Optimal sampling schedules for $h_2$ and $h_\infty$ state-feedback control

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Abstract—We consider a discrete-time linear system for which the control input is updated at every sampling time, but the state is measured at a slower rate. We allow the state to be sampled according to a periodic schedule, which dictates when the state should be sampled over a period. Given a desired average sampling interval, our goal is to determine sampling schedules that are optimal in the sense that they minimize the $h_2$ or the $h_\infty$ closed-loop norm, under an optimal state-feedback control law. Our results show that, when the desired average sampling interval is an integer, the optimal state sampling turns out to be evenly spaced. This result indicates that, for the $h_2$ and $h_\infty$ performance metrics, there is relatively little benefit to going beyond constant-period sampling.

I. INTRODUCTION

The standard paradigm in digital control is to periodically sample the system’s output, compute the control action, and update the system’s input. Digital to analog and analog to digital converters often dictate these operations to occur at evenly spaced times, even if, occasionally, at different rates [1]. However, when, for example, the controller and sensor processing units run on shared processors or control signals are transmitted over shared networks, different sensor and control update schedules are imposed or can be selected [2]. Here we address how to optimize the sampling schedules.

We consider a discrete-time linear system for which the control input is updated at every time step and the state is sampled according to an arbitrary periodic schedule; each schedule is characterized by the intervals between consecutive samples in a period $h$. The cost of a sampling schedule is measured by the $h_2$ or $h_\infty$ closed-loop system norms under an optimal control law. We tackle the problem of picking optimal sampling schedules with a desired rational average sampling interval.

In the $h_2$ framework, we start by establishing a key result stating that the expected value of a quadratic cost in the interval between two samples is a convex function of the length of the interval (in a natural sense for functions with discrete domains). Moreover, we show that the $h_2$ norm can be written as a weighted average of samples of this convex function at the lengths of the intervals characterizing the periodic schedule. These two facts lead to a simple way to modify a schedule in order to decrease the associated $h_2$ norm: take two arbitrary intervals characterizing the schedule and reduce the largest by the same amount that the smallest is increased. This implies that:

1) An $h$-periodic schedule with $m$ intervals between sampling is optimal if (and only if under mild conditions) all of these intervals are either equal to $\lfloor h/m \rfloor$ or $\lceil h/m \rceil$ where $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$ denote the floor and the ceil.
2) When $h/m$ is an integer, then evenly spaced sampling is optimal and, in fact, it is the unique optimal schedule (under mild assumptions).
3) The plot optimal achievable $h_2$ norm versus average rate is a continuous piece-wise affine function connecting the pairs $(1/h, J_{2,h})$ where $J_{2,h}$ is the $h_2$ norm of periodic control with integer average sampling interval $h$ (see Figure 3 below).

The results are briefly connected to existing results in the literature for continuous-time sampled data systems (see Remarks 2, 3 below).

In the $h_\infty$ framework, we show that the $h_\infty$ norm only depends on the longest interval in a schedule, and it is a non-decreasing function of this longest interval. This implies that:

1) A sampling schedule guarantees the smallest attenuation bound ($h_\infty$ norm) achievable for a given rational average sampling time $\frac{h}{m}$ if (and only if under mild assumptions) the largest interval does not exceed $\lfloor h/m \rfloor$. Note that the optimal schedules (in this sense) are in general different from the ones for the $h_2$ case.
2) Also here, when $\frac{h}{m}$ is an integer, then evenly spaced sampling is optimal and it is the unique optimal schedule (under mild assumptions).
3) The plot optimal achievable $h_\infty$ norm versus average sampling time (or average rate) is a discontinuous piece-wise constant function, where these constants are equal to the $h_\infty$ norms corresponding to evenly spaced sampling with integer average sampling interval $h$ (see Figure 4 below).

A numerical example illustrates the results.

There are some related results in the literature, reviewed next. The co-design of the control and scheduling of tasks has been proposed in several papers, see e.g., [3]. The superiority of evenly spaced sampling in the context of continuous-time output feedback sampled linear systems has been established in [4], both in the $H_2$ and $H_\infty$ senses, using arguments based on the Youla parameterization. The results in [5], a discrete-time extension of [4] considering the $h_\infty$ setting, do not.
explicitly handle evenly spaced sampling. In the $h_2$ setting, the impact of the variability of the sampling sequence has been studied in [6], which implies the optimality of evenly spaced sampling for sampled data systems. However, note that our results are different from the ones in [5], [6]. In particular, in the $h_2$ setting, neither the above mentioned convexity properties nor the comparison between arbitrarily schedules differing by two intervals appear in [6]. In particular, while [6] shows that the $h_2$ norm is not necessarily a monotone function of the variance of the sampling intervals, from the properties established here we can provide a simple method to find sampling schedules that monotonically improve the $h_2$ norm (see Remark 1 below). In the $h_\infty$ sense besides providing a discrete-time result analogous to the continuous-time provided in [4], we address some of its implications not addressed in the literature. Besides the tools we use to derive our results are very different from the tools used in [4]–[6]. The paper [7] goes beyond the present case for the $h_\infty$ problem and searches for state-dependent (event-triggered) scheduling policies that can outperform periodic control.

The paper is organized as follows. Section II states the problem and Sections III, IV provide the main results pertaining to $h_2$ and $h_\infty$ respectively. Section V provides numerical examples and Section VI gives concluding remarks. The proofs of some auxiliary results are given in the appendix.

II. Problem Formulation and Problem Statement

Consider a linear system

$$x_{t+1} = Ax_t + B_2 u_t + B_1 w_t, \quad t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

(1)

with the following output of interest

$$z_t = C_2 x_t + D_{21} u_t,$$

(2)

where $x_t \in \mathbb{R}^{n_x}$, $u_t \in \mathbb{R}^{n_u}$, $z_t \in \mathbb{R}^p$, $w_t \in \mathbb{R}^{n_w}$ for $t \in \mathbb{N}_0$.

Without loss of generality, we assume that $C_2^T D_{21} = 0$, and define $Q = C_2^T C_2$ and $R = D_{21}^T D_{21}$. Furthermore, we consider $B_1 = I$ and use the notation $B = B_2$. We assume that $(A, B)$ is controllable, $(A, C_2)$ is observable and $R > 0$.

We assume that the sensors provide the full state $x_t$. However, not necessarily at every time $t$. In fact, we assume the following measured output

$$y_t = \begin{cases} x_t, & \text{if } \sigma_t = 1, \\ 0, & \text{if } \sigma_t = 0, \end{cases}$$

where $\sigma_t$ is a periodic binary function with period $h \in \mathbb{N}$. If $y_t = 0$ means that the state is not available at time $t$. When

$$\sigma_t = \begin{cases} 1 & \text{if } t \text{ is zero or an integer multiple of } h, \\ 0 & \text{otherwise.} \end{cases}$$

(3)

we have evenly spaced sampling. Let $s_t$ be the sampling times defined by $s_{t+1} = s_t + \tau_t$, $s_0 = 0$ with $\tau_t = \min\{j \in \mathbb{N} | s_{t+j} = 1\}$. With a periodic schedule, the sampling intervals $\tau_t$ eventually repeat themselves, i.e., $\tau_t = \tau_{t+j}$ for some $j > 0$. Note that the first sampling intervals $T := (\tau_0, \ldots, \tau_{j-1})$, characterize the periodic schedule. Let the average sampling interval be denoted by $\bar{h} = \frac{1}{j} (\sum_{t=0}^{j-1} \tau_t)$ and the average rate be denoted by $r = \frac{1}{\bar{h}}$.

Note that both $r$ and $h$ are rational numbers. Examples of periodic sampling schedules with the same average rate are $((\tau_0, \sigma_1, \sigma_2, \ldots), (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots))$ and $((\tau_0, \sigma_1, \sigma_2, \ldots), (0, 1, 0, 0, 0, 0, 0, 0, 0, \ldots))$.

The control input $u_t$ can be updated at every time $t$ as a function of the information set $J_t = \{x_k | k \in \{0, \ldots, t\}, \sigma_k = 1\}$, that is, $u_t = \mu_{u,t}(J_t)$ for some functions $\mu_{u,t}$. Performance is measured by either the $h_2$ norm or the $h_\infty$ norm.

The $h_2$ norm is defined as follows. Assume that $\{w_t | t \in \mathbb{N} \cup \{0\}\}$ is a sequence of zero-mean independent and identically distributed random variables with $\mathbb{E}[w_tw_t^T] = W \geq 0$. Then the $h_2$ norm is defined as the average cost

$$J_2 = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\sum_{t=0}^{T-1} \|z_t\|^2\right]$$

(4)

We use the notation $J_{2,h}$ to denote $J_2$ for evenly spaced sampling schedules (3).

To define the $h_\infty$ norm let $w = (w_0, w_1, w_2, \ldots)$, $z = (z_0, z_1, z_2, \ldots)$, define the inner product $\langle w, z \rangle = \sum_{t=0}^{\infty} w_t z_t$ and norm $\|w\| = \sqrt{\langle w, w \rangle}$, and let $\ell_2$ be the Hilbert space of sequences with bounded norm. The system provides an attenuation bound $\gamma$ from the input disturbances to the output of interest if

$$\|z\| \leq \gamma \|w\|, \quad \forall w \in \ell_2, \quad \text{for } x_0 = 0.$$  \hspace{1cm} (5)

We are interested in ensuring that (5) holds for the smallest possible $\gamma$. The initial condition $x_0$ may be non-zero provided that we redefine (5) along the lines discussed, e.g., in [8]. The disturbances depend on the information set $I_t = \{x_k | k \in \{0, \ldots, t\}\}$ that is $w_t = \mu_{w,t}(I_t)$, for some functions $\mu_{w,t}$. We define $\pi_u = (\mu_{w,0}, \mu_{w,1}, \ldots)$ as the policy of the controller and $\pi_w = (\mu_{w,0}, \mu_{w,1}, \ldots)$ as the policy of the disturbances. Then, for a given periodic sampling sequence characterized by $T$, the $h_\infty$ norm coincides with the smallest attenuation bound and is given by

$$\gamma_T := \inf \{\gamma | \exists \pi_u \text{ such that (5) holds when the scheduler is } h_\text{periodic with sampling intervals } T\}.$$  \hspace{1cm} (6)

For evenly spaced sampling (3) we use the alternative notation

$$\gamma_h := \inf \{\gamma | \exists \pi_u \text{ such that (5) holds when the scheduler is given by (3)}\}.\hspace{1cm} (7)$$

Naturally if $\tau_i = h$ for every $i \in \{0, \ldots, j-1\}$, we have $\gamma_h = \gamma_T$.

A sampling schedule characterized by $\tau_0, \tau_1, \ldots, \tau_{j-1}$ for some $j_1$ is said to be (strictly) superior to another sampling schedule $\tau_0', \tau_1', \ldots, \tau_{j_1-1}'$ in the $h_2$ ($h_\infty$) sense if the corresponding optimal controller achieves a non-larger (strictly smaller) $h_2$ ($h_\infty$) cost. It is said to be optimal if there is not a different strictly superior schedule.

We are interested in finding optimal sampling schedules with average sampling interval $\frac{h}{m}$ in the $h_2$ and $h_\infty$ sense.
Fig. 1: A non-standard sampled feedback system consists of a controller $C$ and a periodic scheduler $S$, which sends measurement/state data to the controller; $G$ represents the plant.

III. MAIN RESULTS FOR $h_2$ CONTROL

The following standard result shows that the optimal $h_2$ norm, associated with the optimal controller can be written in terms of a key function $\beta$. Let $P$ be the unique positive definite (since $Q$ is positive definite) solution to

$$P = A^TPA + Q - A^TPB(R + B^TPB)^{-1}B^TPA,$$

and let $Z = A^TPB(B^TPB + R)^{-1}B^TPA$ and $K = -(B^TPB + R)^{-1}B^TPA$. Let $\text{tr}(X)$ denote the trace of a matrix $X$.

**Proposition 1:** For given $T = (\tau_0, \ldots, \tau_{j-1})$, the cost (4) of an optimal control policy is given by

$$J_2 = \text{tr}(PW) + \frac{1}{h} \sum_{i=0}^{j-1} \beta(\tau_i),$$

where $\beta(1) = 0$ and, for $p > 1$,

$$\beta(p) = \text{tr}(Z\left(\sum_{s=1}^{p-1} Y(s)\right)),  \tag{9}$$

where $Y(s) = \sum_{r=0}^{s-1} A^rWA^r$, for $s \in \mathbb{N}$. Moreover, an optimal control policy that minimizes (4) is

$$u_t = K\hat{x}_{t|t}, \quad t \in \mathbb{N}_0, \tag{10}$$

where

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t} + Bu_t,$$

$$\hat{x}_{t|t} = \begin{cases} x_t, & \text{if } \sigma_t = 1, \\ \hat{x}_{t|t-1}, & \text{if } \sigma_t = 0. \end{cases}$$

The proof follows from standard optimal control arguments (see, e.g., [9]) and it is omitted for the sake of brevity.

A function with discrete domain $f(i), i \in \mathbb{N}$, is said to be convex if

$$f(i) \leq \frac{f(i+1) + f(i-1)}{2}, \quad \forall i \in \mathbb{N} \setminus \{1\}. \tag{11}$$

A key result of the present paper is the following observation that $\beta$ is convex.

**Theorem 1:** The function $\beta(i), i \in \mathbb{N}$, defined in (9), is convex.

**Proof:** We have, for $i \in \mathbb{N}$, $\beta(i+1) - \beta(i) = \text{tr}(ZY(i))$ and, for $i \in \mathbb{N} \setminus \{1\}$

$$\beta(i+1) - \beta(i) - (\beta(i) - \beta(i-1)) = \text{tr}(A^tAWA^t) \geq 0.$$ \hfill \blacksquare

The convexity of $\beta$ is used in the next theorem to improve upon a periodic schedule, by replacing two of its sampling intervals by two alternative sampling intervals that are closer to their means.

**Theorem 2:** Consider a given periodic schedule with period $h$ and characterized by $j$ intervals $T = (\tau_0, \ldots, \tau_{j-1})$. Given any two distinct $\tau_i > \tau_l$ construct a modified schedule $\bar{T} = (\bar{\tau}_0, \ldots, \bar{\tau}_{j-1})$ with

$$\bar{\tau}_\ell = \begin{cases} \tau_i - p, & \text{if } \ell = i \\ \tau_l + p, & \text{if } \ell = l \\ \tau_\ell, & \text{otherwise}. \end{cases}$$

for some $p \in \{0, \ldots, [(\tau_i - \tau_l)/2]\}$. Then, $\bar{T}$ is superior to $T$ in the $h_2$ sense.

**Proof:** Let $J_2$ and $\bar{J}_2$ denote the optimal costs associated with the original and modified schedules, respectively. Note that both schedules have the same period denoted by $h$. Due to (8) suffices to prove that

$$J_2 - \bar{J}_2 \leq \frac{1}{h}(\beta(\tau_i) + \beta(\tau_l) - (\beta(\tau_i - p) + \beta(\tau_l + p))).$$ \tag{12}

Since $\beta(i)$ is convex we have, for any $p \in \mathbb{N}$,

$$\beta(i + p + 1) - \beta(i + p) \geq \beta(i + p) - \beta(i + p - 1) \geq \beta(i + 1) - \beta(i) \geq 0, \quad \forall i \in \mathbb{N}. \tag{13}$$

From this fact we conclude that

$$\beta(\tau_i) - \beta(\tau_l - p) = \sum_{k=0}^{p-1} \beta(\tau_i - k) - \beta(\tau_l - 1 - k)$$

$$\geq \sum_{k=0}^{p-1} \beta(\tau_i + p - k) - \beta(\tau_l + p - 1 - k) = \beta(\tau_l + p) - \beta(\tau_l) \tag{14}$$

which implies (12), concluding the proof. \hfill \blacksquare

This results has several implications given next.

**Corollary 1:** An $h$-periodic schedule with $m$ intervals between sampling is optimal if $m_1$ of these intervals equal $h_1 = \lfloor \frac{h}{m_1} \rfloor$ and $m_2$ equal $h_2 = \lceil \frac{h}{m_2} \rceil$ where $m_1 + m_2 = m$. Moreover, the corresponding $h_2$ norm is equal to

$$\text{tr}(PW) + \frac{m_1}{h}\beta(h_1) + \frac{m_2}{h}\beta(h_2) \tag{15}$$

Furthermore, if $\beta(i+1) > \beta(i)$ for every $i \in \mathbb{N}$, then these optimal schedules are unique in the class of schedules with period $h$.

**Proof:** We can list all possible schedules with period $h$ and $m$ sampling intervals and compute the associated $h_2$ norm. Note that all the schedules that meet the form in the present corollary have the same $h_2$ norm due to (8). If such
an $h_2$ norm is minimal we conclude the sufficiency part. In turn, if we would have a schedule with minimal $h_2$ norm that does not take the form stated in the present corollary, due to Theorem 2, we could modify it without increasing the cost so that it does meet the mentioned form, so that it is equally optimal. If $\beta(i + 1) > \beta(i)$ for every $i \in \mathbb{N}$ this leads to a contradiction meaning that only the schedules of the form stated in the present corollary are optimal.

It is immediate from (9) that a sufficient condition for $\beta(i + 1) > \beta(i)$ for every $i \in \mathbb{N}$ is $KW \neq 0$.

Corollary 1 implies the following:

i) In general there might be more than one optimal schedule, e.g., if $h = 10$, $m = 4$, 1010100100 are both optimal.

ii) Given a desired rational average sampling time $\frac{h}{m}$, the optimal schedules are the ones that have $m_1$ intervals equal to $\frac{h}{m}$ and $m_2$ equal to $\lfloor \frac{h}{m} \rfloor$.

(iii) Note that we can write (15) as follows

$$\frac{\beta(h_1)}{h_1 + 1 - \zeta} + \frac{\beta(h_1 + 1)(\zeta - h_1)}{\zeta}$$

since $h = m_1h_1 + m_2h_1 + m_2$ and

$$\frac{(h_1 + 1 - h/m)}{h/m} = \frac{h_1m_1 + h_1m_2 + m_2 - h + m_1}{h}$$

$$\frac{(h/m - h_1)}{h/m} = \frac{h - h_1m_1 - h_2m_2}{h}$$

Rewriting (15) in terms of the rate $r = 1/\xi$ we have

$$\beta(h_1)(r(h_1 + 1) - 1) + \beta(h_1 + 1)(1 - rh_1)|_{r=\xi}$$

Thus, the plot optimal achievable $h_2$ norm versus average rate is (a restriction to the rational numbers of) a continuous piecewise affine function connecting the pairs $(1/h, J_{2,h})$ where $J_{2,h}$ is the $h_2$ norm of periodic control with integer average sampling interval $h$ (see Figure 3 below).

Corollary 1 also implies the following result.

**Corollary 2:** Evenly spaced sampling is optimal in the class of periodic schedulers with integer average sampling time $h \in \mathbb{N}$. This is the unique optimal schedule if $\beta(h-1) < \beta(h) < \beta(h+1)$ when $h > 1$ and if $\beta(1) < \beta(2)$ when $h = 1$.

□

**Remark 1:** One simple method to arrive at an optimal sampling schedule as given in Corollary 1 from an arbitrary scheduled is to recursively reduce the largest interval by the same amount that the smallest is increased, and set this amount to the largest possible according to Theorem 2. The resulting sequence of sampling schedules monotonically improves the $h_2$ norm.

**Remark 2:** The paper [10] considers a continuous-time version of sampled data periodic control with average intersampling time $h_c \in \mathbb{R}_{>0}$. This reference provides the following expression for the cost of periodic control with average sampling period $h_c$:

$$J_c(h_c) = \delta_c + \frac{1}{h_c} \beta_c(h_c), \quad \beta_c(h_c) = \int_0^{h_c} \text{tr}(Z_c) \int_0^s V(r) dr ds$$

where $V(s) = \int_0^s e^{A^r} W_c e^{A^r T} dr$, $A_c$ is the system matrix, $W_c$ is a positive semi-definite matrix proportional to the covariance of the stochastic disturbances and the expressions for $\delta_c$ and $Z_c$ are omitted here. Note that analogously to the discrete-time case $\beta_c$ is a convex function since $\beta''_c(h_c) = \text{tr}(Z_c Z_c)$ $\geq 0$. Due to this convexity property, many results of the previous section can be extended to the sampled-data case; however we do not pursue this here.

**Remark 3:** A different result, shown in [11], states that (16) as a function of the average rate $r = 1/h$, i.e., $J_c(1/r) = r \beta_c(\frac{1}{r})$ is convex. Note, however, that neither $J_c(h)$ nor $J_2(h)$ are in general convex functions of $h$.

**IV. MAIN RESULTS FOR $h_\infty$ CONTROL**

We start by considering evenly spaced sampling (3) and by providing a method to compute $\gamma_h$, given by (7). Let us first define three matrix transformations:

$$F_o(P) := P + P(\gamma^2 I - P)^{-1} P$$

$$F_r(P) := A^T P A + Q - A^T P B (B^T P B + R)^{-1} B^T P A$$

$$F_o(P) := A^T P A + Q.$$

for a given $\gamma \in \mathbb{R}_{>0}$. When $h = 1$ the following iteration $P_{t+1} = F_o(P_o(P_t))$ with $P_0 = 0$ is monotone in the sense that $P_{t+1} \geq P_t$ converges for every $t \in \mathbb{N}_0$ to the unique positive definite solution $\bar{P}_\gamma$ of the algebraic Riccati equation

$$\bar{P}_\gamma = F_o(F_o(P_o)))$$

see [8] (although the expressions in [8] appear in a different but equivalent form). Due to monotonicity, $\gamma^2 I > \bar{P}_\gamma$ implies that $\gamma^2 I > P_t$ for every $t \in \mathbb{N}_0$. Provided that this condition holds, (5) holds for a policy $\pi_x$ specified by $u_t = K_{\gamma, x_t}$, where

$$K_{\gamma} = -(R + B^T F_o(\bar{P}_\gamma)) B^{-1} B^T F_o(\bar{P}_\gamma) A.$$  

If $\gamma$ is such that $\gamma^2 I > P_t$ does not hold for some $t$, then (5) does not hold for any $\pi_u$.

However, if $h > 1$ the conditions on $\gamma$ for the existence of such a control policy are stricter, i.e., $\gamma$ needs to be larger [12]. They actually become stricter as $h$ increases leading to non-decreasing sequence of $\gamma_h$, as stated in the next lemma.

**Lemma 1:** Suppose that $\gamma^2 I > \bar{P}_\gamma$, and consider the following iteration, with $M_1 = \bar{P}_\gamma$,

$$M_{k+1} = F_o(F_o(M_k)) \quad k \in \{1, 2, \ldots, h\}.$$  

which can be run as long as $\gamma^2 I - M_k$ is not singular. Then:

(i) if $\gamma^2 I - M_k > 0$, for all $k \in \{1, \ldots, h\}$, then, for all $k \in \{1, \ldots, h - 1\}$, $M_{k+1} \geq M_k$.

(ii) for every $h \in \mathbb{N}$, $\inf \{\gamma | \gamma^2 I - M_h > 0 \} = \gamma_h$, where $\gamma_h$ is given by (7).
(iii) for every $h \in \mathbb{N}$, $\gamma_{h+1} \geq \gamma_h$.

The proof is given in the appendix.

We turn now to general schedules. The following result provides a simple way of computing the $h_\infty$ norm associated with a general schedule from the $h_\infty$ norm associated with an evenly spaced schedule.

**Theorem 3:** Consider a periodic schedule characterized by the intervals $T := (\tau_0, \ldots, \tau_{\ell-1})$. Let $\bar{\tau} = \max\{\tau_i | i \in \{0, 1, \ldots, \ell - 1\}\}$. Then, $\gamma_T = \gamma_{\bar{\tau}}$.

**Proof:** Suppose that $\gamma_T > \gamma_{\bar{\tau}}$. Pick a $\gamma$ such that $\gamma_T > \gamma > \gamma_{\bar{\tau}}$. Then for any given arbitrary control policy and for schedules $T$ there exists a $w \in \ell_2$ such that $\|z\|^2 - \gamma^2 \|w\|^2 > 0$ i.e., $\lim_{T \to \infty} \sum_{t=0}^{T-1} z_t^* z_t - \gamma^2 w_t^* w_t > 0$. This implies that there exists a $\ell \in \mathbb{N}$ and a $w \in \ell_2$ such that, for $\ell \geq \ell$,

$$\sum_{t=0}^{\ell-1} z_t^* z_t - \gamma^2 w_t^* w_t > 0 \quad (20)$$

Suppose that we pick $\ell$ to be a multiple of $h$. Using Lemma 2 in the appendix, we conclude that if we pick the control policy $u_t = K_t x_t$, $t \in \{0, 1, \ldots, \bar{\tau}\}$ where the $K_t$ are obtained from the iteration (25) initialized with $P_\bar{\tau} = Y_0 = 0$ and $\tau = \bar{\tau}$, we get

$$\sum_{k=0}^{\ell-1} z_k^* z_k - \gamma^2 w_k^* w_k = \sum_{k=0}^{\ell-1} (w_k - L_k(Ax_k + Bu_k))^\top (\gamma^2 I - P_{k+1})(w_k - L_k(Ax_k + Bu_k)) \geq 0 \quad (21)$$

provided that the $\gamma^2 I - P_t$ are invertible. This is indeed the case since as we now argue $\gamma^2 I - P_t > 0$ for every $t \in \{0, 1, \ldots, \ell - 1\}$. To see this it suffices to establish that $P_t \leq P_{\bar{\tau}}$ for every $t \in \{0, 1, \ldots, \ell - 1\}$ since $\gamma^2 I - P_{\bar{\tau}} > 0$ by hypothesis. If we run the iteration (25) for $\tau = \bar{\tau}$ with $Y_0 = P_{\bar{\tau}}$ we obtain $P_t = P_{\bar{\tau}}$ for every $t \in \{0, 1, \ldots, \ell - 1\}$. From the monotonicity property of Lemma 1 we conclude that the $P_t$ obtained when $Y_0 = 0$ satisfy $P_t \leq P_{\bar{\tau}}$ for every $t \in \{0, 1, \ldots, \ell - 1\}$ as desired. Note that (21) implies $\sum_{t=0}^{\ell-1} z_t^* z_t - \gamma^2 w_t^* w_t \leq 0$ for every disturbance sequence which contradicts (20).

Suppose now that $\gamma_T < \gamma_{\bar{\tau}}$ so that for the schedule $T$ we can guarantee that

$$\|z\|^2 - \gamma^2 \|w\|^2 < 0 \quad (22)$$

for some $\gamma$, $\gamma_T < \gamma < \gamma_{\bar{\tau}}$ and for every $w \in \ell^2$. Note that in this case $\gamma^2 I - M_\ell$ has an eigenvalue which is negative. Let $\ell_m \in \arg\max\{\tau_i | i \in \{0, \ldots, \ell - 1\}\}$ so that $\tau_{\ell_m} = \bar{\tau}$ and consider the following disturbance policy

$$w_t = \begin{cases} \\
\xi, & t \in \{0, \ldots, s_{\ell_m} - 1\} \\
L_{\bar{\tau}}(Ax_t + Bu_t) + \eta & t = s_{\ell_m} \\
L_{\bar{\tau} - s_{\ell_m}}(Ax_t + Bu_t), & t \in \{s_{\ell_m} + 1, \ldots, s_{\ell_m} + \ell - 1\} \\
L_{\bar{\tau}}(Ax_t + Bu_t) & t \in \{s_{\ell_m} + \bar{\tau}, \ldots, s_{\ell_m} + \bar{\tau} + q\} \\
0, & t \geq s_{\ell_m} + q + 1 \end{cases}$$

where:

- $\xi \in \mathbb{R}^{n_x}$ is an arbitrary constant;
- $\eta \in \mathbb{R}^{n_y}$ will be chosen in the sequel;
- $L_k = (\gamma^2 I - M_k)^{-1} M_k$, $k \in \{1, \ldots, \bar{\tau} - 1\};$
- $\bar{\tau}, t \in \{s_{\ell_m} + \bar{\tau}, \ldots, s_{\ell_m} + \bar{\tau} + q\}$ are given by (29) in Lemma 4 below (with $k = s_{\ell_m} + \bar{\tau}$) and $q$ is such that $\|x_{s_{\ell_m} + \bar{\tau} + \bar{\eta} G_{x_{s_{\ell_m} + \bar{\tau}}} x_{s_{\ell_m} + \bar{\tau}} + x_{s_{\ell_m} + \bar{\tau} + \bar{\eta} P_{x_{s_{\ell_m} + \bar{\tau}}} x_{s_{\ell_m} + \bar{\tau}} - 1) \| < \alpha$ for a given and arbitrary $\alpha > 0$. Such a $q$ depends on $x_{s_{\ell_m} + \bar{\tau}}$ and exists due to Lemma 4.

Then

$$\|z\|^2 - \gamma^2 \|w\|^2 = \sum_{t=0}^{s_{\ell_m} - 1} z_t^* z_t - \gamma^2 w_t^* w_t + \sum_{t=s_{\ell_m}}^{s_{\ell_m} + \bar{\tau} - 1} z_t^* z_t - \gamma^2 w_t^* w_t$$

$$+ \sum_{t=s_{\ell_m} + \bar{\tau}}^{\infty} z_t^* z_t - \gamma^2 w_t^* w_t \geq \gamma^2 \|x_{s_{\ell_m} + \bar{\tau} + \bar{\eta} G_{x_{s_{\ell_m} + \bar{\tau}}} x_{s_{\ell_m} + \bar{\tau}} + x_{s_{\ell_m} + \bar{\tau} + \bar{\eta} P_{x_{s_{\ell_m} + \bar{\tau}}} x_{s_{\ell_m} + \bar{\tau}} - 1) \| \geq 0$$

provided that $\gamma^2 I - P_t$ are invertible. This is indeed the case since as we now argue $\gamma^2 I - P_t > 0$ for every $t \in \{0, 1, \ldots, \ell - 1\}$. To see this it suffices to establish that $P_t \leq P_{\bar{\tau}}$ for every $t \in \{0, 1, \ldots, \ell - 1\}$ since $\gamma^2 I - P_{\bar{\tau}} > 0$ by hypothesis. If we run the iteration (25) for $\tau = \bar{\tau}$ with $Y_0 = P_{\bar{\tau}}$ we obtain $P_t = P_{\bar{\tau}}$ for every $t \in \{0, 1, \ldots, \ell - 1\}$. From the monotonicity property of Lemma 1 we conclude that the $P_t$ obtained when $Y_0 = 0$ satisfy $P_t \leq P_{\bar{\tau}}$ for every $t \in \{0, 1, \ldots, \ell - 1\}$ as desired. Note that (21) implies $\sum_{t=0}^{\ell-1} z_t^* z_t - \gamma^2 w_t^* w_t \leq 0$ for every disturbance sequence which contradicts (20).

Suppose now that $\gamma_T < \gamma_{\bar{\tau}}$ so that for the schedule $T$ we can guarantee that

$$\|z\|^2 - \gamma^2 \|w\|^2 < 0 \quad (22)$$

for some $\gamma$, $\gamma_T < \gamma < \gamma_{\bar{\tau}}$ and for every $w \in \ell^2$. Note that in this case $\gamma^2 I - M_\ell$ has an eigenvalue which is negative. Let $\ell_m \in \arg\max\{\tau_i | i \in \{0, \ldots, \ell - 1\}\}$ so that $\tau_{\ell_m} = \bar{\tau}$ and consider the following disturbance policy

$$w_t = \begin{cases} \\
\xi, & t \in \{0, \ldots, s_{\ell_m} - 1\} \\
L_{\bar{\tau} - s_{\ell_m}}(Ax_t + Bu_t) + \eta & t = s_{\ell_m} \\
L_{\bar{\tau} - s_{\ell_m} - s_{\ell_m}}(Ax_t + Bu_t), & t \in \{s_{\ell_m} + 1, \ldots, s_{\ell_m} + \ell - 1\} \\
L_{\bar{\tau}}(Ax_t + Bu_t) & t \in \{s_{\ell_m} + \bar{\tau}, \ldots, s_{\ell_m} + \bar{\tau} + q\} \\
0, & t \geq s_{\ell_m} + q + 1 \end{cases}$$

Note that also here there can be multiple optimal schedules (in the sense of this corollary), but are in general different from the ones for the $h_2$ case.

Due to this corollary, the plot optimal achievable $h_\infty$ norm versus average sampling time (or average rate) is
(a restriction to the rational numbers of) a discontinuous piecewise constant function, where these constants are equal to the $h_\infty$ norms of evenly spaced sampling with integer average sampling interval $h$ (see Figure 4 below).

Note that we can have schedules corresponding to arbitrarily poor $h_\infty$ norm and maximum sampling rate. In fact, the schedule characterized by $h = \omega b + b$

$$\tau_i = \begin{cases} 1, & \text{if } i \in \{0, \ldots, \omega b\} \\ b, & \text{if } i = \omega b + 1 \end{cases}$$

leads to an $h_\infty$ norm $\gamma_h$ and average rate $\frac{\omega b}{(\omega b + 1)^2}$. For systems for which $\gamma_h \to \infty$ as $b \to \infty$, the $h_\infty$ norm becomes arbitrary poor while the average rate converges to 1 as $b \to \infty$ and $\omega \to \infty$.

V. NUMERICAL EXAMPLE

Suppose that

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = I_3, \quad R = 1,$$

where $I_3$ is the identity matrix. The values of $\beta(h)$, $J_{2,h}$, and $\gamma_h$ are given in Table I and plotted in Figures 2, 3, 4, respectively.

In Figures 3 the optimal $h_2$ value achievable with the corresponding rational average sampling time is plotted and in Figure 4 plotted and the optimal $h_\infty$ value achievable with the corresponding rational average rate is plotted.

VI. CONCLUSIONS

In this paper we have characterized sampling schedules that are optimal for $h_2$ and $h_\infty$ feedback control. We have shown that if the desired average intersampling time is $h$ then evenly spaced sampling is optimal in both senses. However, when the desired average intersampling time is not an integer, then the class of optimal schedules is different in the $h_2$ and $h_\infty$ senses. While for $h_2$ schedules close to evenly spaced sampling are still optimal, in the $h_\infty$ framework the $h_\infty$ norm is only dictated by the largest sampling interval.

APPENDIX

PROOF OF LEMMA 1

(i) When $k = 1$, $M_1 = P_1 = F_e(F_a(P_h)) = M_2 - A^T F_a(P_h) B (B^T F_a(P_h) B + R)^{-1} B^T F_a(P_h) A \leq M_2$. We now prove that, $F_a(F_a(P_h))$ is a monotone map in the sense that $F_a(F_a(P_1)) \leq F_a(F_a(P_2))$ when $P_1 \leq P_2$. This follows from the fact that, for $P < \gamma^2 I$ and that for an arbitrary $x$

$$x^T F_a(F_a(P)) x = x^T Q x + \max(Ax + w, w)^T P(Ax + w)$$

Using induction, suppose that $M_{k-1} \leq M_k$ for some $k \in \{2, \ldots, h-1\}$. Then $F_a(F_a(M_{k-1})) \leq F_a(F_a(M_k))$ and, concluding the proof.

(ii) In [12] it is shown that $\gamma_h = \inf \{ \gamma | \gamma^2 I - \tilde{D}_h \text{diag}(I \otimes Q, P_h) \tilde{D}_h > 0 \}$, with

$$\tilde{D}_h = \begin{bmatrix} I & 0 & \ldots & 0 \\ A & I & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{T-1} & A^{T-2} & \ldots & I \end{bmatrix}$$

and the condition $\gamma^2 I - \tilde{D}_h \text{diag}(I \otimes Q, P_h) \tilde{D}_h > 0$ is equivalent to the following function being concave in $w_0, \ldots, w_{h-1}$:

$$\sum_{t=0}^{h-1} z_t^\top z_t - \gamma^2 w_t^\top w_t + x_h^\top P_h x_h,$$
where \( x_{t+1} = A x_t + w_t \) for \( t \in \{ 0, 1, \ldots, h - 1 \} \).
Applying dynamic programming to maximize this function with respect to \( w_{t+1}, \ldots, w_{h-1} \), for \( t \in \{ 0, 1, \ldots, h - 1 \} \), we obtain that this cost is equal to

\[
\sum_{t=0}^{\ell} z_t^T z_t - \gamma^2 w_t^T w_t + x_{\ell+1}^T M_{\ell} x_{\ell+1}.
\]

We can rewrite this expression as

\[
\sum_{t=0}^{\ell-1} z_t^T z_t - \gamma^2 w_t^T w_t + x_t^T (Q + A^T M_t A) x_t + w_t^T (M_t - \gamma^2 I) w_t + 2w_t^T M_t Ax_t
\]

from which clear that this function is concave in \( w_t \) if and only if \( \gamma^2 I - M_t > 0 \) where \( \ell \in \{ 1, \ldots, h \} \) is arbitrary.

(iii) Follows from (i) and (ii).

**Lemma 2:** If \( 0 \leq P_1 \leq P_2 \) then

\[
F_c(F_a(P_1)) \leq F_c(F_a(P_2)).
\]

**Proof:** The proof follows by noticing that, for every \( x \in \mathbb{R}^{n_x} \),

\[
x^T F_c(F_a(P_1)) x = \min_{x_t, w_t} x^T Q x + w^T R w - \gamma^2 w^T w + (A x_t + B u_t + w)^T P (A x_t + B u_t + w)^T.
\]

**Lemma 3:** Consider (1), (2), and the following iteration

\[
P_{k-1} = F_c(F_a(P_k)), k \in \{ \tau, \tau - 1, \ldots, 0 \} \tag{25}
\]

with\( P_\tau = Y_0 \) for some \( Y_0 \geq 0 \) and where \( \gamma \) is such that \( \gamma^2 I - P_k \) is invertible for every \( k \in \{ \tau, \tau - 1, \ldots, 0 \} \). Then

\[
\sum_{k=0}^{\tau-1} z_k^T z_k - \gamma^2 w_k^T w_k + x_k^T P_k x_k = x_0^T P_0 x_0 + \sum_{k=0}^{\tau-1} (u_k - K_k x_k)^T (R + B^T F_a(P_{k+1})B)(u_k - K_k x_k) - \sum_{k=0}^{\tau-1} (w_k - L_k(A x_k + B u_k))^T (\gamma^2 I - P_{k+1}) (w_k - L_k(A x_k + B u_k))
\]

where \( K_k \) and \( L_k \) are given by

\[
K_k = - (R + B^T F_a(P_{k+1})B)^{-1} B^T F_a(P_{k+1}) A,
\]

\[
L_k = (\gamma^2 I - P_{k+1})^{-1} F_a(P_{k+1}) A,
\]

for \( k \in \{ 0, 1, \ldots, \tau - 1 \} \)

\[\text{(26)}\]

The proof is similar to the one in [7, Lemma] and it is omitted here for the sake of brevity.

**Lemma 4:** Consider (1) and \( \gamma \) such that \( \gamma^2 I - \tilde{P}_\tau > 0 \). Then

\[
\sum_{t=k}^{\infty} z_t^T z_t - \gamma^2 w_t^T w_t \geq x_k^T G_q x_k,
\]

when

\[
w_t = \begin{cases}
\tilde{L}_{q-(t-k)}(A x_t + B u_t), & \text{if } k \leq t < k + q \\
0, & \text{if } t \geq k + q
\end{cases}
\]

where, for \( k \in \{ 1, \ldots, q \} \),

\[
\tilde{L}_k = (\gamma^2 I - G_{k-1})^{-1} G_{k-1}
\]

and, for \( k \in \{ 0, \ldots, q - 1 \} \),

\[
G_{k+1} = F_c(F_a(G_k))
\]

with \( G_0 = P_{\ell Q} \) where \( P_{\ell Q} \) is the unique positive definite solution to the algebraic Riccati equation

\[
P_{\ell Q} = A^T P_{\ell Q} A + P_{\ell Q} - A^T P_{\ell Q} B (R + B^T P_{\ell Q} B)^{-1} B^T P_{\ell Q} A.
\]

Moreover, for any \( x \in \mathbb{R}^{n_x} \) and \( \alpha \in \mathbb{R}_{>0} \), there exists \( q \in \mathbb{N} \), denoted by \( q = \zeta(x, \alpha) \), such that

\[
\| x^T G_q x - x^T P_{\ell Q} x \| < \alpha.
\]

(31)

Such \( q \) can be found by running (30) until (31) is met. □

This result is simply restates [7, Lemma 3], where the proof can be found.

**References**


