2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Optimal sampling schedules for h_2 and h_∞ state-feedback control

Duarte J. Antunes and João P. Hespanha

***Abstract*— We consider a discrete-time linear system for which the control input is updated at every sampling time, but the state is measured at a slower rate. We allow the state to be sampled according to a periodic schedule, which dictates when the state should be sampled over a period. Given a desired average sampling interval, our goal is to determine sampling schedules that are optimal in the sense that they minimize the h_2 or the h_∞ closed-loop norm, under an optimal state-feedback control law. Our results show that, when the desired average sampling interval is an integer, the optimal state sampling turns out to be evenly spaced. This result indicates that, for the h_2 and h_∞ performance metrics, there is relatively little benefit to going beyond constant-period sampling.**

I. Introduction

The standard paradigm in digital control is to periodically sample the system's output, compute the control action, and update the system's input. Digital to analog and analog to digital converters often dictate these operations to occur at evenly spaced times, even if, occasionally, at different rates [1]. However, when, for example, the controller and sensor processing units ran on shared processors or control signals are transmitted over shared networks, different sensor and control update schedules are imposed or can be selected [2]. Here we address how to optimize the sampling schedules.

We consider a discrete-time linear system for which the control input is updated at every time step and the state is sampled according to an arbitrary periodic schedule; each schedule is characterized by the intervals between consecutive samples in a period h. The cost of a sampling schedule is measured by the h_2 or h_∞ closed-loop system norms under an optimal control law. We tackle the problem of picking optimal sampling schedules with a desired rational average sampling interval.

In the h_2 framework, we start by establishing a key result stating that the expected value of a quadratic cost in the interval between two samples is a convex function of the length of the interval (in a natural sense for functions with discrete domains). Moreover, we show that the h_2 norm can be written as a weighted average of samples of this convex function at the lengths of the intervals characterizing the periodic schedule. These two facts lead to a simple way to modify a schedule in order to decrease the associated h_2 norm: take two arbitrary intervals characterizing the schedule and reduce the largest by the same amount that the smallest is increased. This implies that:

1) An h-periodic schedule with m intervals between sampling is optimal if (and only if under mild conditions) all of these intervals are either equal to $\lfloor \frac{h}{m} \rfloor$ or $\lceil \frac{h}{m} \rceil$ where $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$ denote the floor and the ceil.
2) When $\frac{h}{m}$ is an integer, then evenly spaced sampling is optimal and, in fact, it is the unique optimal schedule (under mild assumptions).
3) The plot optimal achievable h_2 norm versus average rate is a continuous piece-wise affine function connecting the pairs $(1/h, J_{2,h})$ where $J_{2,h}$ is the h_2 norm of periodic control with integer average sampling interval h (see Figure 3 below).

The results are briefly connected to existing results in the literature for continuous-time sampled data systems (see Remarks 2, 3 below).

In the h_∞ framework, we show that the h_∞ norm only depends on the longest interval in a schedule, and it is a non-decreasing function of this longest interval. This implies that:

1) A sampling schedule guarantees the smallest attenuation bound (h_∞ norm) achievable for a given rational average sampling time $\frac{h}{m}$ if (and only if under mild assumptions) the largest interval does not exceed $\lceil \frac{h}{m} \rceil$. Note that the optimal schedules (in this sense) are in general different from the ones for the h_2 case.
2) Also here, when $\frac{h}{m}$ is an integer, then evenly spaced sampling is optimal and it is the unique optimal schedule (under mild assumptions).
3) The plot optimal achievable h_∞ norm versus average sampling time (or average rate) is a discontinuous piece-wise constant function, where these constants are equal to the h_∞ norms corresponding to evenly spaced sampling with integer average sampling interval h (see Figure 4 below).

A numerical example illustrates the results.

There are some related results in the literature, reviewed next. The co-design of the control and scheduling of tasks has been proposed in several papers, see e.g., [3]. The superiority of evenly spaced sampling in the context of continuous-time output feedback sampled linear systems has been established in [4], both in the H_2 and H_∞ senses, using arguments based on the Youla parameterization. The results in [5], a discrete-time extension of [4] considering the h_∞ setting, do not

Duarte J. Antunes is with the Control Systems Technology Group, Dep. of Mechanical Eng., Eindhoven University of Technology, the Netherlands. d.antunes@tue.nl. João P. Hespanha is with the Dep. of Electrical and Computer Eng., University of California, Santa Barbara, CA 93106-9560, USA. hespanha@ece.ucsb.edu. The research is carried out as part of the ITEA4 20216 ASIMOV project. The ASIMOV activities are supported by the Netherlands Organisation for Applied Scientific Research TNO and the Dutch Ministry of Economic Affairs and Climate (project number: AI211006). The research leading to these results is partially funded by the German Federal Ministry of Education and Research (BMBF) within the project ASIMOV-D under grant agreement No. 01IS21022G [DLR], based on a decision of the German Bundestag.

explicitly handle evenly spaced sampling. In the h_2 setting, the impact of the variability of the sampling sequence has been studied in [6], which implies the optimality of evenly spaced sampling for sampled data systems. However, note that our results are different from the ones in [5], [6]. In particular, in the h_2 setting, neither the above mentioned convexity properties nor the comparison between arbitrarily schedules differing by two intervals appear in [6]. In particular, while [6] shows that the h_2 norm is not necessarily a monotone function of the variance of the sampling intervals, from the properties established here we can provide a simple method to find sampling schedules that monotonically improve the h_2 norm (see Remark 1 below). In the h_∞ sense besides providing a discrete-time result analogous to the continuous-time provided in [4], we address some of its implications not addressed in the literature. Besides the tools we use to derive our results are very different from the tools used in [4]–[6]. The paper [7] goes beyond the present case for the h_∞ problem and searches for state-dependent (event-triggered) scheduling policies that can outperform periodic control.

The paper is organized as follows. Section II states the problem and Sections III, IV provide the main results pertaining to h_2 and h_∞ respectively. Section V provides numerical examples and Section VI gives concluding remarks. The proofs of some auxiliary results are given in the appendix.

II. Problem formulation and Problem Statement

Consider a linear system

$$x_{t+1} = Ax_t + B_2 u_t + B_1 w_t, \quad t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{1}$$

with the following output of interest

$$z_t = C_2 x_t + D_{21} u_t, \tag{2}$$

where $x_t \in \mathbb{R}^{n_x}$, $u_t \in \mathbb{R}^{n_u}$, $z_t \in \mathbb{R}^p$, $w_t \in \mathbb{R}^{n_w}$ for $t \in \mathbb{N}_0$. Without loss of generality, we assume that $C_2^\top D_{21} = 0$, and define $Q = C_2^\top C_2$ and $R = D_{21}^\top D_{21}$. Futhermore, we consider $B_1 = I$ and use the notation $B = B_2$. We assume that (A, B) is controllable, (A, C_2) is observable and $R > 0$.

We assume that the sensors provide the full state x_t. However, not necessarily at every time t. In fact, we assume the following measured output

$$y_t = \begin{cases} x_t, \text{ if } \sigma_t = 1, \\ \emptyset \text{ if } \sigma_t = 0, \end{cases}$$

where σ_t is a *periodic* binary function with period $h \in \mathbb{N}$ and $y_t = \emptyset$ means that the state is not available at time t. When

$$\sigma_t = \begin{cases} 1 \text{ if } t \text{ is zero or an integer multiple of } h, \\ 0 \text{ otherwise.} \end{cases} \tag{3}$$

we have *evenly spaced sampling*. Let s_ℓ be the sampling times defined by $s_{\ell+1} = s_\ell + \tau_\ell$, $s_0 = 0$ with $\tau_\ell = \min\{j \in \mathbb{N} | \sigma_{s_\ell + j} = 1\}$. With a periodic schedule, the sampling intervals τ_ℓ eventually repeat themselves, i.e., $\tau_\ell = \tau_{\ell+j}$ for some $j > 0$. Note that the first sampling intervals $\mathcal{T} := (\tau_0, \dots, \tau_{j-1})$, characterize the periodic schedule. Let the average sampling interval be denoted by $\bar{h} = \frac{1}{j}(\sum_{\ell=0}^{j-1} \tau_\ell)$ and the average rate be denoted by $r = \frac{1}{\bar{h}}$. Note that both r and h are rational numbers. Examples of periodic sampling schedules with the same average rate are $(\sigma_0, \sigma_1, \sigma_2, \dots,) = (1,0,0,1,0,0,1,0,0,1,0,0,\dots)$ and $(\sigma_0, \sigma_1, \sigma_2, \dots,) = (1,0,1,0,0,0,1,0,1,0,0,0,\dots)$. The control input u_t can be updated at every time t as a function of the information set $\mathcal{J}_t = \{x_\kappa | \kappa \in \{0, \dots, t\}, \sigma_\kappa = 1\}$, that is, $u_t = \mu_{u,t}(\mathcal{J}_t)$ for some functions $\mu_{u,t}$. Performance is measured by either the h_2 norm or the h_∞ norm.

The h_2 norm is defined as follows. Assume that $\{w_t | t \in \mathbb{N} \cup \{0\}\}$ is a sequence of zero-mean independent and identically distributed random variables with $\mathbb{E}[w_t w_t^\intercal] = W \geq 0$. Then the h_2 norm is defined as the average cost

$$J_2 = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}[\sum_{t=0}^{T-1} \|z_t\|^2] \tag{4}$$

We use the notation $J_{2,h}$ to denote J_2 for evenly spaced sampling schedules (3).

To define the h_∞ norm let $w = (w_0, w_1, w_2, \dots)$, $z = (z_0, z_1, z_2, \dots)$, define the inner product $\langle w, z \rangle = \sum_{t=0}^{\infty} w_t^\top z_t$ and norm $\|w\|^2 := \sqrt{\langle w, w \rangle}$, and let ℓ_2 be the Hilbert space of sequences with bounded norm. The system provides an attenuation bound γ from the input disturbances to the output of interest if

$$\|z\|^2 \leq \gamma^2 \|w\|^2, \quad \forall w \in \ell_2, \text{ for } x_0 = 0. \tag{5}$$

We are interested in ensuring that (5) holds for the smallest possible γ. The initial condition x_0 may be non-zero provided that we redefine (5) along the lines discussed, e.g., in [8]. The disturbances depend on the information set $\mathcal{I}_t = \{x_\kappa | \kappa \in \{0, \dots, t\}\}$ that is $w_t = \mu_{w,t}(\mathcal{I}_t)$, for some functions $\mu_{w,t}$. We define $\pi_u = (\mu_{u,0}, \mu_{u,1}, \dots)$ as the policy of the controller and $\pi_w = (\mu_{w,0}, \mu_{w,1}, \dots)$ as the policy of the disturbances. Then, for a given periodic sampling sequence characterized by $\mathcal{T}$, the h_∞ norm coincides with the smallest attenuation bound and is given by

$$\gamma_{\mathcal{T}} := \inf\{\gamma | \exists \pi_u \text{ such that (5) holds when the scheduler is } h\text{-periodic with sampling intervals } \mathcal{T}\}. \tag{6}$$

For evenly spaced sampling (3) we use the alternative notation

$$\gamma_h := \inf\{\gamma | \exists \pi_u \text{ such that (5) holds when the scheduler is given by (3)}\}. \tag{7}$$

Naturally if $\tau_i = h$ for every $i \in \{0, \dots, j-1\}$, we have $\gamma_h = \gamma_{\mathcal{T}}$.

A sampling schedule characterized by $\tau_0^1, \dots, \tau_{j_1-1}^1$ for some j_1 is said to be (strictly) superior to another sampling schedule $\tau_0^2, \dots, \tau_{j_2-1}^2$ in the h_2 (h_∞) sense if the corresponding optimal controller achieves a non-larger (strictly smaller) h_2 (h_∞) cost. It is said to be optimal if there is not a different strictly superior schedule.

We are interested in finding optimal sampling schedules with average sampling interval $\frac{h}{m}$ in the h_2 and h_∞ sense.

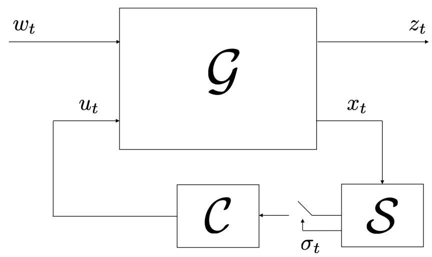

Fig. 1: A non-standard sampled feedback ystem consists of a controller $\mathcal{C}$ and a periodic scheduler $\mathcal{S}$, which sends measurement/state data to the controller; $\mathcal{G}$ represents the plant.

III. Main results for h_2 control

The following standard result shows that the optimal h_2 norm, associated with the optimal controller can be written in terms of a key function β. Let P be the unique positive definite (since Q is positive definite) solution to

$$P = A^\intercal PA + Q - A^\intercal PB(R + B^\intercal PB)^{-1}B^\intercal PA.$$

and let $Z = A^\intercal PB(B^\intercal PB + R)^{-1}B^\intercal PA$ and $K = -(B^\intercal PB + R)^{-1}B^\intercal PA$. Let $\mathrm{tr}(X)$ denote the trace of a matrix X.

Proposition 1: For given $\mathcal{T} := (\tau_0, \dots, \tau_{j-1})$, the cost (4) of an optimal control policy is given by

$$J_2 = \mathrm{tr}(PW) + \frac{1}{h}(\sum_{\ell=0}^{j-1}\beta(\tau_\ell)) \tag{8}$$

where $\beta(1) = 0$ and, for $p > 1$,

$$\beta(p) = \mathrm{tr}\big(Z(\sum_{s=1}^{p-1}Y(s))\big), \tag{9}$$

where $Y(s) = \sum_{r=0}^{s-1} A^r W A^{\intercal r}$, for $s \in \mathbb{N}$. Moreover, an optimal control policy that minimizes (4) is

$$u_t = K\hat{x}_{t|t}, \quad t \in \mathbb{N}_0, \tag{10}$$

where

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t} + Bu_t,$$

$$\hat{x}_{t|t} = \begin{cases} x_t, \text{ if } \sigma_t = 1, \\ \hat{x}_{t|t-1}, \text{ if } \sigma_t = 0. \end{cases}$$

□

The proof follows from standard optimal control arguments (see, e.g., [9]) and it is omitted for the sake of brevity.

A function with discrete domain $f(i)$, $i \in \mathbb{N}$, is said to be convex if

$$f(i) \leq \frac{f(i+1) + f(i-1))}{2}, \quad \forall i \in \mathbb{N} \setminus \{1\}. \tag{11}$$

A key result of the present paper is the following observation that β is convex.

Theorem 1: The function $\beta(i)$, $i \in \mathbb{N}$, defined in (9), is convex.

Proof: We have, for $i \in \mathbb{N}$, $\beta(i+1) - \beta(i) = \mathrm{tr}(ZY(i))$ and, for $i \in \mathbb{N} \setminus \{1\}$

$$(\beta(i+1) - \beta(i)) - (\beta(i) - \beta(i-1)) = \mathrm{tr}(A^i W A^{\intercal i}) \geq 0.$$

■

The convexity of β is used in the next theorem to improve upon a periodic schedule, by replacing two of its sampling intervals by two alternative sampling intervals that are closer to their means.

Theorem 2: Consider a given periodic schedule with period h and characterized by j intervals $\mathcal{T} = (\tau_0, \dots, \tau_{j-1})$. Given any two distinct $\tau_i > \tau_l$ construct a modified schedule $\bar{\mathcal{T}} = (\bar{\tau}_0, \dots, \bar{\tau}_{j-1})$ with

$$\bar{\tau}_\ell = \begin{cases} \tau_i - p, & \text{if } \ell = i \\ \tau_l + p, & \text{if } \ell = l \\ \tau_\ell, & \text{otherwise.} \end{cases}$$

for some $p \in \{0, \dots, \lfloor(\tau_i - \tau_l)/2\rfloor\}$. Then, $\bar{\mathcal{T}}$ is superior to $\mathcal{T}$ in the h_2 sense.

Proof: Let J_2 and $\bar{J}_2$ denote the optimal costs associated with the original and modified schedules, respectively. Note that both schedules have the same period denoted by h. Due to (8) is suffices to prove that

$$J_2 - \bar{J}_2 = \frac{1}{h}(\beta(\tau_i) + \beta(\tau_l) - (\beta(\tau_i - p) + \beta(\tau_l + p))) \geq 0. \tag{12}$$

Since $\beta(i)$ is convex we have, for any $p \in \mathbb{N}$,

$$\begin{aligned} \beta(i+p+1) - \beta(i+p) &\geq \beta(i+p) - \beta(i+p-1) \geq \dots \\ &\geq \beta(i+1) - \beta(i) \geq 0, \qquad \forall i \in \mathbb{N}. \end{aligned} \tag{13}$$

From this fact we conclude that

$$\begin{aligned} \beta(\tau_i) - \beta(\tau_i - p) &= \sum_{k=0}^{p-1}\beta(\tau_i - k) - \beta(\tau_i - 1 - k) \\ &\geq \sum_{k=0}^{p-1}\beta(\tau_l + p - k) - \beta(\tau_l + p - 1 - k) \\ &= \beta(\tau_l + p) - \beta(\tau_l) \end{aligned} \tag{14}$$

which implies (12), concluding the proof. ■

This results has several implications given next.

Corollary 1: An h-periodic schedule with m intervals between sampling is optimal if m_1 of these intervals equal $h_1 = \lfloor\frac{h}{m}\rfloor$ and m_2 equal $h_2 = \lceil\frac{h}{m}\rceil$ where $m_1 + m_2 = m$ $h_1 m_1 + h_2 m_2 = h$. Moreover, the corresponding h_2 norm is equal to

$$\mathrm{tr}(PW) + \frac{m_1}{h}\beta(h_1) + \frac{m_2}{h}\beta(h_2) \tag{15}$$

Furthermore, if $\beta(i+1) > \beta(i)$ for every $i \in \mathbb{N}$, then these optimal schedules are unique in the class of schedules with period h.

Proof: We can list all possible schedules with period h and m sampling intervals and compute the associated h_2 norm. Note that all the schedules that meet the form in the present corollary have the same h_2 norm due to (8). If such an h_2 norm is minimal we conclude the sufficiency part. In turn, if we would have a schedule with minimal h_2 norm that does not take the form stated in the present corollary, due to Theorem 2, we could modify it without increasing the cost so that it does meet the mentioned form, so that it is equally optimal. If $\beta(i+1) > \beta(i)$ for every $i \in \mathbb{N}$ this leads to a contradiction meaning that only the schedules of the form stated in the present corollary are optimal. ■

It is immediate from (9) that a sufficient condition for $\beta(i+1) > \beta(i)$ for every $i \in \mathbb{N}$ is $KW \neq 0$.

Corollary 1 implies the following:

i) In general there might be more than one optimal schedule, e.g., if $h = 10$, $m = 4$, 1010100100 are 1001010010 are both optimal.

ii) Given a desired rational average sampling time $\frac{h}{m}$, the optimal schedules are the ones that have m_1 intervals equal to $\lfloor \frac{h}{m} \rfloor$ and m_2 equal to $\lceil \frac{h}{m} \rceil$.

(iii) Note that we can write (15) as follows

$$\frac{\beta(h_1)(h_1+1-\zeta)+\beta(h_1+1)(\zeta-h_1)}{\zeta}\Big|_{\zeta=\frac{h}{m}}$$

since $h = m_1h_1 + m_2h_1 + m_2$ and

$$\frac{(h_1+1-h/m)}{h/m} = \frac{\overbrace{h_1m_1+h_1m_2+m_2-h}^{=0}+m_1}{h}$$

$$\frac{(h/m-h_1)}{h/m} = \frac{\overbrace{h-h_1m_1-h_2m_2}^{m_2}}{h}$$

Rewriting (15) in terms of the rate $r = 1/\xi$ we have

$$\beta(h_1)(r(h_1+1)-1)+\beta(h_1+1)(1-rh_1)|_{r=\frac{m}{h}}$$

Thus, the plot optimal achievable h_2 norm versus average rate is (a restriction to the rational numbers of) a continuous piecewise affine function connecting the pairs $(1/h, J_{2,h})$ where $J_{2,h}$ is the h_2 norm of periodic control with integer average sampling interval h (see Figure 3 below).

Corollary 1 also implies the following result.

Corollary 2: Evenly spaced sampling is optimal in the class of periodic schedulers with integer average sampling time $h \in \mathbb{N}$. This is the unique optimal schedule if $\beta(h-1) < \beta(h) < \beta(h+1)$ when $h > 1$ and if $\beta(1) < \beta(2)$ when $h = 1$. □

Remark 1: One simple method to arrive at an optimal sampling schedule as given in Corollary 1 from an arbitrary scheduled is to recursively reduce the largest interval by the same amount that the smallest is increased, and set this amount to the largest possible according to Theorem 2. The resulting sequence of sampling schedules monotonically improves the h_2 norm.

Remark 2: The paper [10] considers a continous-time version of sampled data periodic control with average intersampling time $h_c \in \mathbb{R}_{>0}$. This reference provides the following expression for the cost of periodic control with average sampling period h_c:

$$J_c(h_c) = \delta_c + \frac{1}{h_c}\beta_c(h_c),\ \beta_c(h_c) = \int_0^{h_c} \mathrm{tr}(Z_c \int_0^s V(r)dr)ds \tag{16}$$

where $V(s) = \int_0^s e^{A_c r} W_c e^{A_c^\intercal r}$, A_c is the system matrix, W_c is a positive semi-definite matrix proportional to the covariance of the stochastic disturbances and the expressions for δ_s and Z_c are omitted here. Note that analogously to the discrete-time case β_c is a convex function since $\beta_c''(h_c) = \mathrm{tr}(Z_c V(h_c)) \geq 0$. Due to this convexity property, many results of the previous section can be extended to the sampled-data case; however we do not pursue this here.

Remark 3: A different result, shown in [11], states that (16) as a function of the average rate $r = 1/h$, i.e., $J_c(1/r) = r\beta_c(\frac{1}{r})$ is convex, Note, however, that neither $J_c(h)$ nor $J_2(h)$ are in general convex functions of h.

IV. Main results for h_∞ control

We start by considering evenly spaced sampling (3) and by providing a method to compute γ_h, given by (7). Let us first define three matrix transformations:

$$F_a(P) := P + P(\gamma^2 I - P)^{-1}P$$
$$F_c(P) := A^\top PA + Q - A^\top PB(B^\top PB + R)^{-1}B^\top PA$$
$$F_o(P) := A^\top PA + Q.$$

for a given $\gamma \in \mathbb{R}_{>0}$. When $h = 1$ the following iteration $P_{t+1} = F_c(F_a(P_t))$ with $P_0 = 0$ is monotone in the sense that $P_{t+1} \geq P_t$ converges if $\gamma^2 I > P_t$ for every $t \in \mathbb{N}_0$ to the unique positive definite solution $\bar{P}_\gamma$ of the algebraic Riccati equation

$$\bar{P}_\gamma = F_c(F_a(\bar{P}_\gamma)), \tag{17}$$

see [8] (although the expressions in [8] appear in a different but equivalent form). Due to monotonicity, $\gamma^2 I > \bar{P}_\gamma$ implies that $\gamma^2 I > P_t$ for every $t \in \mathbb{N}_0$. Provided that this condition holds, (5) holds for a policy π_x specified by $u_t = K_\gamma x_t$, where

$$K_\gamma = -(R + B^\top F_a(\bar{P}_\gamma)B)^{-1}B^\top F_a(\bar{P}_\gamma)A. \tag{18}$$

If γ is such that $\gamma^2 I \geq P_t$ does not hold for some t, then (5) does not hold for any π_u.

However, if $h > 1$ the conditions on γ for the existence of such a control policy are stricter, i.e., γ needs to be larger [12]. They actually become stricter as h increases leading to non-decreasing sequence of γ_h, as stated in the next lemma.

Lemma 1: Suppose that $\gamma^2 I > \bar{P}_\gamma$, and consider the following iteration, with $M_1 = \bar{P}_\gamma$,

$$M_{k+1} = F_o(F_a(M_k)), \quad k \in \{1, 2, \dots, h\}, \tag{19}$$

which can be ran as long as $\gamma^2 I - M_k$ is not singular. Then:

(i) if $\gamma^2 I - M_k > 0$, for all $k \in \{1, \dots, h\}$, then, for all $k \in \{1, \dots, h-1\}$, $M_{k+1} \geq M_k$.

(ii) for every $h \in \mathbb{N}$, $\inf\{\gamma | \gamma^2 I - M_h > 0\} = \gamma_h$, where γ_h is given by (7).

(iii) for every $h \in \mathbb{N}$, $\gamma_{h+1} \geq \gamma_h$. □

The proof is given in the appendix.

We turn now to general schedules. The following result provides a simple way of computing the h_∞ norm associated with a general schedule from the h_∞ norm associated with an evenly spaced schedule.

Theorem 3: Consider a periodic schedule characterized by the intervals $\mathcal{T} := (\tau_0, \dots, \tau_{j-1})$. Let $\bar{\tau} = \max\{\tau_i | i \in \{0, 1, \dots, j-1\}\}$. Then, $\gamma_{\mathcal{T}} = \gamma_{\bar{\tau}}$.

□

Proof: Suppose that $\gamma_{\mathcal{T}} > \gamma_{\bar{\tau}}$. Pick a γ such that $\gamma_{\mathcal{T}} > \gamma > \gamma_{\bar{\tau}}$. Then for any given arbitrary control policy and for schedules $\mathcal{T}$ there exists a $w \in \ell_2$ such that $\|z\|^2 - \gamma^2\|w\|^2 > 0$ i.e., $\lim_{T\to\infty} \sum_{t=0}^{T-1} z_t^\intercal z_t - \gamma^2 w_t^\intercal w_t > 0$. This implies that there exists a $\underline{t} \in \mathbb{N}$ and a $w \in \ell_2$ such that, for $\bar{t} \geq \underline{t}$,

$$\sum_{t=0}^{\bar{t}-1} z_t^\intercal z_t - \gamma^2 w_t^\intercal w_t > 0 \tag{20}$$

Suppose that we pick $\bar{t}$ to be a multiple of h. Using Lemma 2 in the appendix, we conclude that if we pick the control policy $u_t = K_t x_t$, $t \in \{0, 1, \dots, \bar{\tau}\}$ where the K_t are obtained from the iteration (25) initialized with $P_\tau = Y_0 = 0$ and $\tau = \bar{t}$, we get

$$\sum_{k=0}^{\bar{t}-1} z_k^\top z_k - \gamma^2 w_k^\top w_k = \underbrace{x_0^\top}_{=0} P_0 x_0 - \sum_{k=0}^{\bar{t}-1} (w_k - L_k(Ax_k + Bu_k))^\top (\gamma^2 I - P_{k+1})(w_k - L_k(Ax_k + Bu_k)) \tag{21}$$

provided that the $\gamma^2 I - P_t$ are invertible. This is indeed the case since as we now argue $\gamma^2 I - P_t > 0$ for every $t \in \{0, 1, \dots, \bar{t}-1\}$. To see this it suffices to establish that $P_t \leq \bar{P}_\gamma$ for every $t \in \{0, 1, \dots, \bar{t}-1\}$ since $\gamma^2 I - \bar{P}_\gamma > 0$ by hypothesis. If we run the iteration (25) for $\tau = \bar{t}$ with $Y_0 = \bar{P}_\gamma$ we obtain $P_t = P_\gamma$ for every $t \in \{0, 1, \dots, \bar{t}\}$. From the monotonicity property of Lemma 1 we conclude that the P_t obtained when $Y_0 = 0$ satisfy $P_t \leq \bar{P}_\gamma$ for every $t \in \{0, 1, \dots, \bar{t}-1\}$ as desired. Note that (21) implies $\sum_{t=0}^{\bar{t}-1} z_t^\intercal z_t - \gamma^2 w_t^\intercal w_t \leq 0$ for every disturbance sequence which contradicts (20).

Suppose now that $\gamma_{\mathcal{T}} < \gamma_{\bar{\tau}}$ so that for the schedule $\mathcal{T}$ we can guarantee that

$$\|z\|^2 - \gamma^2\|w\|^2 < 0 \tag{22}$$

for some γ, $\gamma_{\mathcal{T}} < \gamma < \gamma_{\bar{\tau}}$ and for every $w \in \ell^2$. Note that in this case $\gamma^2 I - M_{\bar{\tau}}$ has an eigenvalue which is negative. Let $\ell_m \in \arg\max\{\tau_\ell | \ell \in \{0, \dots, j-1\}\}$ so that $\tau_{\ell_m} = \bar{\tau}$ and consider the following disturbance policy

$$w_t = \begin{cases} \xi, \quad t \in \{0, \dots, s_{\ell_m} - 1\} \\ \tilde{L}_{\bar{\tau}}(Ax_t + Bu_t) + \eta \text{ if } t = s_{\ell_m} \\ \tilde{L}_{\bar{\tau}-t-s_{\ell_m}}(Ax_t + Bu_t), \text{if } t \in \{s_{\ell_m}+1, \dots, s_{\ell_m}+\bar{\tau}-1\} \\ \bar{L}_t(Ax_t + Bu_t) \quad t \in \{s_{\ell_m} + \bar{\tau}, \dots, s_{\ell_m} + \bar{\tau} + q\} \\ 0, \text{ if } t \geq s_{\ell_m+1} + q + 1 \end{cases}$$

where:

- $\xi \in \mathbb{R}^{n_x}$ is an arbitrary constant;
- $\eta \in \mathbb{R}^{n_x}$ will be chosen in the sequel;
- $\tilde{L}_k = (\gamma^2 I - M_k)^{-1} M_k$, $k \in \{1, \dots, \bar{\tau} - 1\}$;
- $\bar{L}_t$, $t \in \{s_{\ell_m} + \bar{\tau}, \dots, s_{\ell_m} + \bar{\tau} + q\}$ are given by (29) in Lemma 4 below (with $k = s_{\ell_m} + \bar{\tau}$) and q is such that $\|x_{s_{\ell_m}+\bar{\tau}}^\intercal G_q x_{s_{\ell_m}+\bar{\tau}} - x_{s_{\ell_m}+\bar{\tau}}^\intercal \bar{P}_\gamma x_{s_{\ell_m}+\bar{\tau}}| < \alpha$ for a given and arbitrary $\alpha > 0$. Such a q depends on $x_{s_{\ell_m}+\bar{\tau}}$ and exists due to Lemma 4.

Then

$$\begin{aligned} &\|z\|^2 - \gamma^2\|w\|^2 \\ &= \underbrace{\sum_{t=0}^{s_{\ell_m}-1} z_t^\intercal z_t - \gamma^2 w_t^\intercal w_t}_{:=c} + \sum_{t=s_{\ell_m}}^{s_{\ell_m}+\bar{\tau}-1} z_t^\intercal z_t - \gamma^2 w_t^\intercal w_t \\ &\qquad + \underbrace{\sum_{t=\bar{s}_{\ell_m}+\bar{\tau}}^{\infty} z_t^\intercal z_t - \gamma^2 w_t^\intercal w_t}_{\geq x_{s_{\ell_m}+\bar{\tau}}^\intercal G_q x_{s_{\ell_m}} \text{ due to (27) below}} \\ &\geq c + \sum_{t=s_{\ell_m}}^{s_{\ell_m}+\bar{\tau}-1} z_t^\intercal z_t - \gamma^2 w_t^\intercal w_t + x_{s_{\ell_m}+\bar{\tau}}^\intercal \bar{P}_\tau x_{s_{\ell_m}+\bar{\tau}} \\ &\qquad \underbrace{-x_{s_{\ell_m}+\bar{\tau}}^\intercal \bar{P}_\tau x_{s_{\ell_m}+\bar{\tau}} + x_{s_{\ell_m}+\bar{\tau}}^\intercal G_q x_{s_{\ell_m}+\bar{\tau}}}_{\geq\alpha} \\ &\geq c + \alpha + x_{s_{\ell_m}}^\intercal \bar{P}_\tau x_{s_{\ell_m}} - \\ &\quad \sum_{t=s_{\ell_m}}^{s_{\ell_m}+\bar{\tau}-1} (w_t - \tilde{w}_t)^\top (\gamma^2 I - M_{s_{\ell_m}+\bar{\tau}-t})(w_t - \tilde{w}_t)) \qquad (23) \\ &= c + \alpha + x_{s_{\ell_m}}^\intercal \bar{P}_\tau x_{s_{\ell_m}} - \eta^\intercal (\gamma^2 I - M_{\bar{\tau}})\eta > 0 \end{aligned}$$

where $\tilde{w}_t = \tilde{L}_{\bar{\tau}-t-s_{\ell_m}}(Ax_t + Bu_t)$, in (23) we have used Lemma 3 below with $Y_0 = P_\tau$ and $\tau = \bar{\tau}$, and in the last inequality we have picked η to be aligned with an eigenvector associated with an eigenvalue of $(\gamma^2 I - M_{\bar{\tau}})$ that is negative and multiplied by a constant high enough to make the expression positive. This contradicts (22). ■

Theorem 3 has several consequences discussed next.

Corollary 3: A sampling schedule guarantees the smallest attenuation bound (h_∞ norm) achievable for a given rational average sampling time $\frac{h}{m}$ if its largest interval does not exceed $\tilde{h} = \lceil \frac{h}{m} \rceil$. Moreover, if

$$\gamma_{\tilde{h}-1} > \gamma_{\tilde{h}} > \gamma_{\tilde{h}+1} \tag{24}$$

the schedules satisfying this property are the unique schedules that guarantee the smallest attenuation bound. Furthermore, under (24), the schedule with smallest average rate that guarantees this attenuation bound corresponds to evenly space sampling.

□

Note that also here there can be multiple optimal schedules (in the sense of this corollary), but are in general different from the ones for the h_2 case.

Due to this corollary, the plot optimal achievable h_∞ norm versus average sampling time (or average rate) is

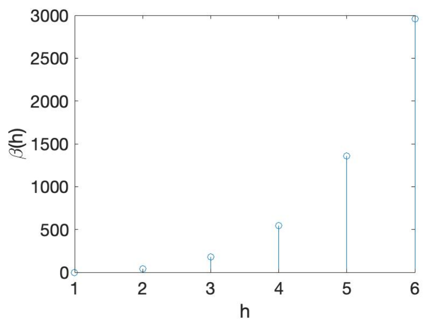

Fig. 2: $\beta(h)$ versus h

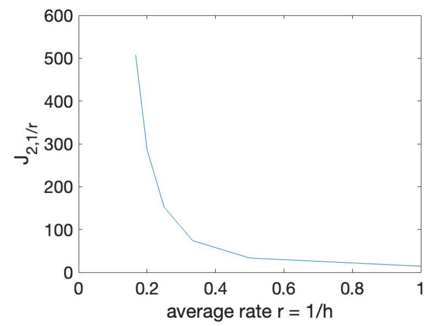

Fig. 3: $J_{2,h}$ versus average rate

(a restriction to the rational numbers of) a discontinuous piecewise constant function, where these constants are equal to the h_∞ norms of evenly spaced sampling with integer average sampling interval h (see Figure 4 below).

Note that we can have schedules corresponding to arbitrarily poor h_∞ norm and maximum sampling rate. In fact, the schedule characterized by $h = \omega b + b$

$$\tau_i = \begin{cases} 1, \text{ if } i \in \{0, \dots, \omega b\} \\ b, \text{ if } i = \omega b + 1 \end{cases}$$

leads to an h_∞ norm γ_b and average rate $\frac{\omega b}{(\omega+1)b}$. For systems for which $\gamma_b \to \infty$ as $b \to \infty$, the h_∞ norm becomes arbitrary poor while the average rate converges to 1 as $b \to \infty$ and $\omega \to \infty$.

V. Numerical example

Suppose that

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad Q = I_3, \quad R = 1,$$

where I_3 is the identity matrix. The values of $\beta(h)$, $J_{2,h}$, and γ_h are given in Table I and plotted in Figures 2, 3, 4, respectively.

In Figures 3 the optimal h_2 value achievable with the corresponding rational average sampling time is plotted and in Figure 4 plotted and the optimal h_∞ value achievable with the corresponding rational average rate is plotted.

VI. Conclusions

In this paper we have characterized sampling schedules that are optimal for h_2 and h_∞ feedback control. We have shown that if the desired average intersampling time is h then evenly spaced sampling is optimal in both senses. However, when the desired average intersampling time is not an integer, then the class of optimal schedules is different in the h_2 and h_∞ senses. While for h_2 schedules close to evenly spaced sampling are still optimal, in the h_∞ framework the h_∞ norm is only dictated by the largest sampling interval.

h	1	2	3	4	5	6
$\beta(h)$	0	38.1	179.1	548.2	1361.3	2960.2
$J_{2,h}$	14.3	33.4	73.9	151.9	286.5	507.6
γ_h	[3.805	7.97	14.55	24.18	37.45	54.58

TABLE I: Values of $\beta(h)$, $J_{2,h}$, and γ_h for $h \in \{1, 2, \dots, 6\}$ for the numerical example.

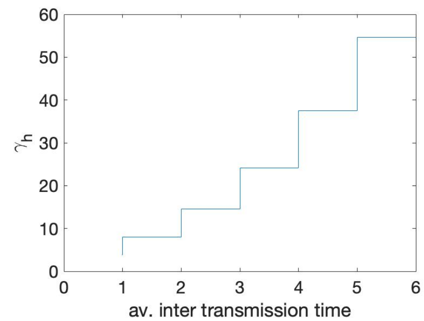

Fig. 4: γ_h versus average sampling interval

Appendix

Proof of Lemma 1

(i) When $k = 1$, $M_1 = \bar{P}_\gamma = F_c(F_a(\bar{P}_\gamma)) = M_2 - A^\top F_a(\bar{P}_\gamma)B(B^\top F_a(\bar{P}_\gamma)B + R)^{-1}B^\top F_a(\bar{P}_\gamma)A \leq M_2$. We now prove that, $F_o(F_a(P))$ is a monotone map in the sense that $F_o(F_a(P_1)) \leq F_o(F_a(P_2))$when $P_1 \leq P_2$. This follows from the fact that, for $P < \gamma^2 I$ and that for an arbitrary x

$$x^\intercal F_o(F_a(P))x = x^\intercal Qx + \max_w (Ax + w)^\intercal P(Ax + w)$$

Using induction, suppose that $M_{k-1} \leq M_k$ for some $k \in \{2, \dots, h-1\}$. Then $F_o(F_a(M_{k-1})) \leq F_o(F_a(M_k))$ and, concluding the proof.

(ii) In [12] it is shown that $\gamma_h = \inf\{\gamma | \gamma^2 I - \bar{D}_h^\top \text{diag}(I \otimes Q, \bar{P}_\gamma)\bar{D}_h > 0\}$, with

$$\bar{D}_h = \begin{bmatrix} I & 0 & \dots & 0 \\ A & I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A^{\tau-1} & A^{\tau-2} & \dots & I \end{bmatrix}$$

and the condition $\gamma^2 I - \bar{D}_h^\top \text{diag}(I \otimes Q, \bar{P}_\gamma)\bar{D}_h > 0$ is equivalent to the following function being concave in $w_0, \dots, w_{h-1}$:

$$\sum_{t=0}^{h-1} z_t^\top z_t - \gamma^2 w_t^\top w_t + x_h^\top \bar{P}_\gamma x_h,$$

where $x_{t+1} = Ax_t + w_t$ for $t \in \{0, 1, \ldots, h-1\}$. Applying dynamic programming to maximize this function with respect to $w_{\ell+1}, \ldots, w_{h-1}$, for $\ell \in \{0, \ldots, h-1\}$, we obtain that this cost is equal to

$$\sum_{t=0}^{\ell} z_t^\top z_t - \gamma^2 w_t^\top w_t + x_{\ell+1}^\top M_\ell x_{\ell+1}.$$

We can rewrite this expression as

$$\sum_{t=0}^{\ell-1} z_t^\top z_t - \gamma^2 w_t^\top w_t + x_\ell^\top (Q + A^\top M_\ell A) x_\ell + w_\ell^\top (M_\ell - \gamma^2 I) w_\ell + 2 w_\ell^\top M_\ell A x_\ell$$

from which clear that this function is concave in w_ℓ if and only if $\gamma^2 I - M_\ell > 0$ where $\ell \in \{1, \ldots, h\}$ is arbitrary.

(iii) Follows from (i) and (ii).

Auxiliary Lemmas

Lemma 2: If $0 \leq P_1 \leq P_2$ then

$$F_c(F_a(P_1)) \leq F_c(F_a(P_2)).$$

□

Proof: The proof follows by noticing that, for every $x \in \mathbb{R}^{n_x}$,

$$x^\intercal F_c(F_a(P_1))x = \min_u \max_w x^\intercal Q x + u^\intercal R u - \gamma^2 w^\intercal w + (Ax + Bu + w)^\intercal P (Ax + Bu + w)^\intercal.$$

■

Lemma 3: Consider (1), (2), and the following iteration

$$P_{k-1} = F_c(F_a(P_k)), k \in \{\tau, \tau-1, \ldots, 0\} \tag{25}$$

with $P_\tau = Y_0$ for some $Y_0 \geq 0$ and where γ is such that $\gamma^2 I - P_k$ is invertible for every $k \in \{\tau, \tau-1, \ldots, 0\}$. Then

$$\begin{aligned}
&\sum_{k=0}^{\tau-1} z_k^\top z_k - \gamma^2 w_k^\top w_k + x_\tau^\top P_\tau x_\tau = x_0^\top P_0 x_0 + \\
&\sum_{k=0}^{\tau-1} (u_k - K_k x_k)^\top (R + B^\top F_a(P_{k+1}) B)(u_k - K_k x_k) - \\
&\sum_{k=0}^{\tau-1} (w_k - L_k(Ax_k + Bu_k))^\top (\gamma^2 I - P_{k+1})(w_k - L_k(Ax_k + Bu_k))
\end{aligned} \tag{26}$$

where K_k and L_k are given by

$$\begin{aligned}
K_k &= -(R + B^\top F_a(P_{k+1}) B)^{-1} B^\top F_a(P_{k+1}) A, \\
L_k &= (\gamma^2 I - P_{k+1})^{-1} P_{k+1} A
\end{aligned}$$

for $k \in \{0, 1, \ldots, \tau-1\}$ □

The proof is similar to the one in [7, Lemma] and it is omitted here for the sake of brevity.

Lemma 4: Consider (1) and γ such that $\gamma^2 I - \bar{P}_\gamma > 0$. Then

$$\sum_{t=k}^{\infty} z_t^\top z_t - \gamma^2 w_t^\top w_t \geq x_k^\top G_q x_k, \tag{27}$$

when

$$w_t = \begin{cases} \bar{L}_{q-(t-k)}(Ax_t + Bu_t), & \text{if } k \leq t < k+q \\ 0, & \text{if } t \geq k+q \end{cases} \tag{28}$$

where, for $k \in \{1, \ldots, q\}$,

$$\bar{L}_k = (\gamma^2 I - G_{k-1})^{-1} G_{k-1} \tag{29}$$

and, for $k \in \{0, \ldots, q-1\}$,

$$G_{k+1} = F_c(F_a(G_k)) \tag{30}$$

with $G_0 = P_{\mathrm{LQ}}$ where P_{LQ} is the unique positive definite solution to the algebraic Riccati equation

$$P_{\mathrm{LQ}} = A^\top P_{\mathrm{LQ}} A + P_{\mathrm{LQ}} - A^\top P_{\mathrm{LQ}} B (R + B^\top P_{\mathrm{LQ}} B)^{-1} B^\top P_{\mathrm{LQ}} A.$$

Moreover, for any $x \in \mathbb{R}^{n_x}$ and $\alpha \in \mathbb{R}_{>0}$, there exists $q \in \mathbb{N}$, denoted by $q = \zeta(x, \alpha)$, such that

$$\|x^\intercal G_q x - x^\intercal \bar{P}_\gamma x\| < \alpha. \tag{31}$$

Such q can be found by running (30) until (31) is met. □

This result is simply restates [7, Lemma 3], where the proof can be found.

References

[1] A. Cuenca, J. Alcaina, J. Salt, V. Casanova, and R. Pizá, "A packet-based dual-rate pid control strategy for a slow-rate sensing networked control system," *ISA Transactions*, vol. 76, pp. 155–166, 2018.

[2] L. Sha, T. Abdelzaher, K.-E. årzén, A. Cervin, T. Baker, A. Burns, G. Buttazzo, M. Caccamo, J. Lehoczky, and A. K. Mok, "Real time scheduling theory: A historical perspective," *Real-Time Systems*, vol. 28, no. 2, pp. 101–155, 2004.

[3] A. Cervin and J. Eker, "Control-scheduling codesign of real-time systems: The control server approach," vol. 1, pp. 209–224, 2005.

[4] L. Mirkin, "Intermittent redesign of analog controllers via the youla parameter," *IEEE Transactions on Automatic Control*, vol. 62, no. 4, pp. 1838–1851, 2017.

[5] M. Braksmayer and L. Mirkin, "On discrete-time h_∞ optimization under intermittent communications," in *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2019, pp. 6418–6423.

[6] ——, "On the dependence of the optimal h2 performance on sampling rate variability," *IEEE Control Systems Letters*, vol. 6, pp. 265–270, 2022.

[7] D. J. Antunes and J. P. Hespanha, "Event-triggered control cannot improve the ℓ_2 gain of h_∞ optimal periodic control and transmit at a smaller average rate," 2023, arXiv:2310.17033. [Online]. Available: https://doi.org/10.48550/arXiv.2310.17033

[8] T. Basar and P. Bernhard, *H-infinity optimal control and related minimax design problems: a dynamic game approach*. Springer Science Business Media, 2008.

[9] D. P. Bertsekas, *Dynamic Programming and Optimal Control*. Athena Scientific, 2005, third Edition, Volume I.

[10] D. J. Antunes and B. A. Khashooei, "Consistent event-triggered methods for linear quadratic control," in *55th IEEE Conference on Decision and Control (CDC)*, Dec 2016, pp. 1358–1363.

[11] B. A. Khashooei, D. J. Antunes, and W. P. M. H. Heemels, "Output-based event-triggered control with performance guarantees," *IEEE Transactions on Automatic Control*, vol. 62, no. 7, pp. 3646–3652, July 2017.

[12] M. Balaghiinaloo, D. J. Antunes, and W. Heemels, "An ℓ_2-consistent event-triggered control policy for linear systems," *Automatica*, vol. 125, p. 109412, 2021.