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Examples of Constructing Exact Reachable Sets for Infinite Dimensional Control System*

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Abstract— **Exact time-dependent reachable sets for dynamic control systems are natural control theory extensions of the exact solutions of Cauchy problem for differential equations. The reachable sets are important in many applications, including optimal control, when analytical expressions are possible. The extensions are usually based on differential-geometric approaches that enable effective methods of analysis and design of nonlinear finite-dimensional control systems. Although a theoretical development for infinite-dimensional systems, including PDEs, requires significantly more advanced mathematics, the analytical technique is of a comparable level of complexity to the finite-dimensional case. The purpose of this paper is to present a set of examples illustrating techniques of obtaining exact analytical expressions of the reachable sets for important classes of linear and nonlinear PDEs: 1-st order, hyperbolic, and parabolic.**

I. Introduction

Exact time dependent reachable set is a natural control theory extension of the exact solution of Cauchy (initial value) problem for differential equations. The construction of reachable sets utilizes a differential-geometric approach. A brief survey [Krener, J. (2014)] reviews the concepts and theorems of differential geometry for control systems in finite-dimensional spaces. Isidori's book [Isidori, A. (1995)] provides a comprehensive and rigorous treatment of the topic with many applications, including feedback linearization. An alternative approach is presented in [Agrachev, A. and Sachov, Yu. (2004)]. Ref. [Schatler, H. and Ledzewick, U. 2012] describes applications of Lie algebras in optimal control.

Differential-geometric approach for infinite-dimensional control systems, including PDEs, is less developed, however. In fact, not all required theorems of finite dimensional differential geometry can be extended to Banach spaces, especially for systems governed by PDEs, i.e. non-continuous operators [Coron, J-M. (2007)]. There are fundamental reasons for this situation discussed in [Arnold, V. (2004)].

However, a proper limited extension applicable to some important classes of control systems was developed by P. Dudnikov and S. Samborskii [Dudnikov, P. and Samborskii, S. (1980)], [Samborskii, S. (1983)].

The theorem of Rashevski-Chow [Rashevski, P. (1938)], [Chow, W. (1939)], [Krener, J. (1974)], [Coron, J-M. (2007), pp. 134-135] is a central result that provides a criterion for complete controllability of symmetric (or driftless) finite-dimensional control systems. This theorem was extended to Banach spaces in [Dudnikov, P. and Samborskii, S. (1980)] and provided the necessary background for development of a differential-geometric approach to control systems described by symmetrical PDEs [Samborskii, S. (1983)], and nonsymmetrical PDEs [Belikov, S. and Samborskii, S. (1983)], [Belikov, S. (1988)], [Belikov, S. (1981)].

In the last two decades there were many publications studying controllability of particular PDEs with high symmetries, e.g. [Coron, J-M. (2007)], [Cerpa, E. and Crepean, E. (2009)]. [Agrachev, A. and Sachov, Yu, (2005)] and [Shirikyan, A. (2006)] used Lie brackets to controllability study for Navier-Stokes equations.

Results of these publications are usually about local, sometimes global, controllability. However, the author did not find publications about the construction of exact analytical expressions of reachable sets for non-controllable PDEs, which is the main topic of this paper.

For brevity, we concentrate on examples of analytical techniques rather than a detailed rigorous formulation (though we provide references for the latter).

Section II of this paper provides an informal background including main limitations necessary for the construction. We assume the reader is familiar with differential geometry applied to control systems - [Isidori, A. (1995)]), [Coron, J-M. (2007)], and PDE theory [Evans, C. (2010)], especially obtaining exact solutions of initial value problems.

Section III (the central section of this paper) presents examples that demonstrate specifics of infinite-dimensional techniques as well as their essential differences from finite-dimensional ones.

Applications are out of scope of this paper; we concentrate on analytical technique. Some practical modelling from first principles can be found in [Corriou, J-P. (2018)] and [Glowinski, R., Lions, J-L. and He, J. (2008)]. The latter classical reference provides numerical approach to controllability based on existence of terminal control problems for PDEs. It covers convex duality, space-time discretization of PDEs, optimization algorithms, etc. that are also applicable to the systems described in this paper when the terminal target belongs to the reachability set.

In contrast to publications about controllability, we consider examples of constructing exact reachable sets, mainly for non-controllable systems. Examples are the main contribution of the paper. According to [Arnold, V. (2004)], "for a student the content of mathematical theory is never larger than the set of examples that are thoroughly understood."

Examples include a first order PDE as well as a wave and a heat equation with control. An example of nonlinear continuous operator (y^2) in $L_2[0,1]$ illustrates an interesting connection with classical approximation theory [Achiezer, N. (2003)]. Our hope is that these examples will narrow the gap between mathematical theory and technical applications.

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II. Background

A. Notations and Assumptions

We consider infinite dimensional nonsymmetrical systems (which become symmetrical for f=0) with m inputs

$$\frac{\partial y(t)}{\partial t} = f(y(t)) + \sum_{i=1}^{m} u_i(t) g_i(y(t)) \tag{1}$$

where $u_i(t)$ are piecewise-continuous scalar controls; state $y(t)$ takes values in Banach space B; and f, g_i are maps into B.

The most important assumption is that *f may not be continuous (and can be a differential operator)*; and the most important limitation is that g_i must be continuous (bounded in linear case).

An even more restrictive assumption is that *iterative Lie brackets of f and* g_i *must be continuous maps***.** This last assumption, however, holds in several important situations: a) when f is continuous; b) when f is unbounded linear and g_i are constant vectors in B (linear systems, including PDEs).

There are many other cases with non-continuous f where this assumption holds, including a differential operator f and integral operators g_i. Examples below will illustrate some cases.

The mathematical constructions necessary for rigorous formulation of the theorems could be found in [Dudnikov, P. and Samborskii, S. (1980)], [Samborskii, S. (1983)], [Belikov, S. and Samborskii, S. (1983)], [Belikov, S. (1988)], [Belikov, S. (1981)].

The main tool for these constructions is a definition of a Banach space B_1 (usually domain of f) that is dense in B, with a continuous map f_1: $B_1 \to B$.

Kuen book [Kuen, C. (2019)] provides an introductory PDE dynamics background with references to [Evans, C. (2010)] for rigorous proofs. Lions's book [Lions, J-L. (1971)] provides rigorous control-specific treatment with Sobolev's spaces and describes numerous examples.

As a simple example, for the differential operator $f=\partial/\partial x$ in the space of continuous functions C, the map f_1:$C_1 \to C$ is continuous, where C_1 is the space of differentiable functions.

We also use the following standard definitions, [Isidori, A. (1995)], [Coron, J-M. (2007)], extended to Banach spaces as suggested in [Samborskii, S. (1983)]:

Lie bracket $[f, g]$*:*

$$[f,g](y) = Df \circ g(y) - Dg \circ f(y) \tag{2}$$

where D is Fréchet derivative, and

$$\mathrm{ad}_f^0\, g = g, \quad \mathrm{ad}_f\, g = \mathrm{ad}_f^1\, g == [f,g], \quad \mathrm{ad}_f^n\, g = \left[f, \mathrm{ad}_f^{n-1}\, g\right] \tag{3}$$

(A rigorous formulation of the sets of densely embedded Banach spaces, where Lie algebras of operators can be rigorously defined, is described in [Samborskii, S. (1983)]).

Remark 1. Definition (2) differs from traditional finite dimensional definition of Lie bracket [Isidori, A. (1995)] by a minus sign. This allows reducing numbers of minus signs in derivations, and is consistent with controllability matrix (B, AB,…,$A^{n-1}B$) for linear control systems, where $[Ay, B]=AB$ rather than $-AB$.

We also define the following Lie algebras [Samborskii, S. (1983)], [Belikov, S. and Samborskii, S. (1983)]:

$$\mathrm{A}_0 = \overline{\mathrm{la}}\{g_j : j = 1..m\}, \quad \mathrm{A}_{i+1} = \overline{\mathrm{la}}\{[f, \mathrm{A}_i], \mathrm{A}_i\} \tag{4}$$

where $\overline{\mathrm{la}}\{\ \}$ is the closure of the Lie algebra generated by the set in the {}, and

$$\mathrm{L} = \overline{\mathrm{la}}\{\mathrm{ad}_f^n\, g_i : i = 1..m, n = 0,1...\} \tag{5}$$

Definitions of finite-dimensional differential-geometric terms, e.g. Lie algebra, with applications to control systems can be found in [Isidori, A. (1995)], and some applications to PDE controllability examples –in [Coron, J-M. (2007)].

We also assume a technical conditions of [Belikov, S. and Samborskii, S. (1983)]. These conditions are usually satisfied.

In contrast to controllability conditions, our focus is on analytical expressions of exact reachable sets for non-controllable systems.

B. Results Needed for Model Examples of Section III

In the examples of Section III we illustrate analytical expressions of reachable sets $\Re_{y_0}(T)$ for system (1) that is the closure of the points in space B that can be reached at time T by the solutions of Eq. (1).

Unfortunately, the exact reachable set can be analytically constructed only for some specific classes of control systems with certain symmetries (this is in fact also the case for finite dimensions). Formulation of the verifiable sufficient conditions that a control system belongs to one of the classes is very important (see *Statement* below).

Hirschorn's condition is an easily verifiable sufficient condition formulated in [Hirschorn, R. (1976)] for finite-dimensional systems:

$$[\mathrm{L}, \mathrm{A}_0] \subseteq \mathrm{A}_0 \tag{6}$$

The following sufficient condition was derived in [Belikov, S. and Samborskii, S. (1983)] for infinite-, as well as finite-, dimensional systems:

$$\forall g \in \mathrm{A}_i : [[f,g],g] \in \mathrm{A}_i,\ i = 0,1,... \tag{7}$$

It is also proved in [Belikov, S. and Samborskii, S. (1983)] that condition (7) is less restrictive than (6). This will be illustrated in the examples.

We now describe the construction of the reachable set assuming that condition (7) is satisfied along with the technical conditions [Belikov, S. and Samborskii, S. (1983)], [Belikov, S. (1981)] outlined in Section II.A. To formulate these conditions, we need a few additional definitions.

Let S_t be the semigroup associated with the operator f (see [Krein, S. (1971), p. 25], [Coron, J-M. (2007), pp. 373-377], [Evans, C. (2010), pp. 433-445] for linear f; and [Evans, C. (2010), pp. 565-570] for nonlinear f), i.e. $S_t(y_0)$ is the solution of Eq. (1) with initial condition y_0 and $u_i(t)\equiv 0$. For $\varphi\in$L let us define the vector field $\psi=S_t\varphi\in$B by the following:

$$\exists\psi\in \mathrm{B}: DS_t\big|_y\,\varphi(y)=\psi(S_t y)\quad \Rightarrow \psi \overset{def}{=} S_t\varphi \tag{8}$$

The technical assumption of the existence of ψ is usually satisfied. Let us use notation $\mathrm{E}^{\mathrm{A}}y$ for the integral manifold of the Lie algebra A (i.e. infinite-dimensional extension of involutive distribution [Isidori, A. (1995)] according to [Samborskii, S. (1983)]) that contains y, and assume another, usually satisfiable, technical condition:

$$\mathrm{E}^{S_t\,\mathrm{L}}\,\mathrm{E}^{\mathrm{L}}\,y=\mathrm{E}^{\mathrm{L}}\,\mathrm{E}^{S_t\,\mathrm{L}}\,y \tag{9}$$

We also need the following notation L_T:

$$\mathrm{L}_T=\overline{\mathrm{la}}\{S_t\,\mathrm{L}:t\in[0,T]\} \tag{10}$$

Statement. If condition (7) is satisfied along with technical conditions of [Belikov, S. and Samborskii, S. (1983)] and (8)-(9), then

$$\Re_{y_0}(T)=\mathrm{E}^{\mathrm{L}_T}\,S_T y_0 \tag{11}$$

We do not call this important statement a theorem because it is an informal non-rigorous summary of the theorems rigorously formulated and proved in [Samborskii, S. (1983)], [Belikov, S. and Samborskii, S. (1983)], [Belikov, S. (1981)]. This summary, however, is sufficient for examples in Section III.

Remark 2. Rigorous formulation requires advanced mathematical constructions that cannot be reproduced here due to space limitation. Section II.A, however, provides an overview and references to reproduce rigorous formulations and proofs. See also Remark 3.

Unfortunately, statement (11) and the related rigorous theorems ([Samborskii, S. (1983)], [Belikov, S. and Samborskii, S. (1983)], [Belikov, S. (1981)]) are far from being the "end of the story". It gives the proven framework, but in any particular example one needs to derive the related Lie algebras; to prove, if possible, condition (6) or (7); and finally, to find analytical solutions for Eq. (1) without control, and for ODEs describing the vector fields of Lie algebra L_T.

Examples presented in this paper demonstrate ideas and techniques to overcome these challenges. Certainly, similar to the exact solution of the Cauchy problem, this may not be possible for "most" of systems (1). However, the symmetry of the "unique" systems, where it is possible, indicate their beauty and importance [Stewart, I. (2007)]. In theoretical physics and engineering the importance of equations that could be solved analytically was well justified and drove scientific research for nearly four centuries (even before rigorous justification; in fact, using infinitesimals as numbers was justified in [Robinson, A. (1996)]). Collection of control systems with analytical expressions for reachable sets may be equally important. Classical results of mathematical physics [Evans, C. (2010)] remain essential for control theory as well.

Remark 3. For finite dimensional systems $\mathrm{L}=\mathrm{L}_T$, and construction of $\mathrm{E}^{\mathrm{L}}y$, based on Frobenius theorem, is described in [Isidori, A. (1995)]. A formal procedure for PDE systems is essentially the same using Fréchet derivative.

However, rigorous constructions are highly nontrivial because, by contrast with ODE, "in theory of PDE the difficulties of communicative algebraic geometry are inextricably bound up with noncommunicative differential algebra, in addition to which the topological and analytical problems that arise are profoundly nontrivial" [Arnold, V. (2004)].

There are also serious limitations outlined in Section II.A. Fortunately, while these nontrivial constructions are essential for rigorous formulations of general results (see Section II.A and Remark 2), they do not prevent applications of the analytical techniques, which we illustrate with examples below.

Simplest Illustrative Example. Let us illustrate the statement on the simplest linear first order PDE:

$$\frac{\partial y(t,x)}{\partial t}=-\frac{\partial y(t,x)}{\partial x}+u(t)p(x),\quad y(0,x)=y_0(x)$$

defined on the loop S=[0,1]mod1, and $y(t,\cdot)\in L_2(S)$.

Assume $p\in C_0^\infty[0,\alpha]\subset C^\infty[S]\subset L_2(S)$, $\alpha<1$, i.e. it is infinitely differentiable and is equal to zero outside the interval $[0,\alpha]$. The semigroup S_t is $S_t y_0(x)=y_0(x-t)$; $\mathrm{A}_0=\overline{\mathrm{sp}}\{p(x)\}$; $[-\partial/\partial x,p(x)]=-p'(x)$. In this example $\mathrm{L}\neq\mathrm{L}_T$. Indeed, $\mathrm{L}=\overline{\mathrm{sp}}\{p^{(k)}(x):k=0,1..\}$; $\mathrm{L}_T=\overline{\mathrm{sp}}\{p^{(k)}(x-\nu):0\le\nu\le T,\,k=0,1..\}$ and $\mathrm{E}^{\mathrm{L}_T}\,y(x)=y(x)+\mathrm{L}_T$.

Condition (6) is always satisfied for linear systems. Then

$$\Re_{y_0(x)}(T)=y_0(x-T)+\overline{\mathrm{sp}}\{p^{(k)}(x-\nu):0\le\nu\le T,\,k=0,1..\}$$

In this example $\mathrm{L}_T\neq\mathrm{L}$. As the result, the system may be controllable for $T\geq 1-\alpha$, but is not controllable for $T<1-\alpha$ (because $p^{(k)}(x-T)\equiv 0,\ \alpha+T<x<1$).

Remark 4. For finite-dimensional systems L_T=L. The difference between L_T and L is an important specific feature of some infinite-dimensional systems that will be illustrated in the following examples.

"Counter example". Let us consider an example where formula (11) does not take place (in this system condition (7) is not satisfied).

We consider control system (1) in R^2 with m=1 [Coron, J-M. (2007), p. 131]:

$$\dot y_1=y_2^2,\quad \dot y_2=u(t)$$

Here $f(y)=(y_2^2,0)^T$, $g_1(y)=(0,1)^T$, $\mathrm{A}_0=\overline{\mathrm{sp}}\{g_1\}$.

Calculating the Lie brackets, we get

$$g_2(y)=[f,g_1](y)=(2y_2,0)^T,\qquad g_3(y)=[g_1,g_2](y)=-(2,0)^T,$$

$\mathrm{A}_1\supseteq\overline{\mathrm{sp}}\{g_1,g_3\}=\mathrm{R}^2$. However, the system is not controllable. Indeed, it cannot be moved in direction g_3 (i.e. $\dot y_1=-2$) because $\dot y_1=y_2^2\geq 0$.

The reason of possibility of the "contradiction" is that condition (7) is not satisfied:

$[[f,g_1],g_1] = -(2,0)^T \notin A_0 = \overline{sp}\{g_1\} = \overline{sp}\{(0,1)^T\}$.

According to the Statement, this "contradiction" cannot take place if (7) is satisfied.

III. Model Examples

Before presenting examples of infinite-dimensional systems we illustrate the concepts on a finite-dimensional one. This will also help to appreciate the difference between finite- and infinite- dimensional techniques. An important technique of verifying condition (6) or (7) for finite-dimensional systems is to compare dimensions of iterated Lie algebras, while in infinite-dimensional cases specific functional-analytical approaches are required. Infinite-dimensional systems may also possess specific properties that are not possible in finite dimensions (see Remarks 3-4).

A. Finite dimensional nonlinear non-controllable system

In this section we demonstrate a construction of an exact reachable set for a nonlinear non-controllable finite-dimensional system.

Let us consider system (1) in R^4 with m=2 and

$$f(y) = (0, y_1 \cdot y_2, y_2, -y_2 + y_3 + y_4)^T \quad (12)$$

$$g_1(y) = (0, y_1, 0, 0)^T; \quad g_2(y) = (1, 0, y_1, -y_1)^T \quad (13)$$

Calculating the Lie brackets, we get:

$$g_3(y) = [g_1, g_2](y) = (0,1,0,0)^T; \quad [g_1, g_3] = [g_2, g_3] = 0 \quad (14)$$

We conclude from (13)-(14) that

$$A_0 = \overline{sp}\{g_1, g_2, g_3\}; \mathrm{rank}\begin{pmatrix} 0 & y_1 & 0 & 0 \\ 1 & 0 & y_1 & -y_1 \\ 0 & 1 & 0 & 0 \end{pmatrix}^T = 2 \Rightarrow \dim A_0 = 2 \quad (15)$$

Continuing calculations, we get:

$$\begin{aligned} g_4(y) &= [f, g_1](y) = (0, y_1^2, y_1, -y_1)^T \\ g_5(y) &= [f, g_2](y) = (0, y_2, 0, 0)^T \\ g_6(y) &= [f, g_3](y) = (0, y_1, 1, -1)^T \end{aligned} \quad (16)$$

We conclude from (12)-(16) that

$$\mathrm{rank}\begin{pmatrix} 0 & y_1 & 0 & 0 \\ 1 & 0 & y_1 & -y_1 \\ 0 & 1 & 0 & 0 \\ 0 & y_1^2 & y_1 & -y_1 \\ 0 & y_2 & 0 & 0 \\ 0 & y_1 & 1 & -1 \end{pmatrix}^T = 3 \Rightarrow \dim \ L \geq 3 \quad (17)$$

We now show that dim L = 3 and

$$L(y) = A_1(y) = H, \quad H = \{y \in R^4 : y_3 + y_4 = 0\} \quad (18)$$

Assertion (18) follows from a stronger assertion

$$\forall g \in L : g(R^4) \subseteq H = \{y \in R^4 : y_3 + y_4 = 0\} \quad (19)$$

To prove (19) we observe that $\forall g \in A_0 : g(R^4) \subseteq H$ and, thus, need to prove that

$$[\forall g \in A_i : g(R^4) \subseteq H] \Rightarrow [\forall g \in A_{i+1} : g(R^4) \subseteq H] \quad (20)$$

Let $g(R^4) \subseteq H$ and consider

$$[f, g](y) = Df|_y g(y) - Dg|_y f(y) \quad (21)$$

We have

$$Df|_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ y_2 & y_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} \Rightarrow Df|_y H \subseteq H \quad (22)$$

and Dg has the following structure

$$g(R^4) \subseteq H \Rightarrow Dg|_y = \begin{pmatrix} \bullet \\ \bullet \\ (\mathrm{grad}\,\alpha(y))^T \\ -(\mathrm{grad}\,\alpha(y))^T \end{pmatrix} \quad (23)$$

with arbitrary first two rows and the fourth row is a negation of the third (α is an arbitrary function). Then $Dg|_y R^4 \subseteq H$ and

$$[f, g](R^4) \subseteq H = \{y \in R^4 : y_3 + y_4 = 0\} \quad (24)$$

Similar way, if vector fields c_1 and c_2 map R^4 to H, their Lie bracket also possesses this property, and this concludes the proof of (20) and, thus, the assertion (18).

As $L(y) = A_1(y)$, the sufficient condition (7) should be verified only for i=0. The condition is satisfied because $[g_4, g_1] = [g_5, g_2] = [g_6, g_3] = 0$. Sufficient condition (6), however, is not satisfied. Indeed, $g_4 \in L$, $g_2 \in A_0$, but $[g_4, g_2](y) = (0, 2y_1, 1, -1)^T$ and $[g_4, g_2] \notin A_0$ because $\dim A_0$=2 and

$$\mathrm{rank}\begin{pmatrix} 0 & y_1 & 0 & 0 \\ 1 & 0 & y_1 & -y_1 \\ 0 & 1 & 0 & 0 \\ 0 & 2y_1 & 1 & -1 \end{pmatrix}^T = 3$$

Thus, the sufficient condition (7) is satisfied and all operators are continuous (as is usually the case for finite-dimensional systems), and

$$\mathfrak{R}_{y_0}(T) = E^{L_T} S_T y_0 = E^{L} S_T y_0 = \{S_T y(0) + H\}$$

where $S_t(y(0))$ is the solution of the following system of ODEs

$$\left\{\begin{array}{l} \dot{y}_1 = 0 \\ \dot{y}_2 = y_1 \cdot y_2 \\ \dot{y}_3 = y_2 \\ \dot{y}_4 = -y_2 + y_3 + y_4 \\ y_i(0) = y_{i0}, i = 1,2,3,4; y_{10} \neq 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} y_1(t) \equiv y_{10} \\ y_2(t) \equiv y_{20}\exp(y_{10}t) \\ y_3(t) \equiv (y_{20}/y_{10})\exp(y_{10}t) \\ + y_{30} - y_{20}/y_{10} \\ y_4(t) \equiv -y_3(t) + (y_{30} + y_{40})e^t \end{array}\right\}$$

Then finally

$$\mathfrak{R}_{y_0}(T) = \{y \in R^4 : y_3 + y_4 = (y_{30} + y_{40})e^T\} \quad (25)$$

B. f is a continuous operator in $L_2[0,1]$

This example illustrates interesting features of infinite-dimensional systems (1) even with continuous operator f. Let us consider the following system in $L_2([0,1])$:

$$\frac{\partial y(t,x)}{\partial t} = (y(t,x))^2 + \sum_{i=1}^{\infty} u_i(t) x^{2^i} \qquad (26)$$

This system with $u_i \equiv 0$ does not generate a continuous semigroup of operators $S(t)$. The solution of the equation $\dot{y} = y^2$, $y(0) = y_0$ is $S(t)y_0 = y_0/(1 - ty_0)$, and for positive y_0 it is defined and satisfies a semigroup property only for $t<1/y_0$, i.e.

$\forall y_0 > 0 \ \forall [t_1, t_2 > 0 : t_1 + t_2 < y_0^{-1}] : S(t_1 + t_2) y_0 = S(t_1) S(t_2) y_0$.

This property allows us to construct reachable sets / controllability for small time.

Let by $(\bullet)^k$ denote the operator (vector field)

$$(\bullet)^k : y(x) \to (y(x))^k$$

Calculating Lie brackets, we have

$$[x^n \cdot (\bullet)^k, x^m \cdot (\bullet)^l](y(x)) = kx^n \cdot y^{k-1}(x) \cdot x^m \cdot y^l(x) - lx^m \cdot y^{l-1}(x) \cdot x^n y^k(x)$$

i.e.

$$[x^n \cdot (\bullet)^k, x^m \cdot (\bullet)^l](y(x)) = (k-l) x^{n+m} y^{k+l-1}(x) \qquad (27)$$

Using (27) let us calculate

$$[x^{2^i}, x^{2^j}] = 0 \Rightarrow \mathrm{A}_0 = \overline{\mathrm{sp}}\{x^{2^i} : i = 1,2,...\} \qquad (28)$$

$[(\bullet)^2, x^{2^i}](y(x)) = 2 \cdot x^{2^i} \cdot y(x)$, $[x^{2^i}(\bullet), x^{2^j}](y(x)) = x^{2^i + 2^j}$,

$[(\bullet)^2, x^{2^i + 2^j}](y(x)) = 2 \cdot x^{2^i + 2^j} \cdot y(x)$,

$[x^{2^i + 2^j}(\bullet), x^{2^k}](y(x)) = 2 \cdot x^{2^i + 2^j + 2^k}$.

Observing that $\left\{ \sum_{l=1}^{N} \alpha_l 2^l : \alpha_l \in \{0,1\}; N = 1,2,... \right\}$ is the set of all even positive numbers, we conclude that

$$\mathrm{A}_1 = \overline{\mathrm{sp}}\{x^{2i}, x^{2j}(\bullet) : i, j = 1,2,...\} \qquad (29)$$

According to the Muntz's theorem [Achiezer, N. (2003), pp. 43-46], the system $\{x^{p_i}\}$ is closed in $L_2([0,1])$ if and only if $\sum_{i=1}^{\infty} (p_i)^{-1} = \infty$. Using this result we conclude that $\mathrm{A}_1 = L_2([0,1])$, (because $\sum_{i=1}^{\infty} (2i)^{-1} = \infty$). Thus, sufficient condition (7) should be verified only for $\mathrm{A}_0 = \overline{\mathrm{sp}}\{x^{2^i} : i = 1,2,...\}$:

$$[[(\bullet)^2, x^{2^i}], x^{2^i}](y(x)) = [2 \cdot x^{2^i} \cdot (\bullet), x^{2^i}](y(x)) = 2 \cdot x^{2^{i+1}} \in \mathrm{A}_0 \qquad (30)$$

Equation (30) shows that (7) is satisfied. Condition (6), however, is not satisfied. Indeed, $x^2(\bullet) \in \mathrm{A}_1 \subseteq \mathrm{L}$, $x^4 = x^{2^2} \in \mathrm{A}_0$, but $[x^2(\bullet), x^4](y(x)) = x^6 \notin \mathrm{A}_0$.

We verified that sufficient condition (7) is satisfied and $\mathrm{L} = L_2([0,1])$. This proves small time controllability of system (26).

C. First Order PDE

Applications of ODEs are often illustrated in textbooks by simplified ecological models [Arnold, V. (1992)], [Taubes, C. (2000)]. In this example we add a spatial component to one of the models. Chemical interpretation is also possible.

Let us consider a control problem for some biological substance (e.g. a school of fish) that slowly moves along a closed loop S. Control actions are applied at certain spatial locations.

Let $y(t, x)$ be the density of the substance at time t and spatial location x. Let $\sigma_1(x)$ be the natural death rate at position x, and $\sigma_2(x)$ is the rate of outcome (fishing). Quantity $\sigma(x) = \sigma_1(x) + \sigma_2(x)$ is the total death rate at position x. This quantity may significantly depend on x, e.g. if the natural death or fishing occurred at certain spatial locations.

Let the birth rate $u(t)p(x)$ has a spatial shape $p(x)$ where its magnitude can be controlled by $u(t)$, e.g. by regulated food supply. $u(t)$ can be positive or negative, (negative means additional outcome).

With these assumptions and with the speed of motion equal to one, the process can be described by the following first order PDE:

$$\frac{\partial y(t,x)}{\partial t} = -\frac{\partial y(t,x)}{\partial x} - \sigma(x) y(t,x) + u(t) p(x) y(t,x) \qquad (31)$$

We assume $y(t,x) \in L_2(S)$, and initial condition is

$$y(0,x) = y_0(x) \qquad (32)$$

Let us formulate a terminal control problem of achieving a desired density $\hat{y}(x)$ at the time $t=T$, e. g. $y(T,x) = \hat{y}(x) = y_0(x)$ at $T=1$ year to keep a sustainable population.

A necessary condition of existence of the solution of this problem is that

$$\hat{y}(x) \in \Re_{y_0(x)}(T) \qquad (33)$$

We have $f(y) = -\frac{\partial y}{\partial x} - \sigma \cdot y$, $g_1(y) = p \cdot y$.

Calculating Lie brackets, obtain: $g_2(y) = [f, g_1](y) =$

$$\left(-\frac{\partial (p \cdot y)}{\partial x} - \sigma \cdot p \cdot y \right) - \left(p \cdot \left(-\frac{\partial y}{\partial x} \right) + p \cdot (-\sigma) \cdot y \right) = -p' \cdot y .$$

We can conclude from this derivation that $\mathrm{A}_0(y) = \overline{\mathrm{sp}}\{p \cdot y\}$, $\mathrm{L}(y) = \overline{\mathrm{sp}}\{p^{(k)} \cdot y : k = 0,1,2,...\}$, and $[\mathrm{L}, \mathrm{A}_0] = 0$, i.e. sufficient condition (6) is satisfied.

Solution of initial value problem (31), (32) with $u(t) \equiv 0$ is

$$y(t,x) = y_0(x-t) \exp\left(-\int_0^t \sigma(x - t + \tau) d\tau \right)$$

Then finally, according to (11),

$$\Re_{y_0(x)}(T) = y_0(x-T) \times$$

$$\exp\left(-\int_0^T \sigma(x - T + \tau) d\tau + \overline{\mathrm{sp}}\{p^{(k)}(x - \nu) : k = 0,1,...; \nu \in [0,T]\} \right) \qquad (34)$$

If the desirable state belongs to the reachable set, $\hat{y}(x) \in \Re_{y_0(x)}(T)$, described by (34), then an approximate solution to the control problem exists. Otherwise, parameters should be changed, e.g. the rate of fishing should be decreased.

The formula for the solution of the Cauchy problem (31),(32), that can be obtained by method of characteristics [Arnold, V. (1992)], is the following:

$$y(t,x) = y_0(x-t) e^{-\int_0^T \sigma((x-t+\tau))d\tau} e^{\int_0^T u(\tau)p((x-t+\tau))d\tau}$$

and if $\hat{y}(x) \in \Re_{y_0(x)}(T)$, control $u(t)$ can be found from the following equation that must hold $\forall x \in \{y_0(x-T) \neq 0\}$

$$\int_0^T u(\tau)p(x-T+\tau)d\tau = f(T,x)$$

where

$$f(T,x) = \ln\frac{\hat{y}(x)}{y_0(x-T)} + \int_0^T \sigma(x-T+\tau)d\tau, \quad x \in \{y_0(x-T) \neq 0\}$$

It is a linear Fredholm integral equation of the first kind. Its regularization can be achieved by minimizing a certain functional $F(u(\cdot))$, e.g.

$$\int_0^T u(t)dt \to \min$$

($u(t)$ can be positive or negative –negative means additional outcome that brings additional profit) with constraints

$$\left|\int_0^T u(\tau)p(x-T+\tau)d\tau - f(T,x)\right| \le \varepsilon, \quad x \in \{y_0(x-T) \neq 0\}$$

and regularization parameter ε. It is a linear programming problem. Additional constraints on u can be also added.

D. Hyperbolic PDE

Let us consider hyperbolic PDE control system in $L_2(\mathrm{R})$:

$$\frac{\partial^2 y(t,x)}{\partial t^2} = \frac{\partial^2 y(t,x)}{\partial x^2} + h(t,x) + u(t)\varphi(x) \qquad 35)$$

with initial conditions

$$y(0,x) = y_0(x), \quad y_t'(0,x) = z_0(x) \qquad (36)$$

and write it as

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} = z(t,x) \\ \dfrac{\partial z(t,x)}{\partial t} = \dfrac{\partial^2 y(t,x)}{\partial x^2} + h(t,x) + u(t)\varphi(x) \\ \dfrac{\partial t}{\partial t} = 1, t(0) = 0 \end{cases} \qquad (37)$$

Remark 5. The artificial last equation in (37) is necessary to formally make the system time-invariant of type (1). Here t is both time as well as artificial space variable (we do not make special notation, e.g. τ=t).

In (37):

$$f(y,z,t) = \begin{pmatrix} z \\ \partial^2 y/\partial x^2 + h \\ 1 \end{pmatrix}, \quad g_1(y,z,t) = \begin{pmatrix} 0 \\ \varphi \\ 0 \end{pmatrix} \qquad (38)$$

and

$$Df|_{(y,z,t)} = \begin{pmatrix} 0 & 1 & 0 \\ \partial^2/\partial x^2 & 0 & \partial h(t,x)/\partial t \\ 0 & 0 & 0 \end{pmatrix}$$

$$g_2(y,z,t) = [f, g_1](y,z,t) = \begin{pmatrix} 0 & 1 & 0 \\ \partial^2/\partial x^2 & 0 & \partial h(t,x)/\partial t \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \varphi(x) \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi(x) \\ 0 \\ 0 \end{pmatrix}$$

$$g_3(y,z,t) = [f, g_2](y,z,t) = \begin{pmatrix} 0 & 1 & 0 \\ \partial^2/\partial x^2 & 0 & \partial h(t,x)/\partial t \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \varphi(x) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi''(x) \\ 0 \end{pmatrix}$$

We can conclude from these calculations that

$$\mathrm{A}_0(y) = \overline{\mathrm{sp}}\left\{\begin{pmatrix} 0 \\ \varphi(x) \\ 0 \end{pmatrix}\right\},$$

$$\mathrm{L}(y) = \overline{\mathrm{sp}}\left\{\begin{pmatrix} \varphi^{(2k)}(x) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi^{(2k)}(x) \\ 0 \end{pmatrix} : k = 0,1,2,\ldots\right\}, [\mathrm{A}_0, \mathrm{L}] = 0 \qquad (39)$$

Thus, sufficient condition (6) is satisfied.

Using d'Alembert's formula [Evans, C. (2010): (8) on p. 68] for initial value problem for a one-dimensional wave equation and formula for solution of a nonhomogeneous problem with zero initial values ($y(0,x) = 0, \quad y_t'(0,x) = 0$) [Evans, C. (2010): (43) on p. 81]:

$$y_h(t,x) = \frac{1}{2}\int_0^t d\tau \int_{x-s}^{x+s} h(t-\tau, s)ds$$

we obtain the following expression for the reachable set:

$$\Re_{(y_0,z_0)}(T) = \begin{pmatrix} y_h(T,x) \\ \partial y_h(t,x)/\partial t|_{t=T} \end{pmatrix} +$$

$$\frac{1}{2}\begin{pmatrix} y_0(x+T) + y_0(x-T) + \int_{-T}^{T} z_0(\tau)d\tau \\ y_0'(x+T) - y_0'(x-T) + z_0(x+T) - z_0(x-T) \end{pmatrix} + \qquad (40)$$

$$\overline{\mathrm{sp}}\left\{\begin{pmatrix} \varphi^{(2k)}(x+\nu) + \varphi^{(2k)}(x-\nu) \\ \varphi^{(2k+1)}(x+\nu) - \varphi^{(2k+1)}(x-\nu) \end{pmatrix}, \begin{pmatrix} \varphi^{(2k-1)}(x+\nu) - \varphi^{(2k-1)}(x-\nu) \\ \varphi^{(2k)}(x+\nu) - \varphi^{(2k)}(x-\nu) \end{pmatrix} : k = 0,1,2,\ldots; \quad \nu \in (0,T)\right\}$$

E. Parabolic PDE

Let us consider bilinear parabolic PDE control system in $L_2(S)$, where S is a unit circle:

$$\frac{\partial y(t,x)}{\partial t} = \nu^2 \frac{\partial^2 y(t,x)}{\partial x^2} + u(t)\int_{-\pi}^{\pi} \mathrm{K}(x,z)y(t,z)dz \qquad (41)$$

where

$$\mathrm{K}(x,z) = \frac{a_0}{2} + \sum_{m=1}^{M}\left(a_m \cdot \cos m(kx+lz) + b_m \cdot \sin m(kx+lz)\right) \qquad (42)$$

with initial condition

$$y(0,x) = y_0(x) = \frac{\alpha_0}{2} + \sum_{j=1}^{\infty}\left(\alpha_j \cdot \cos jx + \beta_j \cdot \sin jx\right) \qquad (43)$$

Solution of the initial value problem for (41)-(43) with $u(t)\equiv 0$ is

$$y(t,x)=\frac{\alpha_0}{2}+\sum_{j=1}^{\infty}\exp\left(-v^2 j^2 t\right)\cdot\left(\alpha_j\cdot\cos jx+\beta_j\cdot\sin jx\right) \tag{44}$$

Let us explore the Lie algebras associated with system (41):

$$f(y(x))=\frac{\partial^2 y(x)}{\partial x^2};\quad g_1(y(x))=\int_{-\pi}^{\pi}\mathrm{K}(x,z)y(z)dz \tag{45}$$

$$g_2(y(x))=[f,g_1](y(x))=\int_{-\pi}^{\pi}\left[\frac{\partial^2\mathrm{K}(x,z)}{\partial x^2}y(z)-\mathrm{K}(x,z)\frac{\partial^2 y(z)}{\partial z^2}\right]dz$$

$$=\int_{-\pi}^{\pi}\left[\frac{\partial^2\mathrm{K}(x,z)}{\partial x^2}-\frac{\partial^2\mathrm{K}(x,z)}{\partial z^2}\right]y(z)dz=\int_{-\pi}^{\pi}\mathrm{K}_1(x,z)y(z)dz,$$

where

$$\mathrm{K}_1(x,z)=\sum_{m=1}^{M}m^2\left(l^2-k^2\right)\left(a_m\cdot\cos m(kx+lz)+b_m\cdot\sin m(kx+lz)\right)$$

Continuing similar way we get

$$g_i(y(x))=[f,g_{i-1}]=\int_{-\pi}^{\pi}\mathrm{K}_{i-1}(x,z)y(z)dz,\ i=2,3,\ \ldots,$$

where

$$\mathrm{K}_i(x,z)=\sum_{m=1}^{M}\left(m^2\left(l^2-k^2\right)\right)^i\left(a_m\cdot\cos m(kx+lz)+b_m\cdot\sin m(kx+lz)\right)$$

, $i=0,1,2,\ldots$ Let us consider two cases:

Case 1. $|k|=|l|$. In this case $[f,g_1]=0$, $\mathrm{L}=\mathrm{A}_0=\overline{\mathrm{sp}}\{g_1\}$, and $\Re_{y_0}(T)=\mathrm{E}^{\mathrm{L}}\,y(T,x)$,

where $y(T,x)$ is calculated by (44), and $\mathrm{E}^{\mathrm{L}}\,y(x)=\{Y(t,x):t\in R\}$, where $Y(t,x)$ is the solution of the following initial value problem:

$$\frac{\partial Y(t,x)}{\partial t}=\int_{-\pi}^{\pi}\mathrm{K}(x,z)Y(t,z)dz,\quad Y(0,x)=y(T,x) \tag{46}$$

Eq. (46) is equivalent to finite system of linear ODEs because the kernel K defined by (42) is a linear combination of $2m+1$ basis functions.

We present derivation for the case $k=1$, $l=-1$. In this case the integral operator in (41) is a convolution.

We have from (44)

$$y(T,x)=\frac{\alpha_0}{2}+\sum_{j=1}^{\infty}e_j^T\cdot\left(\alpha_j\cdot\cos jx+\beta_j\cdot\sin jx\right) \tag{47}$$

where

$$e_j^T=\exp\left(-v^2 j^2 T\right) \tag{48}$$

We are looking the solution of (46) in the form

$$Y(t,x)=\frac{\alpha_0(t)}{2}+\sum_{j=1}^{\infty}\alpha_j(t)\cdot\cos jx+\beta_j(t)\cdot\sin jx \tag{49}$$

Then (46), after tedious trigonometry, is equivalent to the following system of linear ODEs:

$$\dot{\alpha}_0(t)=\pi a_0\alpha_0(t),\quad \alpha_0(0)=\alpha_0 \tag{50.m=0}$$

$$\begin{cases}\begin{pmatrix}\dot{\alpha}(t)\\ \dot{\beta}(t)\end{pmatrix}=\begin{pmatrix}\pi a_m & -\pi b_m\\ \pi b_m & \pi a_m\end{pmatrix}\begin{pmatrix}a_m(t)\\ \beta_m(t)\end{pmatrix},\\ \begin{pmatrix}a_m(0)=\alpha_m e_m^T\\ \beta_m(t)=\beta_m e_m^T\end{pmatrix},\quad m=1,\ldots M\end{cases} \tag{50.m=1..M}$$

$$\begin{cases}\dot{\alpha}_m(t)=0,\quad \alpha_0(0)=\alpha_m e_m^T\\ \dot{\beta}_m(t)=0,\quad \beta_0(0)=\beta_m e_m^T\end{cases}, \tag{50.m>M}$$
$$m=M+1,\ldots$$

Then the reachable set is the following one-dimensional manifold:

$$\Re_{y_0}(T)=\Big\{\frac{\alpha_0}{2}\exp(a_0 t)+\sum_{m=1}^{M}\left(\alpha_m(t)\cos mx+\beta_m(t)\sin mx\right)+\sum_{m=M+1}^{\infty}\left(\alpha_m\cos mx+\beta_m\sin mx\right),\ -\infty<t<\infty\Big\} \tag{51}$$

where

$$\begin{cases}\alpha_m(t)=e_m^T\exp(a_m t)\cdot\left[\alpha_m\cos b_m t-\beta_m\sin b_m t\right]\\ \beta_m(t)=e_m^T\exp(a_m t)\cdot\left[\alpha_m\sin b_m t+\beta_m\cos b_m t\right]\end{cases} \tag{52}$$
$$m=1,\ldots,M$$

Case 2. $|k|\neq|l|$. We construct exact reachable sets for the non-resonant case (general position):

$$k\geq 0;\quad \forall 1\leq m_1,m_2\leq M:\left[m_1 k\neq m_2 l\right] \tag{53}$$

Analyzing formulas for K_i, $i=0,\ldots,M$, we can write

$$\begin{pmatrix}g_1\\ g_2\\ \ldots\\ g_{i+1}\\ \ldots\\ g_{M+1}\end{pmatrix}=\begin{pmatrix}1 & 1 & \ldots & 1 & \ldots & 1\\ 0 & (l^2-k^2) & \ldots & j^2(l^2-k^2) & \ldots & M^2(l^2-k^2)\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & (l^2-k^2)^i & \ldots & \left(j^2(l^2-k^2)\right)^i & \ldots & \left(M^2(l^2-k^2)\right)^i\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & (l^2-k^2)^M & \ldots & \left(j^2(l^2-k^2)\right)^i & \ldots & \left(M^2(l^2-k^2)\right)^M\end{pmatrix}\begin{pmatrix}h_1\\ h_2\\ \ldots\\ h_{i+1}\\ \ldots\\ h_{M+1}\end{pmatrix} \tag{54}$$

where

$$h_1(y)=\frac{a_0}{2}\int_{-\pi}^{\pi}y(z)dz,$$

$$h_{m+1}(y)=\int_{-\pi}^{\pi}\left(a_m\cdot\cos m(kx+lz)+b_m\cdot\sin m(kx+lz)\right)y(z)dz$$

$$=\left(a_m\cos(mkx)+b_m\sin(mkx)\right)\int_{-\pi}^{\pi}\cos(mlz)y(z)dz$$

$$+\left(b_m\cos(mkx)-a_m\sin(mkx)\right)\int_{-\pi}^{\pi}\sin(mlz)y(z)dz,\quad m=1,\ldots,M$$

Matrix in (54) is nonsingular (its determinant is proportional to Vandermonde's determinant) and, thus,

$$\overline{\mathrm{sp}}\{g_i:i=1,2,\ldots,M+1\}=\overline{\mathrm{sp}}\{h_i:i=1,2,\ldots,M+1\}$$

It is easy to verify that the following equality holds for any two kernels $R_1(x,z)$ and $R_2(x,z)$:

$$\int_{-\pi}^{\pi} R_1(x,z)\left(\int_{-\pi}^{\pi} R_2(z,p)y(p)dp\right)dz - \int_{-\pi}^{\pi} R_2(x,z)\left(\int_{-\pi}^{\pi} R_1(z,p)y(p)dp\right)dz$$

$$\equiv \int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\left(R_1(x,z)R_2(z,p) - R_1(z,p)R_2(x,z)\right)dz\right)y(p)dp \tag{55}$$

Due to (53) and orthogonality of the trigonometric system, internal integral in RHS of (55) vanishes for the following pairs of the kernels (i=1,2):

$$\int_{-\pi}^{\pi}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(m_1 kx)\cdot\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(m_2 lz)dz = 0,\ m_1 \neq m_2, m_i = 1,...M,\ i = 1,2$$

that define $h_1,\ldots,h_{M+1}$. Then we conclude that $[h_i, h_j]=0$. We also have $\left[f, h_{m+1}\right] = m^2\left(l^2 - k^2\right)h_{m+1}$. Now we calculate the Lie algebras and verify the sufficient condition (6):

$$\mathrm{A}_0 = \overline{\mathrm{sp}}\left\{\sum_{m=1}^{M+1} h_m\right\},\quad \mathrm{L} = \overline{\mathrm{sp}}\{h_i : i = 1,...,M+1\}\quad \Rightarrow [\mathrm{L}, \mathrm{A}_0] = 0$$

Then L_T=L because L is finite-dimensional. To calculate $\Re_{y_0}(T) = \mathrm{E}^{\mathrm{L}} S_T y_0$ we should solve the Cauchy problems for the equations

$$\frac{\partial Y(t,x)}{\partial t} = h_i\left(Y(t,x)\right),\ i = 1,...,M+1 \tag{56}$$

We are looking for a solution in the form of Fourier series (49). Equation (56) for m=1 is equivalent to

$$\begin{cases}\dot{\alpha}_0(t) = \pi a_0 \alpha_0(t), & \alpha_0(0) = \alpha_0\\ \dot{\alpha}_j(t) = 0, & \alpha_j(0) = \alpha_j\\ \dot{\beta}_j(t) = 0, & \beta_j(0) = \beta_j\end{cases} \tag{57}$$

Solution of (57), assuming $y(0,x)=y_0(x)$, is

$$y(t,x) = y_0(x) - \frac{\alpha_0}{2} + \frac{\alpha_0}{2}\exp(\pi a_0 t) \tag{58}$$

Equation (56) for $i\geq 2$ ($m=i-1=2,...,M$) is equivalent to

$$\begin{cases}\dot{\alpha}_0(t) = 0, & \alpha_0(0) = \alpha_0\\ \dot{\alpha}_j(t) = 0, & \alpha_j(0) = \alpha_j,\ j \neq mk\\ \dot{\beta}_j(t) = 0, & \beta_j(0) = \beta_j,\ j \neq mk\\ \dot{\alpha}_{mk}(t) = \pi\left(a_m \alpha_{m|l|}(t) + b_m \operatorname{sgn} l \cdot \beta_{m|l|}(t)\right), & \alpha_{mk}(0) = \alpha_{mk}\\ \dot{\beta}_{mk}(t) = \pi\left(b_m \alpha_{m|l|}(t) - a_m \operatorname{sgn} l \cdot \beta_{m|l|}(t)\right), & \beta_{mk}(0) = \beta_{mk}\end{cases} \tag{59}$$

From (58) and (59) we conclude that

$m|l| \neq mk \Rightarrow \dot{\alpha}_{m|l|}(t) \equiv \dot{\beta}_{m|l|}(t) \equiv 0 \Rightarrow \alpha_{m|l|}, \beta_{m|l|}$=const, and

$$y(t,x) = y_0(x) + \pi t \cdot \xi_m(x) \tag{60}$$

where

$$\xi_m(x) = \left(a_m \alpha_{m|l|} + b_m \operatorname{sgn} l \cdot \beta_{m|l|}\right)\cos mkx + \left(b_m \alpha_{m|l|} - a_m \operatorname{sgn} l \cdot \beta_{m|l|}\right)\sin mkx \tag{61}$$

Analyzing these solutions and formula (11), we conclude that

$$\Re_{y_0(x)}(T) = \sum_{j=1}^{\infty}\exp\left(-v^2 j^2 T\right)\cdot\left(\alpha_j \cdot \cos jx + \beta_j \cdot \sin jx\right) + \left\{\alpha_0 t_0 + \sum_{m=1}^{M} t_m \xi_m(x):\quad t_0 \geq 0, t_m \in R\right\} \tag{62}$$

Thus, the reachable set (62) of parabolic PDE control system (41) is (M+1) parametric set of functions and a desirable solution is characterized by these parameters.

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