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On a Discrete-Time Networked SIV Model with Polar Opinion Dynamics

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Abstract— **This paper investigates novel epidemic spreading problems under the influence of opinion evolution in social networks, where the opinions reflect the public health concerns toward the epidemic. A coupled bilayer network is proposed, where the epidemics propagate over several communities through a physical network layer while the opinions evolve over the same communities through a social network layer. Specifically, the epidemic spreading process is described by a susceptible-infected-vigilant (SIV) model, which introduces opinion-dependent epidemic vigilance state compared with classical epidemic models. Additionally, a polar opinion dynamics model is adopted on the social network, which incorporates the infection prevalence and human stubbornness into the opinion evolution. By introducing an opinion-dependent reproduction number, we provide the stability analysis of disease-free and endemic equilibria and derive sufficient conditions for their global asymptotic stability. Simulations are conducted to verify the theoretical results.**

I. Introduction

Mathematical modeling of infectious diseases has a long history, dating back to Daniel Bernoulli's work on smallpox in 1760 [1]. The primary objectives of such modeling are to comprehend disease transmission mechanisms and to predict epidemic outcomes [2]. Over the past century, epidemic models have played a vital role in guiding public health policies [3]. The COVID-19 pandemic has highlighted the necessity of continued research in this field [4].

Compartmental models, such as susceptible-infected-susceptible (SIS) [2] and susceptible-infected-recovered (SIR) [5], have aided in grasping epidemic dynamics. However, these models fall short in capturing the complexities of disease traits and human actions. To address these issues, previous works [6], [7] have proposed an expanded model integrating temporary immunity sources reflecting social awareness and protective behaviors [8]. In this study, we adopt this model, denoting it as the susceptible-infected-vigilant (SIV) model following [6]. Specifically, we focus on the discrete-time networked SIV model.

In epidemiology, the reproduction number indicates the expected cases generated by one case in the population and is a crucial parameter [9]. It can be influenced by environmental conditions and population behaviors [10]. In sociology, the health belief model [11] implies that the opinion spreading in social networks may affect the reproduction number, and further, the epidemic spreading. It can be observed that the influence of social awareness, reflected by the vigilant state in the SIV mentioned above, is consistent with the health belief model. Thus, we are interested in the coupling between the networked SIV model and opinion dynamics.

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Previous study [12] has linked epidemic models with the basic DeGroot opinion model [13]. However, the DeGroot model always leads the opinions to a consensus, while persistent disagreements often take place in reality. Altafini's model [14] captures cooperative and antagonistic interactions in opinion exchange dynamics, reflecting disagreement among people. This model has been coupled with a networked SIS model in [15]. However, this model still ignores the personal preferences of individuals, which usually manifest as stubbornness or prejudice in practice [16]. To this end, we employ the polar opinion dynamics [17], which considers personal cognitive factors and stubbornness, to better model people's beliefs and awareness regarding epidemics.

Our contributions include proposing a networked SIV epidemic model coupled with polar opinion dynamics, where health opinions depend on peer influence, individual stubbornness, and infection levels. We introduce an SIV-opinion reproduction number (R_o^V) to measure epidemic severity. We demonstrate that when $R_o^V \leq 1$, the epidemic converges to a healthy state, and opinions reach a consensus that the epidemic is not a threat. For $R_o^V > 1$, we identify conditions for endemicity and dissensus. Numerical simulations on a large-scale real world network verify our results.

This paper is organized as follows: Section II introduces the coupled epidemic-opinion model preliminaries. Section III defines equilibria and the reproduction number, and analyzes the dynamics. A numerical example on a network of Japan's prefectures is presented in Section IV. Finally, in Section V we present some concluding remarks. The proofs are omitted due to space limitations.

Notation: Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ for any positive integer n. Denote by $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ the n-dimensional Euclidean space and the set of $n \times n$ real matrices, respectively. Denote by $A \succ 0$ and $A \prec 0$ that matrix A is positive definite and negative definite, respectively. Let $A^\top$, $\rho(A)$, and $\|A\|_\infty$ be the transpose, spectral radius, and infinity norm of matrix A, respectively. The $n \times n$ identity matrix is given by I_n, and $\mathbf{1}_n$ represents the all-ones vector in $\mathbb{R}^n$. For any vector $x \in \mathbb{R}^n$, x_i denotes the i-th entry of x, and $\mathrm{diag}(x) \in \mathbb{R}^{n\times n}$ denotes a diagonal matrix with x_i being the ith diagonal entry. For any two vectors $x, y \in \mathbb{R}^n$, we simply write $x > y$ if $x_i > y_i, \forall i \in [n]$.

II. Modelling and Problem Formulation

This section introduces a networked epidemic model coupled with a polar opinion dynamics model. We consider a situation where an epidemic is spreading among a group

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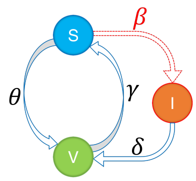


Fig. 1: SIV epidemic model with three states and various transition parameters.

of communities. The spreading process is affected by the topology of the physical network and the attitudes of the communities towards the disease affect the spreading process. Moreover, the opinion of each community changes over time depending on its infection status and the opinions of other communities in the social network.

A. Epidemic Dynamics

In this paper, we extend the widely studied SIS model to a generalized variant known as the SIV model [6]. In the SIV model, individuals can be in one of the three classes of states: Susceptible class S, infected class I^p with $p \in \{1,\dots,m_I\}$, and vigilant class V^q with $q \in \{1,\dots,m_V\}$. Susceptible individuals can be infected by their infected neighbors. Infected individuals recover with a certain rate and become vigilant. Vigilant individuals are not susceptible or infected; they may adopt protective measures like wearing masks, social distancing, or becoming immune from infection or vaccination. In the infected class and vigilant class, an individual can be classified into any of the m_I/m_V states; this allows us to model various disease characteristics, including disease severity, sources of vigilance, and so on. For simplicity, we consider the case with $m_I = m_V = 1$ in this paper, but other cases can be analyzed using similar methods.

We consider a physical interaction network represented by the directed graph $\mathcal{G}_D = (\mathcal{V}, \mathcal{E}_D)$. Here, $\mathcal{V} = [n]$ represents the communities, and $\mathcal{E}_D \subseteq \mathcal{V} \times \mathcal{V}$ represents disease spreading interactions. A directed edge (j,i) indicates that community j can infect community i. Denote by $\mathcal{N}_i^D = \{j \mid (j,i) \in \mathcal{E}_D\}$ is the set of the neighbors of community i.

Fig. 1 shows the three-state SIV epidemic transmission model where the transitions between different compartments are shown with arrows. The proportions of the susceptible, infected, and vigilant population in community i at continuous time t are denoted, respectively, by $x_i^S(t)$, $x_i^I(t)$, and $x_i^V(t)$. Note that for all $i \in [n]$ and $t \geq 0$, it holds that $x_i^S(t), x_i^I(t), x_i^V(t) \in [0,1]$ and $x_i^S(t) + x_i^I(t) + x_i^V(t) = 1$. Then similar to the construction of the networked SIS model in [2], the SIV epidemic dynamics to capture the evolution of the n communities is given by

$$\begin{aligned}
\dot{x}_i^S(t) &= \gamma_i x_i^V(t) - \theta_i x_i^S(t) - x_i^S(t) \sum_{j \in \mathcal{N}_i^D} \beta_{ij} x_j^I(t), \\
\dot{x}_i^I(t) &= x_i^S(t) \sum_{j \in \mathcal{N}_i^D} \beta_{ij} x_j^I(t) - \delta_i x_i^I(t), \\
\dot{x}_i^V(t) &= \delta_i x_i^I(t) + \theta_i x_i^S(t) - \gamma_i x_i^V(t).
\end{aligned} \tag{1}$$

The transition parameters are given as follows: $\beta_{ij} \in [0,1]$ denotes the average infection rate from community j to community i, $\delta_i \in [0,1]$ denotes the average recovery rate of the infected population in community i, $\gamma_i \in [0,1]$ is the average susceptibility rate of the vigilant population in community i after despising protective measures or losing immunity, and $\theta_i \in [0,1]$ is the average vigilance rate of the susceptible population in community i to become vigilant.

In this paper, we deal with the system in the discrete-time domain as in the SIS model case in [18]. Taking the sampling period as $\Delta T = 1$ without loss of generality, we describe (1) in the discretized form as

$$\begin{aligned}
x_i^S(k+1) &= x_i^S(k) + \gamma_i x_i^V(k) - \theta_i x_i^S(k) \\
&\quad - x_i^S(k) \sum_{j \in \mathcal{N}_i^D} \beta_{ij} x_j^I(k), \\
x_i^I(k+1) &= x_i^I(k) + x_i^S(k) \sum_{j \in \mathcal{N}_i^D} \beta_{ij} x_j^I(k) - \delta_i x_i^I(k), \\
x_i^V(k+1) &= x_i^V(k) + \delta_i x_i^I(k) + \theta_i x_i^S(k) - \gamma_i x_i^V(k).
\end{aligned} \tag{2}$$

B. Opinion Dynamics

For community i at time k, its opinion towards the epidemic severity is denoted as $o_i(k) \in [0,1]$. With $o_i(k) = 1$, the community believes that the epidemic is extremely serious, and with $o_i(k) = 0$, the community perceives that the epidemic is not a threat.

Opinions within a social network can evolve as communities interact with their neighbors. We model this opinion evolution across a network of n connected communities using the directed graph $\mathcal{G}_O = (\mathcal{V}, \mathcal{E}_O)$. Similar to the physical network $\mathcal{G}_D$, the neighbor set of community i in the social network $\mathcal{G}_O$ is defined as $\mathcal{N}_i^O$.

In this paper, we focus on the following polar opinion dynamics with stubborn positives [17]:

$$o_i(k+1) = o_i(k) + (1 - o_i(k)) \sum_{j \in \mathcal{N}_i^O} w_{ij}(o_j(k) - o_i(k)), \tag{3}$$

where w_{ij} measures the amount of relative influence of community j upon community i. Assume that $\sum_j w_{ij} = 1$ for all $i \in [n]$. Then let $W = [w_{ij}] \in \mathbb{R}^{n \times n}$ be the row-stochastic adjacency matrix of the social network, and let $L = I_n - W$ be the network's Laplacian matrix.

Model (3) incorporates human stubbornness into opinion dynamics to be more realistic. The key distinction from the classical DeGroot opinion model [13] lies in the term $1 - o_i(k)$. This term signifies that extreme opinions at one end of the spectrum resist changes more than those at the opposite end. It models scenarios where communities at one negative extreme of the opinion spectrum may be more open to alternative views, while communities with positive opinions are motivated to maintain their positions [17]. In this paper, we assume that $o_i(k) = 1$ represents stubborn positives based on the theory of mass panic [19].

C. Coupled Epidemic-Opinion Dynamics

After introducing the networked SIV epidemic model and the polar opinions model spreading over the same set of n

communities, it is natural to consider a network dynamical model that couples the two models together.

First, based on the health belief model in health behavior research [11], it is reasonable to expect that for a community, its opinion or attitude toward the serverity of an epidemic will affect its actions of adopting protective behaviors. For example, a community being very serious about the epidemic may propagate the dangers of the epidemic more widely and frequently, and tend to make stricter policies to prevent the epidemic. People in such a community may also be more likely to adhere to protective behaviors, follow the instructions given by scientific institutions, and get vaccinated actively. These actions may lead to higher vigilance rate θ and lower susceptibility rate γ in the community. To describe such dependence more explicitly, we take the rates θ_i and γ_i to be functions of opinion $o_i(k)$ as $\theta_i(o_i(k))$ and $\gamma_i(o_i(k))$.

The community's perception of epidemic severity can also be influenced by its infection rate. We present an opinion dynamics model for community i, integrating its original opinion model (3) with current infection level as follows:

$$\begin{aligned} o_i(k+1) = {} & \phi_i x_i^I(k) + (1-\phi_i)\left[o_i(k)\right. \\ & \left. + (1-o_i(k)) \sum_{j\in\mathcal{N}_i^O} w_{ij}(o_j(k)-o_i(k))\right], \end{aligned} \tag{4}$$

where $\phi_i \in (0,1)$ is a given constant. The second term on the right-hand side of (4) is from (3). The neighbors of community i influence its opinion following the polar model with stubborn positives. The first term captures how the infection level of community i affects its opinion. If, for example, community i has a low opinion but experiences a severe infection ($x_i^I(k)$ is high), $o_i(k+1)$ in (4) will increase. This model is consistent with the health belief model in [11].

III. Analysis of SIV-Opinion Dynamical Model

This section considers well-posedness and the equilibria of the SIV-opinion dynamical model. Furthermore, we analyze stability conditions of our model and discuss the mutual influence between epidemic spreading and opinion evolution.

A. Well-Posedness

In order for our coupled epidemic-opinion model to be well posed, its solutions must remain in the state space $[0,1]^n$. To this end, we pose three assumptions related to the graphs, transition parameters, and initial states.

Assumption 1. *Both the physical interaction graph $\mathcal{G}_D$ and the social graph $\mathcal{G}_O$ are strongly connected.*

Assumption 2. *For all $i,j\in[n]$ and $k\geq 0$, it holds that $\delta_i,\beta_{ij}\in(0,1),\theta_i(o_i(k)),\gamma_i(o_i(k))\in[0,1]$, $\sum_{j\in\mathcal{N}_i^D}\beta_{ij}+\theta_i(o_i(k))\leq 1$, $\theta_i(o_i(k))+\gamma_i(o_i(k))\geq c$ for some constant $c\in(0,1)$, and $\theta_i(o_i(k)),\gamma_i(o_i(k))$ take the boundary value iff $o_i(k)$ takes the boundary value.*

Assumption 3. *For all $i\in[n]$, it holds that $x_i^S(0),x_i^I(0)$, $x_i^V(0),o_i(0)\in[0,1]$ and $x_i^S(0)+x_i^I(0)+x_i^V(0)=1$.*

Under these assumptions, we obtain our first result.

Proposition 1. *For the model defined in (2) and (4), the states satisfy $x_i^S(k),x_i^I(k)$, $x_i^V(k),o_i(k)\in[0,1]$ for all $i\in[n]$ and $k\geq 0$.*

B. Equilibria of the Coupled Model

Due to the constraint that $x_i^S(k)+x_i^I(k)+x_i^V(k)=1$ for all $i\in[n]$ and $k\geq 0$, one of the equations in (2) is redundant. By setting $x_i^S(k)=1-x_i^I(k)-x_i^V(k)$, the coupled SIV-opinion model can be described by

$$\begin{aligned} x_i^I(k+1) &= x_i^I(k) - \delta_i x_i^I(k) \\ &\quad + (1-x_i^I(k)-x_i^V(k)) \sum_{j\in\mathcal{N}_i^D} \beta_{ij} x_j^I(k), \\ x_i^V(k+1) &= x_i^V(k) + \delta_i x_i^I(k) - \gamma_i(o_i(k)) x_i^V(k) \\ &\quad + \theta_i(o_i(k))(1-x_i^I(k)-x_i^V(k)), \\ o_i(k+1) &= \phi_i x_i^I(k) + (1-\phi_i)\left[o_i(k)\right. \\ &\quad \left. + (1-o_i(k)) \sum_{j\in\mathcal{N}_i^O} w_{ij}(o_j(k)-o_i(k))\right]. \end{aligned} \tag{5}$$

To study the system (5), let ($x_i^{I^*}$, $x_i^{V^*}$, o_i^*) denote an equilibrium of the three equations.

Definition 1. *An equilibrium state $z^*=(x^{I^*},x^{V^*},o^*)$ of the coupled SIV-opinion model (5) is said to be*

1) *a healthy state if $x^{I^*}=0$, and an endemic state otherwise;*
2) *a consensus state if $o_i^*=o_j^*,\forall i,j\in[n]$, and a dissensus state otherwise.*

C. Stability Analysis of Disease-Free Equilibrium

In practice, achieving the healthy state, which means to reach the disease-free equilibrium, should be the most worth exploring scenario. To further analyze stability conditions of disease-free equilibria, we state a few preliminaries.

In epidemiology, the basic reproduction number, denoted by R_0, is a critical parameter to measure epidemic spreading [9]. It can be affected by various factors such as pathogen types and population behaviors. Thus, we define a specific reproduction number R_o^V to characterize the infectivity of the SIV-opinion model (5).

Definition 2. *For the coupled SIV-opinion model in (5), the reproduction number is defined as*

$$R_o^V = \rho\left(I_n - \Delta + (I_n-\Psi)B\right),$$

where

$$\Psi = \mathrm{diag}\left(\min_{o_i\in[\underline{o}_i,\overline{o}_i]} \psi_i(o_i)\right), \tag{6}$$

$$\psi_i(o_i) = \frac{\theta_i(o_i)}{\gamma_i(o_i)+\theta_i(o_i)}, i\in[n], \tag{7}$$

$B=[\beta_{ij}]\in\mathbb{R}^{n\times n}$, $\Delta=\mathrm{diag}(\delta_1,\ldots,\delta_n)$, with $\underline{o}_i$ and $\overline{o}_i$ being the lower and upper bounds of $o_i(k)$ for $k\geq 0$, respectively.

The next lemma shows the relation between the infection rate and the opinions when the disease eradicates.

Lemma 1. *For the coupled SIV-opinion model in (5), $o(k)$ asymptotically converges to 0 if $x^I(k)$ asymptotically converges to 0.*

Remark 1. *Lemma 1 identifies a social state representing epidemic extinction. When the epidemic vanishes for any reason, the society goes to a consensus-healthy equilibrium $z^* = (0, x^{V^*}, 0)$. This signifies that if the epidemic is eradicated, all communities will agree that it poses no threat.*

Now we are ready to prove the stability of the disease-free equilibrium. First, we define a particular equilibrium of the vigilant state as

$$\hat{V}^* = \Theta(0)(\Gamma(0) + \Theta(0))^{-1}\mathbf{1}_n, \tag{8}$$

where $\Theta(o) = \operatorname{diag}(\theta_1(o_1), \ldots, \theta_n(o_n))$, $\Gamma(o) = \operatorname{diag}(\gamma_1(o_1), \ldots, \gamma_n(o_n))$. Then in the following theorem, a sufficient condition for the global stability of the disease-free equilibrium will be established. To simplify notations, $\gamma_i(o_i(k))$, $\theta_i(o_i(k))$, $\Gamma(o(k))$, $\Theta(o(k))$ and $\gamma_i(k)$, $\theta_i(k)$, $\Gamma(k)$, $\Theta(k)$ can substitute each other in the rest of this paper.

Theorem 1. *If $R_o^V \leq 1$, the healthy-consensus state $z^* = (0, \hat{V}^*, 0)$ of the system in (5) is globally asymptotically stable.*

Theorem 1 shows the role of the reproduction number R_o^V, or more specifically, the lower bound of $\psi_i(o_i(k))$ defined in (7), in epidemic eradication. In practice, when R_o^V is large and the epidemic cannot disappear spontaneously, administrations of the communities can lead the population so that the lower bound of $\psi(o(k))$ becomes larger, which will make R_o^V smaller than 1. Further, it will be interesting to analyze optimal control strategies theoretically to realize effective epidemic suppression under some budget constraints in a future work.

D. Stability Analysis of Endemic Equilibrium

Since we analyzed the stability of the disease-free equilibrium, the next step is to study the endemic equilibrium, which reveals the impact of opinions under situations with more severe epidemics.

We have seen in Lemma 1 that all communities reach consensus when the epidemic disappears. Now we consider opinion states of the endemic equilibrium. We have the following proposition.

Proposition 2. *For the coupled SIV-opinion system in (5), an consensus-endemic state ($x^{I^*} \neq 0$, $o^* = a\mathbf{1}_n$) is an equilibrium only if $a \in (0, 1]$ and*

$$\delta_i = \frac{1 - a - \frac{\theta_i(a)(1-a)}{\gamma_i(a)+\theta_i(a)}}{\frac{a}{\gamma_i(a)+\theta_i(a)} + \frac{1}{\sum_{j \in \mathcal{N}_i^D} \beta_{ij}}}, \ \forall i \in [n]. \tag{9}$$

The following corollary is a direct result from the system model of (5) and Proposition 2.

Corollary 1. *If z^* is an endemic equilibrium of the coupled SIV-opinion model, then $x^{I^*} > 0$, $x^{V^*} > 0$, and $o^* > 0$.*

Proposition 2 and Corollary 1 state that as long as the epidemic persists, no community can be completely disease-free or agree that the epidemic does not pose a threat. Furthermore, the communities cannot reach a consensus on the severity of the epidemic except for some systems with particular transition parameters.

We now study the stability of the endemic equilibrium. The following result characterizes the condition under which the endemic equilibrium is globally asymptotically stable.

Theorem 2. *Suppose that $R_o^V > 1$ and $z^* = (x^{I^*}, x^{V^*}, o^*)$ is an endemic equilibrium of the coupled SIV-opinion model (5). If*

$$-2w_{ii} - \frac{\phi_i}{1-\phi_i} < (Lo^*)_i < \frac{\phi_i}{1-\phi_i}, \forall i \in [n], \tag{10}$$

and there exists a matrix $P \succ 0$ such that

$$F^\top(o, x^S) P F(o, x^S) - P \prec 0, \tag{11}$$

where

$$\begin{aligned} &F(o, x^S) \\ &= \begin{bmatrix} I_n - \Delta - H + \operatorname{diag}\left(x^S\right) B & -H \\ \Theta(o) - \Delta & I_n - \Gamma(o) - \Theta(o) \end{bmatrix}, \\ &H = \operatorname{diag}\left(Bx^{I^*}\right), \end{aligned}$$

for all $o_i \in [\underline{o}_i, \overline{o}_i]$ and $x_i^S \in [0, 1]$, $i \in [n]$, then the equilibrium z^* is asymptotically stable for all disease-nonzero initial conditions, i.e., $x^I(0) \neq 0$.

The above theorem demonstrates that when $R_o^V > 1$, i.e., the disease is severe, the state of system (5) will converge to an endemic equilibrium under certain conditions. Note that Theorem 1 gives a sufficient condition of the convergence of disease-free equilibrium, instead of a sufficient and necessary one. Therefore, when $R_o^V > 1$, the coupled SIV-opinion system (5) may have healthy or endemic equilibria. Analysing the existence of a larger healthy/endemic boundary or lack thereof (e.g., locally stable healthy and endemic equilibria coexist in one system with their own attractive region) remains a future research direction.

IV. Simulations

In this section, we employ the coupled SIV-opinion model (5) to simulate epidemic spread and illustrate the theoretical findings using a real-world large-scale network structure.

A. Real-world Network

We analyze an epidemic process over a network of 46 communities, each representing a prefecture in Japan (except for Kumamoto, due to the lack of statistics). Both the physical network for disease spreading and the social network for opinion evolution adhere to Assumption 1, but they differ in their link structures due to real-world variations.

The physical network models human mobility and migration between prefectures, based on data from the Eighth National Survey on Migration of Japan [20]. To simplify this network, we remove low-weight edges by setting entries in

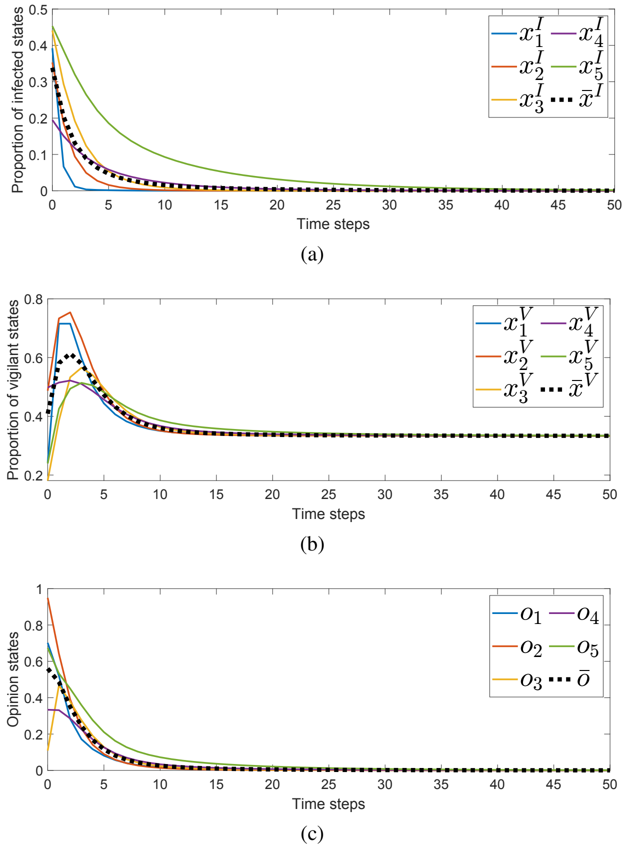


Fig. 2: Under a mild epidemic with $R_o^V = 0.9956$, the evolution of the coupled SIV-opinion system for the $n = 46$ network. (a) The infected states converge to zero. (b) The vigilant states converge to 0.3333. (c) The opinion states reach consensus and converge to zero.

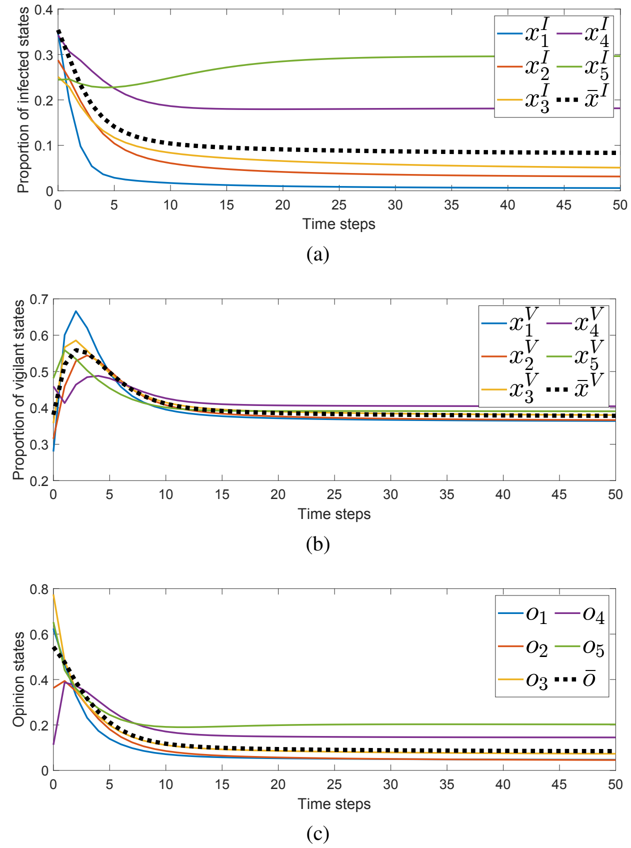


Fig. 3: Under a severe epidemic with $R_o^V = 1.1827$, the evolution of the coupled SIV-opinion system for the $n = 46$ network. (a) The infected states reach an endemic equilibrium. (b) The vigilant states converge to an equilibrium. (c) The opinion states reach dissensus.

the adjacency matrix B below a threshold to zero, ensuring irreducibility while preserving communication patterns in Japan. The recovery rate matrix Δ is derived from the Physician Maldistribution Index of Japan in 2022, provided by the Ministry of Health, Labour and Welfare [21].

The social network represents opinion communication between prefectures. Since individuals across regions can communicate easily thanks to the Internet, we use the Watts-Strogatz model [22] to generate a small-world network with parameters n, d, and c denoting network size, average degree, and clustering coefficient, respectively. In this section, we set $n = 46$, $d = 10$, and $c = 0.5$.

To ensure well-posed simulation parameters, we employ monotonic functions for $\theta_i(o_i(k))$ and $\gamma_i(o_i(k))$, choosing $\theta_i(o_i(k)) = 0.2 + 0.3o_i(k)$ and $\gamma_i(o_i(k)) = 0.4 - 0.4o_i(k)$, where $o_i(k) \in [0, 1]$, for all $i \in [n]$ and $k \geq 0$. We normalize the original physical adjacency matrix B to satisfy $B\mathbf{1}_n = 0.5\mathbf{1}_n$, ensure that the normalized recovery rate matrix Δ falls within $[0, 1]^n$, and normalize the social adjacency matrix W to be row stochastic. For clarity, we present the dynamics of five randomly selected communities alongside the average values for all communities (represented by thick black dotted line) in the figures showing simulation results below.

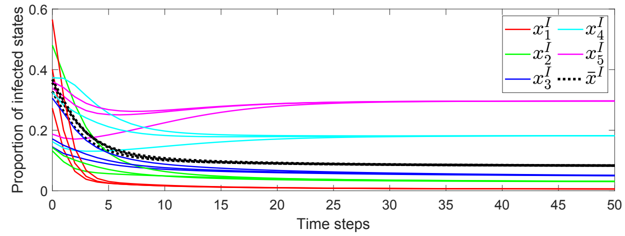
B. Mild Epidemics

Firstly, we simulate the evolution of a mild epidemic with low infectivity, using an adjacency matrix of $0.4B$. Then according to Definition 2, we obtain $R_o^V = 0.9956$. The initial epidemic-opinion states are generated randomly following Assumption 3. As Lemma 1 implies, the opinions of all communities finally converge to a consensus that the epidemic is not serious when the epidemic fades away, as shown in Figs. 2a and 2c. Moreover, when $R_o^V \leq 1$, Theorem 1 states that all the communities converge to a health-consensus equilibrium, which can be computed as $(0, 0.3333\mathbf{1}_n, 0)$ for this example. From the plots in Fig. 2, we confirm this theoretical result.

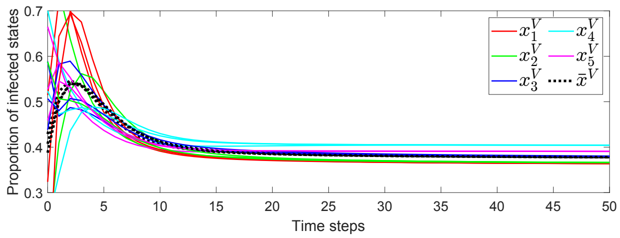
C. Severe Epidemics

The evolution of a severe epidemic with an adjacency matrix B is illustrated in Fig. 3. We have $R_o^V = 1.1827$ following Definition 2. As shown in Figs. 3a and 3c, the coupled SIV-opinion system converges to a dissensus-endemic equilibrium. That is, none of the communities reaches a disease-free state ($x_i^{I*} = 0$) or think the epidemic is not a threat ($o_i{}^* = 0$), which is consistent with Corollary 1.

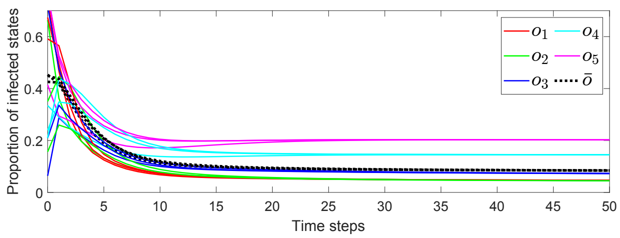
Further, the dissensus-endemic equilibrium appears to



(a)



(b)



(c)

Fig. 4: Under the same condition as Fig. 3, the evolution of the coupled SIV-opinion system for 3 different initial conditions. The 3 trajectories of each community are depicted by same color. The system states converge to the same endemic-dissensus equilibrium, independent of the initial condition. (a) Infected states. (b) Vigilant states. (c) Opinion states.

be unique under different initial conditions in simulation. In Fig. 4, using the same parameters as in Fig. 3, we start the system with 3 different initial conditions. We can observe that the states converge to the same dissensus-endemic equilibrium, which implies that this equilibrium may have a large region of attraction. We can verify that the equilibrium z^* in Fig. 3 is locally exponentially stable by substituting z^* into (11). Based on 10^5 Monte Carlo simulations, the stability radius of equilibrium z^* is not less than 0.14. A challenging question is to find out the specific conditions for the uniqueness and the region of attraction of dissensus-endemic equilibria theoretically. Theorem 2 provides a sufficient condition, which may be conservative. Tighter conditions remain to be explored in future work.

V. Conclusion

This paper has addressed a discrete-time networked SIV epidemic model coupled with polar opinion dynamics, examining how epidemics are influenced by both physical and social factors. We have introduced an SIV-opinion reproduction number and established stability conditions for disease-free and endemic equilibria. Our findings highlight the impact of opinion dynamics on epidemic spread. Numerical simulations have validated our theoretical results. Future research will extend this study to other epidemic models and analyze optimal control strategies for eradicating epidemics. We are also interested in sufficient and necessary conditions for the disease-free and endemic equilibria, or more precise boundaries between them.

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