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Koopman Resolvents of Nonlinear Discrete-Time Systems: Formulation and Identification

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Abstract— **The Koopman operator framework is a promising direction of analysis and synthesis of systems with nonlinear dynamics based on (linear) Koopman operators. In this paper, we address the resolvent of a Koopman operator for a nonlinear autonomous discrete-time system, which we call the Koopman resolvent, and its identification problem. First, we show that for the nonlinear system with a scalar-valued output, the z-transform of the output is represented by the action of Koopman resolvent. Second, we describe an identification method of the Koopman resolvent directly from time-series data of the output, in which we estimate parameters of the resolvent as well as poles and residues of the z-transform of the output. By combining the so-called frequency-domain Prony method with the Vandermonde-Cauchy form in the Dynamic Mode Decomposition (DMD), we propose the method which we call the frequency-domain DMD, in which all the unknowns can be estimated in the frequency domain.**

Index Terms— **Nonlinear system, Koopman operator, Resolvent, Dynamic mode decomposition, Prony method, Frequency domain**

I. INTRODUCTION

The Koopman operator framework is a promising direction of analysis and synthesis of systems with nonlinear dynamics based on (linear) Koopman operators: see, e.g., [1]–[5]. The Koopman operator is a composition operator based on a (possibly nonlinear) map [6]. It can capture the complete information of the underlying nonlinear map with a linear (but infinite-dimensional) setting, mirroring the classical approach to linear systems. Of particular interest here is the spectral analysis of the Koopman operator, which establishes a mathematically and computationally tractable way of solving a nonlinear problem without lacking any information. For instance, the point spectrum, called Koopman eigenvalues, imply intrinsic modal frequencies embedding in multivariate time series generated by a nonlinear system, which was discovered in [7], [8] as the Koopman Mode Decomposition (KMD). Its numerical algorithm is generally termed the Dynamic Mode Decomposition (DMD): see, e.g., [9].

This paper focuses on utilizing the Koopman operator to study the z-domain representation of nonlinear autonomous discrete-time systems. The z-domain is a classical approach to linear systems that provides a systematic method to characterize the systems through complex analysis techniques: see, e.g., [10]. If a similar approach is established, it will provide the principles and methodology for analyzing and synthesizing nonlinear discrete-time systems in the frequency domain. In [11], we presented a theory of the Laplace-domain representation of nonlinear autonomous continuous-time systems, proving that for a nonlinear continuous-time system with a scalar-valued output, its Laplace transform is represented by the action of the so-called *resolvent* (operator) of a Koopman generator. This shows that the Laplace transformation can completely capture properties of the nonlinear system through the Koopman generator. In the present paper, to establish its parallel theory to discrete-time systems, we address the resolvent of a Koopman operator for a nonlinear autonomous discrete-time system, which as in [11] we term the *Koopman resolvent*.

The contributions of this paper are two. First, we show that for a nonlinear discrete-time system with a scalar-valued output, its z-transform is represented by the action of Koopman resolvent. This is a parallel result to [11] and implies that the z-transformation can completely capture properties of the nonlinear system through the Koopman operator. The idea is summarized in Table I and will be explained in further detail later. Also, we provide several formulae describing how the Koopman resolvent is connected to the z-transform of the nonlinear output. Technically, parameterizations of the Koopman resolvent and z-transform are derived under an assumption of invariant subspace. By this derivation, functional properties of the z-transform, precisely speaking, its poles and residues, are connected to the KMD, which correspond to the Koopman eigenvalues and Koopman modes [8], respectively.

Second, we describe an identification method of the Koopman resolvent in the framework of DMD. Many variants of the DMD are reported in the literature: see, e.g., recently [12], [13]. The identification problem is to estimate the parameters of the resolvent, including poles and residues of the z-transform, where all the unknowns are related to the frequency domain (precisely, the z-domain). Here, we intend to develop a method of DMD suitable to the frequency-domain (z-domain) theory of nonlinear systems. The standard DFT (Discrete Fourier Transform) is a well-established approach in signal processing. Still, it poses a fundamental issue in the frequency resolution that might be critical to the estimation accuracy of the poles. In this paper, to estimate the parameters and poles, we utilize the so-

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TABLE I

z-DOMAIN REPRESENTATION OF DISCRETE-TIME SYSTEMS

	Linear	Nonlinear
System models	$\boldsymbol{x}_{t+1} = \mathbf{A}\boldsymbol{x}_t$	$\boldsymbol{x}_{t+1} = \boldsymbol{T}(\boldsymbol{x}_t)$
Scalar outputs	$y_t = \boldsymbol{c}^\top \boldsymbol{x}_t$	$y_t = f(\boldsymbol{x}_t)$
z-transforms of outputs y_t	$z\boldsymbol{c}^\top (z\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{x}_0$	$zR(z;U)f(\boldsymbol{x}_0)$

called *Frequency-Domain Prony Method* (FDPM) developed by Ando [14], where he resolved the issue by combining the DFT with the Prony formulation to estimate parameters of sinusoidal signals. The Prony formulation (or generally, time-delay embedding) has been utilized for the DMD [15]–[17], and the FDPM was also utilized to estimate the Koopman eigenvalues [18]. Furthermore, to estimate the residues in the frequency domain, we utilize the Vandermonde-Cauchy form [19] in DMD. By combining the FDPM with the Vandermonde-Cauchy form, we propose the method which we call the *frequency-domain DMD*, in which all the unknowns are estimated in the frequency domain.

Notation.—The sets of all real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, and the imaginary unit by $\mathrm{i} := \sqrt{-1}$. The m-dimensional identity matrix is denoted by $\mathbf{I}_m$, and the identity operator acting on a function space by I. The transpose operation of vectors is denoted by $\top$, the diagonal matrix by $\mathrm{diag}(\cdots)$, the Kronecker product of two matrices by $\otimes$. The linear hull in a vector space is denoted by $\mathrm{span}\{\cdots\}$. The notion $\|\cdot\|$ stands for the standard vector norm and the operator norm.

II. Koopman Resolvents for Discrete-Time Systems

A. System Models

Consider a discrete-time dynamical system on a finite-dimensional state space $\mathbb{X}$, described by a nonlinear continuous map $\boldsymbol{T} : \mathbb{X} \to \mathbb{X}$ as

$$\boldsymbol{T} : \boldsymbol{x} \mapsto \boldsymbol{T}(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{X}. \tag{1}$$

A scalar-valued function defined on $\mathbb{X}$, called the *observable*, is denoted by $f : \mathbb{X} \to \mathbb{C}$, and its linear space by $\mathbb{F}$. Then, the so-called *Koopman operator* $U : \mathbb{F} \to \mathbb{F}$ is defined to map f to another observable in the following composition ($\circ$) manner:

$$U : f \mapsto f \circ \boldsymbol{T}, \quad \forall f \in \mathbb{F}.$$

This U is linear and bounded with an operator norm $\|U\|$ (it holds under a general choice of $\mathbb{F}$). The Koopman operator framework is to analyze the linear operator U in order to understand the properties of the nonlinear map $\boldsymbol{T}$. As one of such analyses, it is possible to represent the output $\{y_t\}_{t=0,1,\dots}$ generated by the nonlinear map $\boldsymbol{T}$ and observable f as follows:

$$y_t = f(\boldsymbol{T}^t(\boldsymbol{x}_0)) = U^t f(\boldsymbol{x}_0), \quad t = 0, 1, \dots, \tag{2}$$

where $\boldsymbol{x}_0$ is the initial state. This shows that U governs how the output y_t of the nonlinear system (1) through f behaves in time t.

B. z-Transformation and Koopman Resolvent

Next, we consider the z-transformation of the nonlinear output (2). For this, using the Neumann series of the bounded U, i.e., for $z \in \mathbb{C}$ satisfying $|z| > \|U\|$,

$$(zI - U)^{-1} = z^{-1} \sum_{t=0}^{\infty} (z^{-1}U)^t,$$

it is possible to represent the z-transform of (2), denoted by $Y(z; f)$, as follows:

$$\begin{aligned} Y(z; f) &= \sum_{t=0}^{\infty} y_t z^{-t} = \sum_{t=0}^{\infty} U^t f(\boldsymbol{x}_0) z^{-t} \\ &= z(zI - U)^{-1} f(\boldsymbol{x}_0), \quad |z| > \|U\|. \end{aligned} \tag{3}$$

The operator $(zI - U)^{-1}$ is the resolvent operator of U, which we denote by $R(z; U)$ and call the *Koopman resolvent*. This shows that the z-transform of the nonlinear output is represented as the action of the Koopman resolvent to an observable f with respect to initial state $\boldsymbol{x}_0$:

$$Y(z; f) = zR(z; U) f(\boldsymbol{x}_0), \quad |z| > \|U\|. \tag{4}$$

This is a generalization of the classical approach to linear discrete-time systems as shown in Table I and parallels the Laplace-domain representation [11] for nonlinear continuous-time systems.

C. Resolvent-Based Formulae

Following [16], let $\mathbb{F}_n = \mathrm{span}\{f, Uf, \dots, U^n f\}$ be a (finite) k-dimensional subspace of $\mathbb{F}$ which is invariant under the action of U, which is coined in [20] as the Koopman invariant subspace. This is the case, for example, where (possibly damped) sinusoidal components are dominant in the output y_t as for the KMD. Now, consider the first k elements of $\mathbb{F}_n$, that is, $\{f, Uf, \dots, U^{k-1} f\}$, which are linearly independent. If $U^k f$ is nonzero, then there exists k scalars $a_0, a_1, \dots, a_{k-1}$, not all zero such that

$$U^k f = a_0 f + a_1 U f + \cdots + a_{k-1} U^{k-1} f. \tag{5}$$

Equation (5) is an Auto-Regressive (AR) model with order k for the time evolution of the observable f and associated with the Arnoldi-type formulation [8] in DMD. The scalars $a_0, a_1, \dots, a_{k-1}$ then become elements of the companion matrix (see Section III-C) that is a finite-dimensional approximation of the action of $U|_{\mathbb{F}_n}$ (U restricted on $\mathbb{F}_n$).

Now, we derive several resolvent-based formulae describing how the Koopman resolvent in the invariant subspace is connected to the z-transform $Y(z; f)$. First, acting the resolvent $R(z; U)$ to both sides of (5), we have the z-domain version of (5) as

$$\begin{aligned} R(z; U) U^k f &= a_0 R(z; U) f + a_1 R(z; U) U f + \cdots \\ &\quad + a_{k-1} R(z; U) U^{k-1} f, \quad |z| > \|U\|. \end{aligned} \tag{6}$$

Second, multiplying by z both sides of (6) and using (4) with initial state $\boldsymbol{x}_0$, we have

$$\begin{aligned} Y(z; U^k f) &= a_0 Y(z; f) + a_1 Y(z; U f) + \cdots \\ &\quad + a_{k-1} Y(z; U^{k-1} f), \quad |z| > \|U\|. \end{aligned} \tag{7}$$

Equation (7) will appear when we think of a connection with the Prony formulation for the identification problem. Lastly, by using[1]

$$R(z;U)U^j f = z^j R(z;U)f - z^{j-1}f - z^{j-2}Uf - \cdots - U^{j-1}f,$$

we rewrite (6) as

$$p_k(z)R(z;U)f = \sum_{j=0}^{k-1} p_j(z)U^j f, \quad |z| > \|U\|,$$

with

$$\left.\begin{array}{l} p_k(z) := z^k - (a_0 + a_1 z + \cdots + a_{k-1}z^{k-1}) \\ p_0(z) := z^{k-1} - (a_1 + a_2 z + \cdots + a_{k-1}z^{k-2}) \\ p_1(z) := z^{k-2} - (a_2 + a_3 z + \cdots + a_{k-1}z^{k-3}) \\ \vdots \\ p_{k-2}(z) := z - a_{k-1} \\ p_{k-1}(z) := 1 \end{array}\right\}. \quad (8)$$

Because of the linear independency of $\{f, Uf, \ldots, U^{k-1}f\}$ and of the fact $p_{k-1}(z) = 1$, we conclude that $p_k(z) \neq 0$ for $|z| > \|U\|$. Therefore, we have

$$R(z;U)f = \sum_{j=0}^{k-1} \frac{p_j(z)}{p_k(z)} U^j f, \quad |z| > \|U\|, \quad (9)$$

and from (4),

$$Y(z;f) = z\sum_{j=0}^{k-1} \frac{p_j(z)}{p_k(z)} U^j f(\boldsymbol{x}_0), \quad |z| > \|U\|. \quad (10)$$

Equation (9) implies that the resolvent $R(z;U)$ is parametrized by the (at most) k-th order polynomials $p_0(z), \ldots, p_k(z)$ with the k scalars $a_0, \ldots, a_{k-1}$. The scalars are targets of the identification in the next section. Equation (10) implies that the z-transform $Y(z;f)$ can be analyzed with the polynomials $p_0(z), \ldots, p_k(z)$. The *poles* of $Y(z;f)$, which can exist in the disk $|z| \leq \|U\|$, are equivalent to the k roots of $p_k(z)$ that are the eigenvalues of $U|_{\mathbb{F}^n}$, namely, Koopman eigenvalues. For a pole λ of order m, the *residue* of $Y(z;f)$ around $z = \lambda$ can be located, e.g., with the formula using the resolvent

$$\operatorname*{Res}_{z=\lambda} Y(z;f)dz = \frac{1}{(m-1)!} \cdot \cdot \left. \frac{d^{m-1}}{dz^{m-1}} z(z-\lambda)^m R(z;U)f(\boldsymbol{x}_0) \right|_{z=\lambda}.$$

A simple pole λ (with $m = 1$) and associated residue $\operatorname*{Res}_{z=\lambda} Y(z;f)dz$ characterize a single mode embedded in the nonlinear output y_t, represented as

$$\lambda^t \operatorname*{Res}_{z=\lambda} Y(z;f)dz, \quad t = 0, 1, \ldots.$$

where the residue corresponds to the Koopman mode [8], and its finite sum over multiple poles is called the KMD. Note that the residue is connected to the projection operation onto spaces spanned by the Koopman eigenfunctions via the Cauchy integral formula, see [11]. The pole and residue are the other targets of the identification in the next section.

[1]To derive it, we use $zR(z;U) - I = R(z;U)U = UR(z;U)$.

III. Identification of Koopman Resolvents

The identification problem is to estimate the scalars $a_0, \ldots, a_{k-1}$ as the parameters in (9) of the Koopman esolvent $R(z;U)$, as well as the poles λ_i and residues $\operatorname*{Res}_{z=\lambda_i} Y(z;f)dz$ in (10) of the z-transform $Y(z;f)$ as the Koopman eigenvalues and modes. Identifying poles and residues is of great interest in science and technology; see, e.g., [21], [22]. In Section III-A, we formally show that the Prony formulation is capable of the estimation of $a_0, \ldots, a_{k-1}$. In Sections III-B.1 and III-B.2, we introduce the FDPM (Frequency-Domain Prony Method) [14] and the Vandermonde-Cauchy form [19] utilized in this paper. Finally, as their combination, in Section III-C, we propose a method to estimate all the above unknows in the frequency domain.

A. Connection to the Prony Formulation

The resolvent-based formula (7) is connected to the so-called Prony formulation of signal processing. The Prony formulation is used in the DMD [15], [16] and to estimate k scalars $c_0, \ldots, c_{k-1}$ from finite-length time series $\{y_0, y_1, \ldots, y_{k+n-1}\}$ as

$$\begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+n-1} \end{bmatrix} = \begin{bmatrix} y_0 & y_1 & \cdots & y_{k-1} \\ y_1 & y_2 & \cdots & y_k \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1} & y_n & \cdots & y_{k+n-2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix}, \quad (11)$$

where we have assumed k known. As in [16], that is, by using (2), this Prony formulation can be interpreted in terms of the action of the Koopman operator as

$$\begin{bmatrix} U^k f(\boldsymbol{x}_0) \\ U^{k+1} f(\boldsymbol{x}_0) \\ \vdots \\ U^{k+n-1} f(\boldsymbol{x}_0) \end{bmatrix} = \begin{bmatrix} f(\boldsymbol{x}_0) & Uf(\boldsymbol{x}_0) & \cdots & U^{k-1}f(\boldsymbol{x}_0) \\ Uf(\boldsymbol{x}_0) & U^2 f(\boldsymbol{x}_0) & \cdots & U^k f(\boldsymbol{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ U^{n-1}f(\boldsymbol{x}_0) & U^n f(\boldsymbol{x}_0) & \cdots & U^{k+n-2} f(\boldsymbol{x}_0) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix}. \quad (12)$$

Comparing (12) with (5) implies $c_j = a_j$, that is, the Prony formulation is capable of estimating the k parameters $a_0, \ldots, a_{k-1}$. Now, by applying $[1, z^{-1}, z^{-2}, \ldots, z^{-(n-1)}]$ to both sides of (12) from the left, we have

$$\sum_{t=0}^{n-1} U^t(U^k f)(\boldsymbol{x}_0)z^{-t} = \left[\sum_{t=0}^{n-1} U^t f(\boldsymbol{x}_0)z^{-t}, \sum_{t=0}^{n-1} U^t(Uf)(\boldsymbol{x}_0)z^{-t}, \ldots, \sum_{t=0}^{n-1} U^t(U^{k-1}f)(\boldsymbol{x}_0)z^{-t}\right] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix}.$$

By taking the infinite sequence of time series ($n \to \infty$) and using (3), we have

$$zR(z;U)U^k f(\boldsymbol{x}_0) = [zR(z;U)f(\boldsymbol{x}_0), \quad zR(z;U)Uf(\boldsymbol{x}_0), \dots, zR(z;U)U^{k-1}f(\boldsymbol{x}_0)] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix},$$

with $|z| > \|U\|$. This corresponds to (7), also implying $c_j = a_j$. This shows that the Prony formulation (11) is capable of estimating the k parameters $a_0, \dots, a_{k-1}$, namely, the action of the Koopman resolvent $R(z;U)$. This finding is valid for the general case of multi-channel time series $\boldsymbol{y}_t \in \mathbb{R}^m$, $m > 1$.

B. Review of Frequency-Domain Methods

1) Frequency-Domain Prony Method: First, we present an extended version of the FDPM (Frequency-Domain Prony Method) [14] to the m-channel time series $\boldsymbol{y}_t \in \mathbb{R}^m$. Regarding (11), let us identify the AR model with order k as

$$\boldsymbol{y}_t = c_0 \boldsymbol{y}_{t-k} + c_1 \boldsymbol{y}_{t-(k-1)} + \cdots + c_{k-1}\boldsymbol{y}_{t-1} + \boldsymbol{r}_t, \quad (13)$$

where $\boldsymbol{r}_t \in \mathbb{R}^m$ is the residual at time t or noise (driving term) that can appear when applying (11) to real data. The identification is to determine the k coefficients $c_0, \dots, c_{k-1} \in \mathbb{R}$ directly from finite-length time series that provide frequencies and damping coefficients of sinusoidal signals embedded therein. The problem is formulated using finite-length time series $\{\boldsymbol{y}_0, \dots, \boldsymbol{y}_{k+n-1}\}$ ($n > k$) as follows:

$$\begin{bmatrix} \boldsymbol{y}_0 & \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_{k-1} & \boldsymbol{y}_k \\ \boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_k & \boldsymbol{y}_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{y}_{n-1} & \boldsymbol{y}_n & \cdots & \boldsymbol{y}_{k+n-2} & \boldsymbol{y}_{k+n-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \\ -1 \end{bmatrix} = -\tilde{\boldsymbol{r}}_k, \quad (14)$$

where $\tilde{\boldsymbol{r}}_k$ is again the residual $[\boldsymbol{r}_k^\top, \dots, \boldsymbol{r}_{k+n-1}^\top]^\top$. To handle the problem with the n snapshots $\{\boldsymbol{y}_0, \dots, \boldsymbol{y}_{n-1}\}$, by multiplying a matrix $\mathcal{W}_n^{(m)} := \mathcal{W}_n \otimes \mathbf{I}_m$, where $\mathcal{W}_n := \mathrm{diag}(w_0, \dots, w_{n-k-1}, w_{n-k} = 0, \dots, w_{n-1} = 0)$, to both hand sides of (14) (saying in [14], *windowing the AR model*), we have the following circulant matrix-based formulation:

$$\mathcal{W}_n^{(m)} \begin{bmatrix} \boldsymbol{y}_0 & \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_{k-1} & \boldsymbol{y}_k \\ \boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_k & \boldsymbol{y}_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{y}_{n-k-1} & \boldsymbol{y}_{n-k} & \cdots & \boldsymbol{y}_{n-2} & \boldsymbol{y}_{n-1} \\ \boldsymbol{y}_{n-k} & \boldsymbol{y}_{n-k+1} & \cdots & \boldsymbol{y}_{n-1} & \boldsymbol{y}_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{y}_{n-1} & \boldsymbol{y}_0 & \cdots & \boldsymbol{y}_{k-2} & \boldsymbol{y}_{k-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \\ -1 \end{bmatrix} = -\mathcal{W}_n^{(m)} \tilde{\boldsymbol{r}}_k. \quad (15)$$

By using $\mathcal{F}_n^{(m)} := \mathcal{F}_n \otimes \mathbf{I}_m$ with the DFT matrix

$$\mathcal{F}_n := \left\{ \frac{\Omega_n^{(i-1)(j-1)}}{\sqrt{n}} \right\}_{i,j=1,\dots,n}, \quad \Omega_n := \mathrm{e}^{-\mathrm{i}2\pi/n}, \quad (16)$$

and by defining $\mathcal{Q}_n^{(m)} := \mathcal{F}_n^{(m)} \mathcal{W}_n^{(m)} (\mathcal{F}_n^{(m)})^{-1} = \mathcal{Q}_n \otimes \mathbf{I}_m$ with $\mathcal{Q}_n := \mathcal{F}_n \mathcal{W}_n \mathcal{F}_n^{-1}$, we can rewrite (15) as follows:

$$\mathcal{Q}_n^{(m)} \mathcal{F}_n^{(m)} \begin{bmatrix} \boldsymbol{y}_0 & \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_k \\ \boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{y}_{n-1} & \boldsymbol{y}_0 & \cdots & \boldsymbol{y}_{k-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ -1 \end{bmatrix} = -\mathcal{Q}_n^{(m)} \mathcal{F}_n^{(m)} \tilde{\boldsymbol{r}}_k,$$

and

$$\mathcal{Q}_n^{(m)} \begin{bmatrix} \boldsymbol{\eta}_0 & \boldsymbol{\eta}_0 & \cdots & \boldsymbol{\eta}_0 \\ \boldsymbol{\eta}_1 & \Omega_n \boldsymbol{\eta}_1 & \cdots & \Omega_n^k \boldsymbol{\eta}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\eta}_{n-1} & \Omega_n^{n-1} \boldsymbol{\eta}_{n-1} & \cdots & \Omega_n^{(n-1)k} \boldsymbol{\eta}_{n-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ -1 \end{bmatrix} = -\mathcal{Q}_n^{(m)} \mathcal{F}_n^{(m)} \tilde{\boldsymbol{r}}_k,$$

with the DFTs $\boldsymbol{\eta}_i$ of $\{\boldsymbol{y}_t\}$, given by

$$[\boldsymbol{\eta}_0, \dots, \boldsymbol{\eta}_{n-1}] := \underbrace{[\boldsymbol{y}_0, \dots, \boldsymbol{y}_{n-1}]}_{\mathbf{Y}_n} \mathcal{F}_n, \quad (17)$$

where i represents the frequency (it corresponds to $i/(nh)$ if $\boldsymbol{y}_t$ is derived under an equally-spaced sampling h of continuous-time signal). The term $\mathcal{F}_n^{(m)} \tilde{\boldsymbol{r}}_k$ above contains the DFTs of the residual. By defining a convolution (see [14] in detail) of $\boldsymbol{\eta}_i$ with $\mathcal{Q}_n$ as

$$\boldsymbol{\eta}_i^{[d]} := \sum_{j=0}^{n-1} [\mathcal{Q}_n]_{ij} \Omega_n^{jd} \boldsymbol{\eta}_j, \quad i = 0, \dots, n-1, d = 0, \dots, k, \quad (18)$$

where d represents the delay, we obtain the equation for determining $a_1, \dots, a_k$ in the frequency domain as

$$\begin{bmatrix} \boldsymbol{\eta}_0^{[k]} \\ \vdots \\ \boldsymbol{\eta}_{n-1}^{[k]} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\eta}_0^{[0]} & \cdots & \boldsymbol{\eta}_0^{[k-1]} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\eta}_{n-1}^{[0]} & \cdots & \boldsymbol{\eta}_{n-1}^{[k-1]} \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_{k-1} \end{bmatrix} + \mathcal{Q}_n^{(m)} \mathcal{F}_n^{(m)} \tilde{\boldsymbol{r}}_k.$$

A solution $\hat{c}_0, \dots, \hat{c}_{k-1}$ of $\min_{c_0,\dots,c_{k-1}} \|\mathcal{Q}_n^{(m)} \mathcal{F}_n^{(m)} \tilde{\boldsymbol{r}}_k\|$ can be located using the n DFTs (17) via (18). It is remarked that the window matrix $\mathcal{W}_n$ is synthesized in the same manner as [14].

2) Vandermonde-Cauchy Form for DMD: Second, we review the Vandermonde-Cauchy form [19] for DMD. For the data matrix $\mathbf{Y}_k = [\boldsymbol{y}_0, \dots, \boldsymbol{y}_{k-1}]$, let us denote the shifted one by $\mathbf{Y}'_k := [\boldsymbol{y}_1, \dots, \boldsymbol{y}_k]$. Then, by virtue of (13) at $t = k$, we have

$$\mathbf{Y}'_k = \mathbf{Y}_k \underbrace{\begin{bmatrix} 0 & 0 & \cdots & \hat{c}_0 \\ 1 & 0 & \cdots & \hat{c}_1 \\ 0 & 1 & \cdots & \hat{c}_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{c}_{k-1} \end{bmatrix}}_{\hat{\mathbf{C}}_k} + \underbrace{[\mathbf{0}, \dots, \mathbf{0}, \boldsymbol{r}_k]}_{\mathbf{R}_k}, \quad (19)$$

where $\hat{\mathbf{C}}_k$ is the companion matrix. It is known in matrix theory that the characteristic polynomial of $\hat{\mathbf{C}}_k$ corresponds to $p_k(\lambda) = 0$ using $\hat{c}_1, \ldots, \hat{c}_k$, for which we denote the k roots by $\hat{\lambda}_1, \ldots, \hat{\lambda}_k$. The $\hat{\mathbf{C}}_k$ can be diagonalized using the Vandermonde matrix $\hat{\mathbf{V}}_k$ as

$$\left.\begin{aligned} \hat{\mathbf{C}}_k &= \hat{\mathbf{V}}_k^{-1} \underbrace{\mathrm{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_k)}_{\hat{\mathbf{\Lambda}}_k} \hat{\mathbf{V}}_k \\ \hat{\mathbf{V}}_k &:= \begin{bmatrix} 1 & \hat{\lambda}_1 & \hat{\lambda}_1^2 & \cdots & \hat{\lambda}_1^{k-1} \\ 1 & \hat{\lambda}_2 & \hat{\lambda}_2^2 & \cdots & \hat{\lambda}_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \hat{\lambda}_k & \hat{\lambda}_k^2 & \cdots & \hat{\lambda}_k^{k-1} \end{bmatrix} \end{aligned}\right\}.$$

Then, by constructing $\hat{\mathbf{W}}_k = [\hat{\boldsymbol{W}}_1, \ldots, \hat{\boldsymbol{W}}_k] := \mathbf{Y}_k \hat{\mathbf{V}}_k^{-1}$, from (19), we have the modal decomposition of $\boldsymbol{y}_t$ as follows:

$$\boldsymbol{y}_t = \sum_{j=1}^{k} \hat{\lambda}_j^t \hat{\boldsymbol{W}}_j + \left\{ \begin{array}{ll} \mathbf{0}, & t = 0, \ldots, k-1, \\ \boldsymbol{r}_t, & t = k, \end{array} \right. \tag{20}$$

where $\hat{\lambda}_j$ is called the j-th Ritz value [8] or Prony value [15] as an estimated Koopman eigenvalue, and $\hat{\boldsymbol{W}}_j$ the associated Ritz vector [8] or Prony vector [15] as an estimated Koopman mode. A numerical difficulty of the decomposition arises from the potentially high condition number of $\hat{\mathbf{V}}_k$, which precludes exploiting it in numerical computations. To avoid $\hat{\mathbf{V}}_k^{-1}$, according to [19], by multiplying $\mathcal{F}_k^{-1}$ in (16) to both sides of (19) from the right, we derive the following formula of (19) in the frequency domain:

$$\begin{aligned} \mathbf{Y}_k' \mathcal{F}_k^{-1} &= (\mathbf{Y}_k \mathcal{F}_k^{-1})(\hat{\mathbf{V}}_k \mathcal{F}_k^{-1})^{-1} \hat{\mathbf{\Lambda}}_k (\hat{\mathbf{V}}_k \mathcal{F}_k^{-1}) + \mathbf{R}_k \mathcal{F}_k^{-1} \\ &= \underbrace{(\mathbf{Y}_k \mathcal{F}_k^{-1})(\mathcal{D}_1 \mathcal{C} \mathcal{D}_2)^{-1}}_{\hat{\mathbf{W}}_k} \hat{\mathbf{\Lambda}}_k (\mathcal{D}_1 \mathcal{C} \mathcal{D}_2) + \mathbf{R}_k \mathcal{F}_k^{-1}, \end{aligned}$$

with

$$[\mathcal{D}_1]_{ii} \mathcal{C}_{ij} [\mathcal{D}_2]_{jj} = \left[\frac{\hat{\lambda}_i^k - 1}{\sqrt{k}} \right] \left[\frac{1}{\hat{\lambda}_i - \Omega_k^{1-j}} \right] [\Omega_k^{1-j}], \quad i, j = 1, \ldots, k,$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are diagonal, $\mathcal{C}$ is a Cauchy matrix, and $\mathcal{D}_2 = \mathrm{diag}(1, \Omega_k^{-1}, \ldots, \Omega_k^{-(k-1)})$ is unitary. The term $\mathbf{Y}_k \mathcal{F}_k^{-1}$ is exactly the DFT of $\mathbf{Y}_k = [\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{k-1}]$. One computation formula of $\hat{\mathbf{W}}_k$ is taken from [19] in the frequency domain as follows:

$$\hat{\mathbf{W}}_k = \left(\left(\mathbf{Y}_k \mathcal{F}_k^{-1} \right) (\mathcal{C} \mathcal{D}_2)^{-1} \right) \mathcal{D}_1^{-1}. \tag{21}$$

The key for the accuracy of (21) is high accuracy computation of the LU and the SVD decompositions of the scaled Cauchy matrix $\mathcal{D}_1 \mathcal{C} \mathcal{D}_2$ (or $\mathcal{C} \mathcal{D}_2$) [23], [24].

C. Proposed Method

Finally, by combining the FDPM with the Vandermonde-Cauchy form, we propose a novel method of DMD consisting of three steps (S1,S2,S3), where all the unknowns are estimated in the frequency domain.

(S1) Estimate the k scalars $\hat{c}_0, \ldots, \hat{c}_{k-1}$ in the Koopman resolvent by using the FDPM.

(S2) Estimate the k poles (Koopman eigenvalues) of the z-transform $\hat{\lambda}_1, \ldots, \hat{\lambda}_k$ by locating the eigenvalues of the companion matrix $\hat{\mathbf{C}}_k$.

(S3) Estimate the associated k residues (Koopman modes) $\hat{\boldsymbol{W}}_1, \ldots, \hat{\boldsymbol{W}}_k$ by using the Vandermonde-Cauchy form, i.e., (21).

Several remarks on the proposed method are made. The existing methods of DMD conduct (S1) to (S3) in the time domain with singular value decomposition as [16] and without it as [15]. We have derived the method of DMD in the frequency domain, that is, using the DFTs $\mathbf{Y}_n \mathcal{F}_n$ and $\mathbf{Y}_k \mathcal{F}_k^{-1}$. As stated in [18], by using the FDPM, it is expected to be robust against observational noise by formulating the estimation procedure in the frequency domain, where around a peak frequency, the parameters are estimated with the best signal-to-noise ratio and isolation from another peak. Also, it can be stated in an equivalent sense that the time-variant noise as a stationary process has a steady Fourier spectrum, which is easily filtered out. Furthermore, by using the Vandermonde-Cauchy form, it becomes possible to accurately estimate the residues even in the case of a high condition number of the Vandermonde matrix $\hat{\mathbf{V}}_k$. We term the proposed method the *frequency-domain DMD*.

D. Numerical Example

Here, as the simplest case, we numerically evaluate the frequency-domain DMD using a synthetic time series (sampled data of the damped oscillation) modeled by

$$\boldsymbol{y}_t = \mathrm{e}^{-\mu h t} \mathrm{Re} \left[\mathrm{e}^{\mathrm{i} \omega h t} \begin{bmatrix} 1 \\ 1.5 \mathrm{e}^{\mathrm{i}\pi/3} \end{bmatrix} \right] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi_t, \tag{22}$$

where $t = 0, \ldots, n-1$ is the discrete-time, $n = 64$ the number of samples, $h = 2/(50\mathrm{Hz})$ is the sampling period, $\mu = 0.1\mathrm{Hz}$ is the decay coefficient, $\omega = 2\pi \times (0.5\mathrm{Hz})$ is the angular frequency, and ξ_t is the observation noise obeying the normal distribution with zero mean and variance σ^2. The Koopman principal eigenvalues in a latent discrete-time system are $\mathrm{e}^{-\mu h} \mathrm{e}^{\pm \mathrm{i} \omega h}$ with modulus $\mathrm{e}^{-\mu h}$ and arguments $\pm \omega h$, and the associated Koopman mode, precisely, its second element is $1.5 \mathrm{e}^{\mathrm{i}\pi/3}$. The frequency $\omega/(2\pi)$, decay μ, modulus 1.5, and argument $\pi/3$ are the targets for the evaluation.[2] For comparison, we apply the Arnoldi method [8] and the Time-Domain Prony Method (TDPM) [15] to the same time series.

Figure 1 shows a comparison result of the estimation, where the horizontal axis is σ^2 and the vertical axis is the medians of squared errors between estimated and true values. 2000 trials of the estimation (i.e., using 2000 sample processes of $\{\xi_t\}_{t=0,\ldots,n-1}$) were performed for each noise variance σ^2. Regarding the frequency and decay, the FDPM outperforms both the Arnoldi method and TDPM, implying the robustness of the FDPM. Regarding the modulus and

[2] Hereafter, we do not discuss the coefficients c_j of the resolvent because their accuracy can be evaluated with the resultant Koopman eigenvalue and mode.

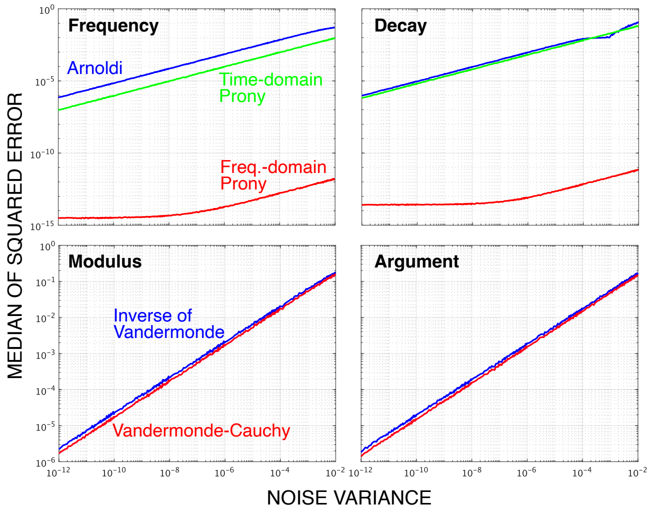

Fig. 1. Comparison of the medians of squared errors for the estimation

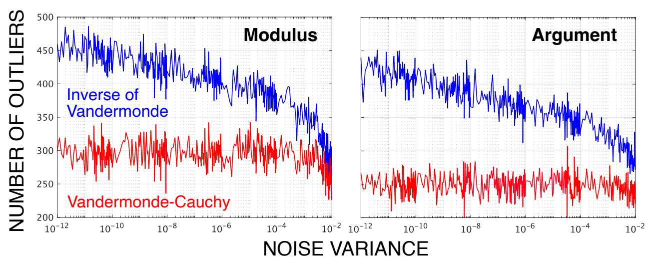

Fig. 2. Comparison of the number of outliers for the estimation

argument, we use the two schemes—using the inverse as $\mathbf{Y}_k\hat{\mathbf{V}}_k^{-1}$ and using the Vandermonde-Cauchy form as (21)—and show that their squared errors are comparable in terms of the median.

Figure 2 shows another comparison result of the estimation, where we show the number of outliers for the estimated modulus and argument, that is, we visualize how they are varied for the 2000 trials. For this, we used the command `isoutlier` with `quartitles` in MATLAB. The number of outliers for the Vandermonde-Cauchy form is smaller than for taking the inverse. Note that for the inverse approach in the figure, there are more outliers for lower levels of noise because the pure (clean) damped oscillation poses the high condition number of $\hat{\mathbf{V}}_k$. This implies that the Vandermonde-Cauchy form effectively avoids less accurate estimation, thereby robustifying the DMD.

IV. Conclusion

We reported the formalism of z-domain representation of nonlinear autonomous systems in the Koopman operator framework. The Koopman resolvent is the key mathematical object for capturing dynamics described by nonlinear systems through the traditional z-transformation. An identification method related to the Koopman resolvent, the frequency-domain DMD, was also reported. Extending the formalism and method to vector-valued observations is possible without significant modification. Extending the z-domain representation to nonlinear systems with inputs is crucial and listed as our next topic. Integrating the signal-processing technique of the FDPM [25] with the DMD is also interesting for real-time implementation. In this, exemplifying it for nonlinear time series and evaluating its computational cost are necessary.

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