Edge-based funnel control for multi-agent systems using relative position measurements

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Abstract—In this work we consider the problem of control under Signal Temporal Logic specifications (STL) that depend on relative position information among neighboring agents. In particular, we consider STL tasks for given pairs of agents whose satisfaction is translated into a set of setpoint output tracking problems with transient and steady-state constraints. Contrary to existing work the proposed framework does not require initial satisfaction of the funnel constraints but can ensure their satisfaction within a pre-specified finite time. Given a tree topology in which agents sharing a STL task form an edge, we show that the resulting control laws ensure the satisfaction of the STL task as well as boundedness of all closed loop signals using only local information.

I. INTRODUCTION

Multi-agent systems have been deployed in a plethora of highly complex environments such as underwater or underground environments, the space or in industrial settings. In such environments communication with central entities responsible for control and planning is often hard to establish or costly. To that end, great emphasis is given on decision making strategies that are based on local information obtained by onboard sensors such as range sensors and/or cameras.

Motivated by such applications, in this work we focus on the design of control strategies under complex, time-constrained tasks that depend on relative-position information among agents. These tasks are expressed in Signal Temporal Logic (STL) [1], a formal language that allow us to code complex spatial tasks that need to performed within given time intervals. Contrary to other logics, STL is evaluated over continuous-time signals and is equipped with a robustness metric [2], [3] that allow us to quantify how well the STL task is satisfied. In the context of multi-agent control, distributed control strategies have been discussed in [4]–[7]. In [4] a hierarchical approach is proposed for control of local motion and safety tasks as well as global communication constraints. In [5] decentralized control laws are designed based on assume-guarantee contracts designed in a centralized manner. In [6] a distributed MPC scheme is proposed for single integrator systems under reach-avoid specifications with recursive feasibility guarantees while [7] proposes an iterative algorithm for control under coupled reach-avoid specifications using MILP programming. Closely to our approach is the work proposed in [8], [9], where prescribed performance control strategies have been applied to nonlinear [8] or interconnected systems [9]. In [8] a funnel-based switching control law is proposed for single-agent relative degree one systems while [9] employs a contract-based funnel control strategy to ensure decentralized control of interconnected systems subject to local STL tasks.

In all the aforementioned works the STL tasks are local and expressed in terms of the absolute position of the agents. Contrary to existing work, in this paper we consider STL tasks that depend on relative position information among neighboring agents. STL satisfaction is enforced by means of a set of output tracking objectives that need to be achieved with a prescribed transient and steady state behavior. Assuming a tree sensing topology, we design a switching control law that ensures the satisfaction of the STL task with a desired robustness as well as boundedness of all closed loop signals based on local information. Contrary to the majority of works in prescribed performance control literature, here agents may initially violate the funnel constraints which are guaranteed to be satisfied after a pre-specified finite time instant thanks to appropriately designed shifting functions.

II. NOTATION AND PRELIMINARIES

True and false are denoted by \( \top \), \( \bot \) respectively. Scalars and vectors are denoted by non-bold and bold letters respectively. \( A \otimes B \) denotes the Kronecker product of \( A, B \). The cardinality of a set \( V \) is denoted by \( |V| \) and the identity matrix of dimension \( n \) by \( I_n \). The block diagonal matrix of \( A_1, \ldots, A_p \) is denoted by \( \text{diag}(A_1, \ldots, A_p) \). The minimum eigenvalue of a matrix \( \lambda_{\min}(A) \) and \( A \succ 0 \) denotes that \( A \) is positive definite. The weighted Euclidean norm of a vector \( x \in \mathbb{R}^n \) is given by \( \|x\| := \sqrt{x^T Q x} \), where \( Q > 0 \). Given subsets \( I_k \subset \mathbb{N} \) with \( k \in M \subset \mathbb{N} \) we define the minimum and maximum of \( i_k \) by \( i_{k,\min} := \min I_k \) and \( i_{k,\max} := \max I_k \), respectively. In addition, for \( i \in I_k \setminus \{i_{k,\min}\} \), we define \( [i]_k := \max \{i' \in I_k : i' < i\} \) and for \( i \in I_k \setminus \{i_{k,\max}\} \), \( [i]_k := \min \{i' \in I_k : i < i'\} \).

A. Signal Temporal Logic (STL)

Signal Temporal Logic (STL) determines whether a predicate \( \mu \) is true or false. The validity of each predicate \( \mu \) is evaluated based on a continuously differentiable function \( h : \mathbb{R}^n \to \mathbb{R} \) as follows: \( \mu = \top \), if \( h(x) \geq 0 \), or \( \mu = \bot \) otherwise. The basic STL formulas are given by the grammar:

\[
\phi := \top \mid \phi \land \varphi \mid \phi \land \varphi \mid G_{[a,b]} \phi \mid F_{[a,b]} \phi \mid P_{[a,b]} \phi \mid U_{[a,b]} \phi,
\]

where \( \phi, \varphi \) are STL formulas and \( G_{[a,b]} \), \( F_{[a,b]} \), \( P_{[a,b]} \) is
the always, eventually and until operator defined over the interval $[a, b]$ with $0 \leq a < b < \infty$. Let $x \models \phi$ denote the satisfaction of the formula $\phi$ by a signal $x : \mathbb{R}_0^+ \to \mathbb{R}^n$. The formula $\phi$ is satisfiable if $\exists x : \mathbb{R}_0^+ \to \mathbb{R}^n$ such that $x \models \phi$. STL is equipped with robustness metrics determining how robustly an STL formula $\phi$ is satisfied at time $t$ by a signal $x$. The STL semantics and robust metrics are defined in [1] and [10], respectively. Note that $x \models \phi$ if $\rho(\phi, x) > 0$.

**B. Graph Theory**

An undirected graph $G$ is defined as a pair $G = (V, E)$, where $V = \{1, \ldots, R\} \subset \mathbb{N}$ is a finite set of nodes and $E \subseteq \{(r, r') \in V \times V : r \neq r'\}$. A path is a sequence of edges connecting two distinct vertices. A graph is connected, if there exists a path between any pair of vertices. Given a numbering $k \in \mathcal{M} = \{1, \ldots, M\}$ of the edges $e_k \in E$, after assigning an orientation to each edge in $G$ we may define the incidence matrix $D = [d_{ij}]$ of $G$ as follows: $d_{ij} = 1$, if the node $i$ is the head of edge $j$, $d_{ij} = -1$, if $i$ is the tail of edge $j$ and $d_{ij} = 0$, otherwise. The edge Laplacian matrix of $G$ is given by $L_e = D^T D$.

**C. Prescribed Performance Control**

Prescribed Performance Control (PPC) is a control method initially proposed in [11] ensuring that the tracking error $e : \mathbb{R}_0^+ \to \mathbb{R}$ remains at all times within a bounded region determined by a-priori known time-varying factors that impose a prescribed transient and steady state performance. Specifically, $-\gamma(t) < e(t) < \gamma(t)$ should hold for every $t \geq t_0$ which is a smooth, bounded, and monotonically decreasing function satisfying $\lim_{t \to +\infty} \gamma(t) = \gamma_\infty > 0$. An example of such function is $\gamma(t) = (\gamma_0 - \gamma_\infty) \exp(-lt) + \gamma_\infty$, where $\gamma_0, \gamma_\infty, l$ are positive parameters chosen such that $|e(t_0)| < \gamma(t_0)$ and $\gamma_\infty < \gamma_0$. The value of $\gamma_\infty$ determines the maximum allowable size of the tracking error at steady state and can be chosen arbitrarily small while the parameter $l$ determines a lower bound on the speed of convergence of the tracking error.

**D. Problem Formulation**

In this work we consider a multi-agent team of $R$ agents whose dynamics are given by:

$$\dot{x}_r = f_r(x_r) + g_r(x_r)u_r + w_r, \quad (1)$$

where $f_r : \mathbb{R}^n \to \mathbb{R}^n, g_r : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are locally Lipschitz functions, $x_r \in \mathbb{R}^n, u_r \in \mathbb{R}^m$ is the state and input of the $r$-th agent, respectively, $w_r \in \mathbb{R}^m$ is a piecewise continuous and bounded disturbance acting on the $r$-th agent and $r \in V := \{1, \ldots, R\}$.

**Assumption 1.** The matrix $g_r : \mathbb{R}^n \to \mathbb{R}^{n \times m}, r \in V$ is full row-rank for every $x_r \in \mathbb{R}^n$.

A direct implication of Assumption 1 is that $n \leq m$, i.e., the number of inputs is at least as equal as the number of states. Examples of $g_r(\cdot)$ satisfying Assumption 1 are constant, full row-rank matrices or $n \times n$ and invertible matrices for every $x_r$. Let $x := [x^T_1 \ldots x^T_R]^T \in \mathbb{R}^{Rn}$, $u := [u^T_1 \ldots u^T_R]^T \in \mathbb{R}^{Rm}$, $w := [w^T_1 \ldots w^T_R]^T \in \mathbb{R}^{Rm}$, denote the stacked vector of the states, inputs and disturbances of the multi-agent system, respectively. Then, the dynamics of the multi-agent team are given as follows:

$$\dot{x} = f(x) + g(x)u + w. \quad (2)$$

where $f(x) := [f^T_1(x_1), \ldots, f^T_R(x_R)]^T$ and $g(x) := \text{diag}(g_1(x_1), \ldots, g_R(x_R))$. Each agent $r$ is assumed to have relative position information with respect to a limited number of its peers. Let $G = (V, E)$ be the sensing graph where $V$ is the set of agents and $E \subseteq V \times V$ is the edge set. Here, an edge $e_k = (r_k, r'_{k}) \in E$ exists iff $r_k, r'_{k}$ have access to the relative position vector $x_{r_k} - x_{r'_{k}}$. In the following we make the following assumption on the sensing graph $G$:

**Assumption 2.** The graph $G = (V, E)$ is a static, undirected tree.

Assumption 2 requires agents to form a tree sensing graph which ensures that the edge Laplacian $L_e$ is positive definite [12], a property which will be later used to ensure boundedness of the error signals.

Here, the multi-agent team is subject to a global STL task described by the following fragment:

$$\psi := T | \mu | \neg \mu | \psi_1 \land \psi_2, \quad (3a)$$

$$\varphi := G_{[a, b]} \psi \land F_{[a, b]} \psi' | \psi_1 U_{[a, b]} \psi_2, \quad (3b)$$

$$\phi := \psi_1 \land \cdots \land \psi_{q'}, \quad (3c)$$

where $\psi_1, \psi_2$ are STL formulas of the form (3a), $\psi_i, i = 1, \ldots, q'$ are STL formulas of the form (3b), $a \leq b < \infty$. In the following we will assume that $\phi$ is defined as a conjunction of always and eventually STL tasks of the form $G_{[a, b]}(h(x) \geq 0)$ and $F_{[a, b]}(h(x) \geq 0)$, respectively, as the satisfaction of more complex formulas of (3a)-(3c) is ensured by conjunctions of such tasks. Given the set $\mathcal{I} := \{1, \ldots, q\}$, where $q > 1$, in this work we consider the global STL task:

$$\phi := \bigwedge_{i \in \mathcal{I}} \psi_i. \quad (4)$$

Let $\beta_k(x) := x_{r_k} - x_{r'_{k}}$ be the relative position among agents $r_k, r'_{k}$ forming the $k$-th edge of $G$, where $k \in \mathcal{M} := \{1, \ldots, M\}, M = |E|$. An example of such task is $G_{[0,5]}(||x_1 - x_2|| \leq 1)$, which requires agents 1,2 to be no more than 1m apart for every $t \in [0, 5]$. Here, we focus on STL tasks $\varphi_i, i \in \mathcal{I}$ that depend on $\beta_k(x)$ for some $k \in \mathcal{M}$. More specifically, we consider STL tasks that satisfy the following assumption:

**Assumption 3.** The predicate functions $h_i : \mathbb{R}^n \to \mathbb{R}, i \in \mathcal{I}$ have the following properties: (i) $h_i(\cdot)$ is a function of the relative position among agents $r_k, r'_{k}$ for some $k \in \mathcal{M}$, i.e., $h_i = h_i(\beta_k)$, (ii) $h_i(\beta_k)$ is continuously differentiable and $h_i(\beta_k) \to +\infty$ as $||\beta_k|| \to +\infty$, and (iii) $\frac{\partial h_i}{\partial \beta_k}$ is bounded with $\frac{\partial h_i}{\partial \beta_k} \neq 0$ in some known set $B_i \subseteq \mathbb{R}^n$ with $B_i := \sup_{\beta_k \in B_i} h_i(\beta_k) > 0$. In addition, given the time intervals $[a_i, b_i]$ corresponding to the formulas $\varphi_i, i \in \mathcal{I}$ the following hold: (i) $a_i - a_{i\min} > 0$, for every $k \in \mathcal{M}$, (ii) $b_i < a_i - \epsilon$, for
every \( i \in \mathcal{I}_k \setminus \{i_k\}, k \in \mathcal{M} \) where \( \mathcal{I}_k := \{ \hat{i}' \in \mathcal{I} : h_i = h_i(\beta_k) \} \), is the index set of the STL formulas involving the agents forming the \( k \)-th edge of \( G \), \( \hat{i}' = [i]_k \) and \( \epsilon > 0 \) is a positive parameter.

Assumption 3 ensures that the STL tasks \( \varphi_i \) involving the agents forming the same edge in \( G \) are successive tasks, i.e., the time intervals within which they need to be satisfied are not overlapping. In this work each predicate function \( h_i(\cdot) \), \( i \in \mathcal{I} \) is subject to prescribed performance constraints given as follows:

\[
-\eta_i \gamma_i(t) < h_i(\beta_k(t)) - \bar{p}_i < \rho_i \gamma_i(t), \quad t \in [c_i, b_i + \epsilon), \quad (5)
\]

where \( \rho_i, \eta_i, \bar{p}_i > 0 \) are positive tuning parameters, \( \bar{p} \in (0, \min_{i \in \mathcal{I}} \rho_i) \) is fixed and given, \( \epsilon > 0 \) is the same as in Assumption 3, and \( c_i \) is a time instant to be chosen such that for \( i \in \mathcal{I}_k \setminus \{i_k\}, k \in \mathcal{M}, \) \( c_i \in (b_i + \epsilon, a_i) \), if \( \varphi_i = \varphi'_{i[a_i, b_i]} \psi_i \) or \( c_i \in (b_i + \epsilon, b_i) \), otherwise. \( \varphi_i = \varphi'_{i[a_i, b_i]} \psi_i \) or \( c_i \in (0, b_i) \), otherwise. In addition, \( \gamma_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) are smooth, positive and strictly decreasing functions defined as:

\[
\gamma_i(t) := (1 - \gamma_i, \infty) \exp(-l_i t) + \gamma_i, \quad (6)
\]

where \( \gamma_i, \infty := \frac{\rho_i}{\max_{\bar{p}_i, \eta_i}} > 0 \) and \( \gamma_i, \infty, l_i \in \mathbb{R}_{\geq 0} \) are tuning parameters determining the desired transient and steady state behavior of \( h_i(\beta_k) - \bar{p}_i \). Intuitively, (5) enforces a desired behavior for the agents forming the \( k \)-th edge towards satisfying \( \varphi_i \) with a desired robustness \( \bar{p}_i \). In particular, choosing \( c_i > 0 \) as mentioned above and after appropriately tuning the parameters of \( \gamma_i(t) \) we can ensure that the predicate function \( h_i(\beta_k) \) can reach close to \( \bar{p}_i \) within \([c_i, b_i + \epsilon)\). Based on the above, the problem considered in this work is expressed as follows:

**Problem 1.** Consider a team of \( R \) agents that is subject to a global STL task defined by (4). The states of each agent evolve over time according to (1). Let Assumptions 1-3 hold. Then, design \( u_i, r \in \mathbb{V} \) (if possible) using only local information such that the satisfaction of (5) for each \( i \in \mathcal{I} \) ensures \( \rho^\phi(x, 0) \geq \bar{p} \), where \( \bar{p} > 0 \) is a designer’s choice.

### III. MAIN RESULTS

In this section we will design a switching control law that ensures \( \rho^\phi(x, 0) \geq \bar{p} \). In Section III-A, we will design control laws that ensure the satisfaction of (5) using only local information and then in Section III-B we propose how to choose the design parameters in order to ensure the desired robustness of satisfaction of \( \phi \).

**A. Control Design**

We begin with the control design assuming that the parameters determining the funnel constraints as well as \( \bar{p}_i, i \in \mathcal{I} \) are given. By Assumption 3 the tasks involving the agents of the \( k \)-th edge of \( G \) are sequential. Since \( a_i \leq b_i \) for every \( i \in \mathcal{I} \), then if \( i \in \mathcal{I}_k \setminus \{i_k\} \), for some \( k \in \mathcal{M}, \) it follows that \( b_i + \epsilon < b_i' \), where \( i' = [i]_k \). Therefore, for every \( k \in \mathcal{M} \) we can define the ordered set \( \Sigma_k := \{b_i + \epsilon: i \in \mathcal{I}_k \setminus \{i_k\}\} \cup \{0, +\infty\} \). Based on \( \Sigma_k \) we can assign to each \( i \in \mathcal{I}_k, k \in \mathcal{M} \) an interval \( \mathcal{T}_i \) representing the time interval at which agents \( r_{ki}, r_{ki}' \) will move towards satisfying \( \varphi_i \) as follows:

\[
\mathcal{T}_i := \begin{cases} 
(0, b_i + \epsilon), & \text{if } i = \hat{i}_{\min} \\
[b_i + \epsilon, b_i + \epsilon), & \text{if } i \in \mathcal{I}_k \setminus \{i_{\min}, i_{\max}\}, \quad (7)
\end{cases}
\]

where \( \hat{i} = [i]_k \). Note that \([c_i, b_i + \epsilon) \subset \mathcal{T}_i \) hold for each \( i \in \mathcal{I} \), where \([c_i, b_i + \epsilon) \) is the time interval considered in (5). This property will be shortly considered towards introducing a modified funnel constraint (given in (8)) for each \( i \in \mathcal{I} \) that needs to be satisfied for every \( t \in \mathcal{T}_i \). Traditional PPC strategies require the state of the system at \( t_0 \) to be within the performance bounds, i.e., \(-\eta_i \gamma_i(t_0) < e_i(t_0) < \rho_i \gamma_i(t_0) \) (see Section II-C). Nevertheless, in case of multiple switches among different objectives this condition may not be met. To that end, we will enforce a modified funnel-based constraint for each \( i \in \mathcal{I} \) given as follows:

\[
-\eta_i \gamma_i(t) < e_i(t) < \rho_i \gamma_i(t), \quad t \in \mathcal{T}_i \quad (8)
\]

where \( \mathcal{T}_i \subset \mathbb{R}_{\geq 0} \) is defined in (7), \( e_i(t) := \omega_i(t)(h_i(\beta_k(t)) - \bar{p}_i) \) for \( t \geq 0 \), and \( \omega_i : \mathbb{R}_{\geq 0} \to (0, 1], \) is a shifting function that satisfies the following assumption:

**Assumption 4.** The functions \( \omega_i : \mathbb{R}_{\geq 0} \to (0, 1], i \in \mathcal{I} \) are continuously differentiable in \( \mathcal{T}_i \) and strictly increasing for \( t \in [\min \mathcal{T}_i, c_i] \) with \( \omega_i(\min \mathcal{T}_i) = \delta_i > 0 \) and \( \omega_i(t) = 1 \), for every \( t \geq c_i \), where \( \delta_i \) is a positive tuning parameter, \( \mathcal{T}_i \subset \mathbb{R}_{\geq 0} \) is defined in (7), and \( \min \mathcal{T}_i = 0, \) if \( i = \hat{i}_{\min} \), or \( \min \mathcal{T}_i = b_i + \epsilon \), otherwise. In addition, \( \omega_i(t), i \in \mathcal{I} \) are bounded for \( t \in \mathcal{T}_i \).

The designed shifting function \( \omega_i(t) \) translates a bounded value of \( h_i(\beta_k(\min \mathcal{T}_i)) - \bar{p}_i \) to \( e_i(\min \mathcal{T}_i) \) such that \(-\eta_i \gamma_i(\min \mathcal{T}_i) < e_i(\min \mathcal{T}_i) < \rho_i \gamma_i(\min \mathcal{T}_i) \) is satisfied when \( \delta_i \) is chosen sufficiently small within \((0, 1)]. \) In addition, since \( \omega_i(t) = 1 \), for every \( t \geq c_i \), it follows that \( e_i(t) = h_i(\beta_k(t)) - \bar{p}_i \), for every \( t \geq c_i \). Due to the latter, and since \([c_i, b_i + \epsilon) \subset \mathcal{T}_i \) holds for every \( i \in \mathcal{I} \), it follows that the satisfaction of (8) ensures the satisfaction of (5). Similar assumptions to Assumption 4 have been made in [13], where shifting functions have been considered for output reference tracking of higher relative degree systems.

**Remark 1.** For \( i = \hat{i}_{\min}, k \in \mathcal{M}, \) the shifting function can be chosen as \( \omega_i(t) = 1, t \geq 0 \) when \( e_i(x(0)) \in (-\eta_i \gamma_i(0), \rho_i \gamma_i(0)) \).

Differentiating \( e_i = \omega_i(h_i(\beta_k) - \bar{p}_i) \) we obtain:

\[
\dot{e}_i = \omega_i(\beta_k) - \bar{p}_i + \omega_i \frac{\partial h_i}{\partial \beta_k}(\dot{x}_{k_i} - \dot{x}_{k_i'}). \quad (9)
\]

Let \( \Sigma := \bigcup_{k \in \mathcal{M}} \Sigma_k \) and \( \mathbf{e} := [e_{i_1} \ldots e_{i_\ell}]^T \). For each time interval [\( \sigma_{p}, \sigma_{p+1} \), where \( \sigma_{p}, \sigma_{p+1} \in \Sigma, p \in \mathcal{P} := \{0, \ldots, |\Sigma| - 1\} \) are consecutive time instants let \( \mathbf{e}_p := [e_{i_{p_1}} \ldots e_{i_{p_\ell}}]^T \in \mathbb{R}^{\ell}, z_p \leq q, \) denote the vector of the error signals \( e_{i}, i \in \mathcal{I} \) for which \( \mathcal{T}_i \cap [\sigma_p, \sigma_{p+1}) \neq \emptyset, \) i.e., the
error signals corresponding to STL tasks for which the funnel constraints (8) are active. Then, based on (9) the derivative of the stacked error vector at each \([\sigma_p, \sigma_{p+1}], p \in P\) is given by:

\[
\dot{e}_p = \Omega_p(t)e_p + F_p(x, t)(D_p^T \otimes I_n)\dot{x},
\]

(10)

where \(\Omega_p(t) := \text{diag}(\omega_{i1}(t), \ldots, \omega_{ip}(t)), D_p \in \mathbb{R}^{R \times z_p}\) is a matrix whose \(i\)-th column is equal to the \((k+1)\)-th column of the incidence matrix \(D\) of \(G_p : \mathbb{R}^{Rn} \times \mathbb{R} \to \mathbb{R}^{z_p \times z_p}\) as defined as:

\[
F_p(x, t) := \begin{bmatrix}
\omega_{i1}(t) \frac{\partial h_{1p}^T}{\partial \sigma_{i1}} \\
\vdots \\
\omega_{ip}(t) \frac{\partial h_{ip}^T}{\partial \sigma_{ip}}
\end{bmatrix}
\]

In particular, if \(z_p = M\) and for every \(i, i' \in \{1, \ldots, i_p\}\) with \(i \neq i'\) it holds \(k_i \neq k_{i'}\), then \(D_p = D_P\) for some permutation matrix \(P \in \{0, 1\}^{M \times M}\). Next, given \(e_i, i \in I\) define the normalized errors with respect to the prescribed performance functions as \(\tau_i(t) := \frac{e_i(t)}{\sqrt{\gamma_i(t)}} \in (-\eta_i, \eta_i)\).

Using the transformation functions \(T_i : (-\eta_i, \eta_i) \to \mathbb{R}, i \in I\) that are smooth, strictly increasing and satisfy \(T_i(0) = 0\), we may define the transformed errors \(\tilde{e}_i := T_i(\tau_i), i \in I\), where

\[
T_i(\tau_i) := \ln \left(1 + \frac{\tau_i}{1 - \frac{\tau_i}{\eta_i}}\right).
\]

(11)

Considering the transformed errors \(\tilde{e}_i, i \in I\) it can be shown that if \(\tilde{e}_i\) is bounded, then \(\tau_i \in (-\eta_i, \eta_i)\), which in turn ensures that (8) is satisfied. Differentiating the transformed errors with respect to time we have \(\dot{\tilde{e}}_i = J_i(\tau_i, t)(\dot{\tau}_i + \alpha_i(t)e_i)\), for every \(i \in I\), where \(J_i(\tau_i, t) := \frac{\partial T_i(\tau_i)}{\partial \tau_i} \frac{1}{\gamma_i(t)} > 0\) and \(\alpha_i(t) := -\frac{\gamma_i}{\gamma_i(1)} > 0\). Let \(\varepsilon := [\varepsilon_1 \ldots \varepsilon_t]^T\) and \(\varepsilon_p := [\varepsilon_1 \ldots \varepsilon_{ip}]^T \in \mathbb{R}^{z_p}, p \in P\).

Then, the derivative of \(\varepsilon_p\) can be written in vector form as:

\[
\dot{\varepsilon}_p = J_p(\varepsilon_p, t)(\dot{\varepsilon}_p + \alpha_p(t)e_p)e_p
\]

(12)

where \(J_p(\varepsilon_p, t) := \text{diag}(J_{i1}(\tau_{i1}, t), \ldots, J_{ip}(\tau_{ip}, t)), \alpha_p(t) := \text{diag}(\alpha_{i1}(t), \ldots, \alpha_{ip}(t))\), and \(\varepsilon_p := [\tau_{i1} \ldots \tau_{ip}]^T\).

Based on the above, we define \(u_r\) for every \(r \in V\) as follows:

\[
u_r = -g_T(x_r) \sum_{k \in M} \sum_{\sigma_p \in P} \tau_k(t) \tilde{C}_i d_{rk} J_k(\tau_k, t) \varepsilon_k(t) \frac{\partial h_k(\beta_k)}{\partial \beta_k},
\]

(13)

where \(D = \{d_{rk}\}, \tilde{C}_i > 0\) are gains to be tuned and \(\tau_1(t) := 1, \) if \(t \in T_i\), or \(\tau_1(t) := 0,\) otherwise. Then, \(u\) can be written in stack vector form as:

\[
u = -g_T(x) \sum_{p \in P} a_p(t)(D_p \otimes I_n)F_p(x, t)J_p(\varepsilon_p, t)G_p e_p,
\]

(14)

where \(G_p := \text{diag}(\tilde{C}_i, \ldots, \tilde{C}_{ip})\), and \(a_p(t) := 1,\) if \(t \in [\sigma_p, \sigma_{p+1}),\) or \(a_p(t) := 0,\) otherwise and \(p \in P\). Under the proposed control law, we can show the satisfaction of (8) for each \(i \in I\) as depicted in the following theorem:

**Theorem 1.** Consider the multi-agent system (2) that is subject to the STL task defined in (4) and let Assumptions 1-4 hold. Assume further that \([f_r(x_r)] \leq +\infty\) for every \(x_r \in \mathbb{R}^n, r \in V = \mathbb{R}^n, \beta_k(\sigma_p)\) satisfies (8) for every \((i_1, \ldots, i_p)\), \(\sigma_p \in \Sigma\) and \(p \in P\). Given \(\omega \geq \frac{1}{(0, 1]}\) and parameters \(\eta_i, \xi, \gamma_i, T_i, z_i, \xi, \eta, \xi, \) let \(\beta_{i1}(\cdot, t) = \varepsilon_{i1}(\min T_i, \xi_{i1} \gamma_i(\min T_i)) \in B_i,\) for every \(t \in T_i\) and \(i \in I\), where \(B_i\) are the same sets as in Assumption 3. Then, the control law (14) ensures that (8) is satisfied for every \(i \in I\) and the closed-loop signals at each \([\sigma_p, \sigma_{p+1}], p \in P\) are bounded.

**Proof.** For each \(p \in P\) we consider the Lyapunov function \(V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}\), defined as \(V_p(\varepsilon_p) = \frac{1}{2} \varepsilon_p^T(\varepsilon_p)G_p e_p(\varepsilon_p),\) where \(D_p := (-\eta_i, \eta_i) \times \cdots \times (-\eta_i, \eta_i)\). Differentiating \(V_p\) and after substitution of (12) we get \(\dot{V}_p = \varepsilon_p^T G_p J_p(\varepsilon_p, t)(\dot{\varepsilon}_p + \alpha_p(t)e_p)\).

Using (10), (2) and after substitution of the proposed control law (14) \(V_p\) becomes:

\[
\dot{V}_p = \varepsilon_p^T G_p J_p(\varepsilon_p, t)(\dot{\varepsilon}_p + \alpha_p(t)e_p)e_p + \frac{1}{2} \varepsilon_p^T G_p J_p(\varepsilon_p, t)\dot{\varepsilon}_p e_p + e_p^T G_p J_p(\varepsilon_p, t)x_v G_p e_p + e_p^T G_p J_p(\varepsilon_p, t)\dot{\varepsilon}_p e_p.
\]

By Assumption 1, \(g(x)\) is full row rank, thus \(g(x)g(x)^T\) is positive definite. In addition, since \(G\) is a tree graph, \(\text{rank}(D) = M\) [12, Lem.1]. Due to Assumption 3 and by definition of \(\Sigma\), within each \([\sigma_p, \sigma_{p+1})\) there exists at most one \(i \in I\) involving the \((k+1)\)-th edge of \(G\), thus \(z_p \leq M\) and the columns of \(D_p\) are linearly independent which in turn implies that \(\text{rank}(D_p) = z_p\). Therefore, by virtue of [14, Obs. 7.1.8] and the properties of the Kronecker product it follows that \((D_p \otimes I_n)g(x)g(x)^T(\bar{D}_p \otimes I_n)\) is positive definite. Furthermore, by Assumption 3 and since \(\omega(t) > 0\) as well as \((\beta_{i1}(\cdot, t) = (-\eta_i, \eta))\) \(g(x)g(x)^T(\bar{D}_p \otimes I_n)\) is positive definite. Let \(\lambda := \inf_{t \in [\sigma_p, \sigma_{p+1})} \lambda_{min}(A(x(t), t))\) and note that \(\lambda > 0\).

Then, \(\dot{V}_p\) satisfies:

\[
\dot{V}_p \leq \varepsilon_p^T G_p J_p(\varepsilon_p, t)(\dot{\varepsilon}_p + \alpha_p(t)e_p)e_p - 2\kappa(\lambda - \varepsilon)\varepsilon_p - \xi \|J_p(\varepsilon_p, t)G_p e_p\|^2,
\]

for a parameter \(0 < \xi < \lambda \) and after adding and subtracting \(\xi \|J_p(\varepsilon_p, t)G_p e_p\|^2\) to the right-hand side of the aforementioned inequality we obtain:

\[
\dot{V}_p \leq \varepsilon_p^T G_p J_p(\varepsilon_p, t)(\dot{\varepsilon}_p + \alpha_p(t)e_p)e_p - 2\kappa(\lambda - \varepsilon)\varepsilon_p - \xi \|J_p(\varepsilon_p, t)G_p e_p\|^2,
\]

where \(\kappa = \frac{\pi}{\max_{p \in P}(G_p)}\) and \(\xi := \min_{t \in [\sigma_p, \sigma_{p+1})} \frac{1}{\gamma_i(1)}\).
as follows:

\[
\gamma_{i,\infty} \in (0, \max(\bar{\gamma}_i, \eta_i)),
\]

\[
\mathcal{P}_i \in (\mathcal{P} + \eta_i \gamma_{i,\infty}, \ln(\bar{\gamma}_i, \eta_i + \bar{\mathcal{P}}))
\]

\[
\lambda_i > -\frac{\ln(\gamma_{i,\infty} + \hat{\gamma}_i, \eta_i)}{t_i}
\]

where \( \hat{\gamma}_i := \bar{\gamma}_i - \bar{\mathcal{P}}, \) for every \( i \in \mathcal{I} \) and \( \bar{\gamma}_i := \sup_{\beta_k \in \mathcal{B}_i} h_i(\beta_k) \) as defined in Assumption 3. Constraint (16a) implies that \( \gamma_{i,\infty} := \frac{\hat{\gamma}_i, \eta_i}{\max(\bar{\gamma}_i, \eta_i)} < 1 \) which in turn guarantees that \( \gamma_{i,\infty} := \lim_{t \to \infty} \gamma_i(t) < \gamma_i(0) \). Constraint (16c) ensures that \( \gamma_i(t_i) > \gamma_i \) which by definition \( \gamma_i \) implies that \( \gamma_i(t_i) > \gamma_i \). Thus, if the funnel constraints (8) are satisfied, then \( h_i(\beta_k_i(x(t_i))) > \bar{\mathcal{P}} \) and since \( \gamma_i(t_i) \) is strictly decreasing, \( h_i(\beta_k_i(x(t_i))) > \bar{\mathcal{P}} \) will hold for every \( t_i \in [t_i^*, b_i] \). Note that by design of \( \gamma_i(t_i) \), \( l_i \) chosen according to (16c) should satisfy \( l_i > 0 \) or equivalently \( \max(\bar{\gamma}_i, \eta_i) > 1 \). Provided that the left hand side of the aforementioned inequality is strictly positive in order for the argument of the logarithm to be well-defined. Given the definition of \( \hat{\gamma}_i \) and due to (16a) these properties are always satisfied when \( \bar{\mathcal{P}} \) is chosen according to (16b). Based on the aforementioned discussion, we can deduce the following:

**Theorem 2.** Consider the multi-agent system (2) and let \( \varphi_i, i \in \mathcal{I} \) be the STL formulas given in (4). Let the assumptions of Theorem 1 hold. Given \( \bar{\mathcal{P}} \in (0, \min_{i \in \mathcal{I}} \bar{\gamma}_i) \), if for every \( i \in \mathcal{I} \) the positive parameters \( \bar{\gamma}_i, \eta_i, \bar{\gamma}_i, \gamma_{i,\infty}, t_i \), determining (8) are chosen according to (16a)-(16c) such that \( \{ \beta_k_i : e_i(\beta_k_i,t_i) \in (-\bar{\eta}_i \gamma_i(\min T_i), \bar{\gamma}_i(\min T_i) ) \} \subseteq \mathcal{B}_i \), then under the proposed control law (14), \( \rho^\phi(x,0) \geq \bar{\mathcal{P}}, \) for every \( t_i \in \mathcal{I} \) and thus \( \rho^\phi(x,0) \geq \bar{\mathcal{P}} \).

**Proof.** In Theorem 1 it has been shown that the control law proposed in (14) ensures the satisfaction of (8). Choosing the parameters determining the funnel constraint according to (16a)-(16c) and since \( t_i^* \) is in the interior of \( T_i \) defined in (7), then \( h_i(\beta_k_i(x(t_i))) > \bar{\mathcal{P}} \), for every \( t_i \in [t_i^*, b_i] \). If \( \varphi_i = \mathcal{P}_{\max(a_i,b_i)} \psi_i \) then for some \( t_i \in [a_i,b_i] \cap [t_i^*, b_i] \) if it holds that \( h_i(\beta_k_i(x(t_i))) > \bar{\mathcal{P}} \), which implies that \( \max_{e_i(a_i,b_i)} h_i(\beta_k_i(x(t_i))) > \bar{\mathcal{P}} \) and the result follows. If \( \varphi_i = \mathcal{P}_{\max(a_i,b_i)} \psi_i \), then choosing the funnel parameters as in (16a)-(16c) as well as by design of \( t_i^*, c_i \), defined in (15) and (5), respectively, we have \( h_i(\beta_k_i(x(t_i))) > \bar{\mathcal{P}} \), for every \( t_i \in [a_i,b_i] \) in which turn implies that \( \min_{e_i(a_i,b_i)} h_i(\beta_k_i(x(t_i))) \geq \bar{\mathcal{P}} \). Finally, \( \rho^\phi(x,0) \geq \bar{\mathcal{P}} \) follows by aforementioned analysis and the definition of \( \rho^\phi(x,0) \).

**IV. Numerical Example**

The control strategy proposed in Section III will be applied to a time-varying formation control problem involving \( R = 4 \) agents with \( f_r(x_r) = \left[ \cos(x_r - r) - 2 \sin(y_r) \cos\left(0.2(x_r - y_r)\right) \right] \), \( r \in V \setminus \{2\} \), \( g_r(x_r) = \left[ 1 + 0.25 x_r^2 - 1 \right] \) for every
In this work a distributed switching control strategy is designed to ensure the satisfaction of a conjunction of STL tasks that are based on relative position information among neighboring agents. The satisfaction of individual STL tasks is enforced by prescribed performance functions designed to ensure a desired level of robustness. Assuming a tree graph topology, we show the satisfaction of the funnel constraints after pre-specified time instants and the boundedness of the closed-loop signals. Future efforts will be directed towards considering more complex graph topologies and STL tasks as well higher-relative degree systems.

**References**


