omega_M^\top A_1 + \Gamma_{1,M} & \cdots & \Omega_M^\top A_M + A_M^\top \Omega_M + \Gamma_{M,M} & \Omega_M^\top \\ \Psi & \Omega_0 & \Omega_1 & \cdots & \Omega_M & -\Phi \end{bmatrix},
$$

$$
\Gamma_{0,0}(\theta, \dot{\theta}) = \sum_{z=1}^{q} \dot{\theta}_z \frac{\partial P(\theta)}{\partial \theta_z} + 2 \sum_{j=1}^{M} \sum_{i=1}^{k_j} W_{g,g}^{i,j} \sum_{z=1}^{q} \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} + \Upsilon_{0,0}(\theta), \tag{6}
$$

$$
\Gamma_{j,j}(\theta, \dot{\theta}) = 2 \sum_{j=1}^{M} \sum_{i=1}^{k_j} W_{j,j}^{i,j} \sum_{z=1}^{q} \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} + \Upsilon_{j,j}(\theta),\ j = \overline{1, M},
$$

$$
\Gamma_{k,s}(\theta, \dot{\theta}) = 2 \sum_{j=1}^{M} \sum_{i=1}^{k_j} \tilde{W}_{k,s}^{i,j} \sum_{z=1}^{q} \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} + \tilde{\Upsilon}_{k,s}
$$

with

$$
\sum_{r=0}^{M} \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{M-1} \sum_{s=z+1}^{M} H_z^\top \tilde{\Upsilon}_{z,s}(\theta) H_s + \rho_3 \sum_{j=1}^{M} H_j^\top \Upsilon_{0,j}(\theta) H_j \succ 0
$$

satisfying for all $\theta \in \Theta$, then the system (2) *is iISS.*

Verification of matrix inequalities formulated in Theorem 3 is more complicated since they depend on two independent vector variables θ and $\dot{\theta}$.

VI. Example

Consider a mechanical system with cubic velocity friction term modeled as:

$$
\dot{x}_1(t) = x_2(t),
$$
$$
\dot{x_2}(t) = -a_1(t)x_1(t) - a_2(t)x_2(t) - a_3(t)x_2^3(t) + d(t),
$$

where $x_1(t), x_2(t) \in \mathbb{R}$ are the system position and velocity, respectively, $d(t) \in \mathbb{R}$ is a bounded disturbance; $a_i(t)$ for $i = 1, 2, 3$ are positive time-varying parameters, whose instantaneous values are unknown, but the sets of admissible deviations have been identified: $a_{i\,\min} \le a_i(t) \le a_{i\,\max}$ for all $t \ge 0$ with some $0 < a_{i\,\min} \le a_{i\,\max} < +\infty$ being the minimal and maximal possible values, respectively, for these parameters. Our goal is to verify robust stability of this system using the proposed conditions. It is easy to check that this model can be rewritten in the form (2) for $M = 1$ and $f^1(s) = s^3$, $H_1 = [0 \quad 1]$, $k_1 = 1$ satisfying sector condition, with

$$
A_g(\theta) = \sum_{k=1}^{q} \theta_k A_{gk},\ g = 0, 1,\ q = 8,
$$
$$
A_{01} = A_{05} = \begin{bmatrix} 0 & 1 \\ -a_{1\,\min} & -a_{2\,\min} \end{bmatrix},
$$
$$
A_{02} = A_{06} = \begin{bmatrix} 0 & 1 \\ -a_{1\,\min} & -a_{2\,\max} \end{bmatrix},
$$
$$
A_{03} = A_{07} = \begin{bmatrix} 0 & 1 \\ -a_{1\,\max} & -a_{2\,\min} \end{bmatrix},
$$

$$
A_{04} = A_{08} = \begin{bmatrix} 0 & 1 \\ -a_{1\,\max} & -a_{2\,\max} \end{bmatrix},
$$
$$
A_{11} = \cdots = A_{14} = \begin{bmatrix} 0 \\ -a_{3\,\min} \end{bmatrix},
$$
$$
A_{15} = \cdots = A_{18} = \begin{bmatrix} 0 \\ -a_{3\,\max} \end{bmatrix},
$$

where $\sum_{k=1}^{q} \theta_k = 1$ and $\theta_k \in [0, 1]$. The compact set $\Theta \subseteq [0, 1]^q$ can be straightforwardly computed, then Assumption 1 is verified. Moreover, it is clear that Assumption 3 is also satisfied, therefore, the LMIs of Corollary 1 can be used to check ISS property of this time-varying nonlinear system.

Let

$$
a_{1\,\min} = 2,\ a_{2\,\min} = 5,\ a_{3\,\min} = 0.1,
$$
$$
a_{1\,\max} = 3,\ a_{2\,\max} = 10,\ a_{3\,\max} = 0.2,
$$

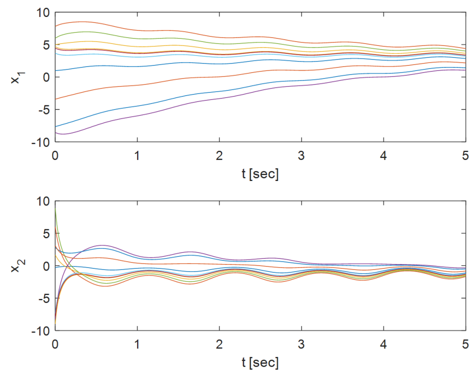

Figure 1. The results of simulation for mechanical system: $x(t)$ versus time t [sec]

then the LMIs of Corollary 1 are verified (up to numerical reliability, with eigenvalues of Q_k lying in the numeric precision bundle). Select

$$d(t) = \begin{bmatrix} 1+\sin(6t) \\ 1+\cos(3t) \end{bmatrix},\ a_1(t) = 2+\sin^2(3t),$$
$$a_2(t) = 5+5\cos^2(3t),\ a_3(t) = 0.1+0.1\sin^2(3t),$$

then in Fig. 1, the results of simulation are shown for ten different initial conditions chosen randomly in the set $[-10, 10]^2$. All trajectories converge to a vicinity of the origin proportional to the amplitude of the perturbation d, which is an illustration of ISS behavior.

VII. Conclusion

In this paper, the class of Parameter-Varying Persidskii (PVP) systems is introduced. This class serves as a framework to model complex dynamics involving multiple nonlinearities and various uncertainties. Within this framework, we present conditions for Input-to-State Stability (ISS) and integral Input-to-State Stability (iISS) in the form of matrix inequalities. These conditions are derived using either parameter-independent or parameter-dependent Lyapunov functions. Future directions for research include the design of stabilizing controls and observers for PVP systems.

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