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Analysis of EMPC schemes without terminal constraints via local incremental stabilizability

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***Abstract*— Economic model predictive control (EMPC) is a popular control methodology that enjoys attention both from practitioners as well as the control research community. Of particular interest are EMPC schemes without terminal constraints in the underlying optimal control problems, and a considerable amount of theoretical analyses are already available. In this work, we derive many of these results using the notion of local incremental stabilizability, a concept that proved to be important in robust model predictive control. We show that this notion can be seamlessly used in the analysis of EMPC, and also derive new continuity results, replacing a corresponding assumption in existing works.**

I. Introduction

Model predictive control (MPC) is one of the most important modern control methodologies, receiving considerable attention in both industrial practice and academic research [1]. The basic idea is that in each time step, an open loop optimal control problem (OCP) is solved, the first part of the resulting trajectory is applied to the plant, and in the next time step the OCP is solved again at the resulting state. Instead of stabilizing a given set point or track a predefined trajectory, in economic model predictive control (EMPC), a generic cost function is optimized, often related to economic performance measures [2]. In this context, MPC schemes without terminal constraints are of particular importance [3], [1]. So far, these schemes are analyzed using dissipativity and turnpike notions [4], establishing various performance and stability guarantees. Concurrently, in the context of robust MPC (RMPC), the concept of local incremental stabilizability has proven to be a very useful concept in the design and analysis of RMPC schemes [5], [6].

In this work, we show that the latter concept can also be used as the foundation of the analysis of EMPC schemes without terminal constraints. In particular, in turns out that standard proof techniques in this context can be implemented using local incremental stabilizability as the starting point. Furthermore, in previous investigations, the continuity of the discrete-time storage function appearing in the definition of dissipativity is posed as an assumption, cf. [1, Assumption 8.24a], but in the present context we can show that this property follows from local incremental stabilizability. Interestingly, while this continuity assumption is central in the analysis of EMPC without terminal constraints, there are few results in the literature to establish this property. One of these is [7, Lemma 6], but it is formulated in continuous time and is based on a rather strong reachability condition. The results in [8] and follow-up work require much weaker conditions, but they depend strongly on the continuous time nature of the system (in particular, reversibility of the underlying ODE).

In summary, by introducing local incremental stabilizability as the central assumption in the analysis of EMPC without terminal constraints, we can connect the theory of RMPC and EMPC, and open up new approaches to tackle and improve the analysis of EMPC schemes.

Outline In Section II we introduce the setup and central assumptions. In Section III, we rederive standard EMPC results using local incremental stabilizability as the starting point. Additionally, in Proposition III.10 we provide a novel continuity result for discrete-time storage functions, and use this in the context of EMPC in Lemma III.12, one of the first continuity results for the storage function in this context. Finally, we conclude in Section IV with a discussion and summary.

II. Preliminaries

We use standard comparison functions. The class $\mathcal{K}$ consists of all functions $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\alpha(0) = 0$ and that are continuous and strictly increasing. The class $\mathcal{K}_\infty$ consists of functions $\alpha \in \mathcal{K}$ such that $\lim_{r\to\infty} \alpha(r) = \infty$, and the class $\mathcal{L}_{\mathbb{N}_+}$ consists of all functions on the positive integers decreasing strictly to zero.

A. Setup

Consider a discrete-time nonlinear control system

$$x_+ = f(x, u), \tag{1}$$

where $f : X \times U \to X$ is the transition function, and the state X and the input space U are normed vector spaces, but everything works mutatis mutandis also for metric spaces. The state trajectory $x(\cdot; x, u)$ starting at $x \in X$ under some control input $u \in U^N$, $N \in \mathbb{N} \cup \{\infty\}$, is recursively defined as usual by

$$\begin{aligned} x(0; x, u) &= x \\ x(n + 1; x, u) &= f(x(n; x, u), u(n)). \end{aligned}$$

Furthermore, let $\mathbb{Z} \subseteq X \times U$ be a constraint set, inducing state constraints $\mathbb{X} = \{x \in X \mid \exists u \in U : (x, u) \in \mathbb{Z}\}$, and input constraints $\mathbb{U}(x) = \{u \in U \mid (x, u) \in \mathbb{Z}\}$ for all $x \in X$. For $N \in \mathbb{N}_+$, let $\mathbb{U}^N(x)$ be the set of $u \in U^N$ s.t. $(x(n; x, u), u(n)) \in \mathbb{Z}$ for all $n = 0, \ldots, N - 1$, and define

$$\mathbb{U}^\infty(x) = \{u \in U^\infty \mid \forall N \in \mathbb{N}_+ : u|_{\{0,\ldots,N-1\}} \in \mathbb{U}^N(x)\},$$

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as well as $\mathbb{X}_N = \{x \in \mathbb{X} \mid \mathbb{U}^N(x) \neq \emptyset\}$. Additionally, given $\mathbb{Y} \subseteq X$, define $\mathbb{U}^N_{\mathbb{Y}}(x) = \{u \in \mathbb{U}^N(x) \mid x(N; x, u) \in \mathbb{Y}\}$.

We can turn (1) into a closed loop system $x_+ = f(x, \kappa(x))$ by using a feedback map $\kappa : X \to U$, and define as usual the state trajectory starting at $x \in X$ recursively by

$$x_\kappa(0; x) = x$$
$$x_\kappa(n+1; x) = f(x_\kappa(n; x), \kappa(x_\kappa(n; x))),$$

and input trajectory by $u_\kappa(n; x) = \kappa(x_\kappa(n; x))$.

For a given stage cost $\ell : X \times U \to \mathbb{R}$, we define for $N \in \mathbb{N}_+ \cup \{\infty\}$ and $u \in U^N$ the total cost

$$J_N(x, u \mid \ell) = \sum_{n=0}^{N-1} \ell(x(n; x, u), u(n)), \tag{2}$$

and the corresponding value function

$$V_N(x \mid \ell) = \inf_{u \in \mathbb{U}^N_{\mathbb{X}}(x)} J_N(x, u \mid \ell). \tag{3}$$

By convention we set $J_0 \equiv 0$. If we have a feedback map $\kappa : X \to U$, define the closed loop total cost for horizon $N \in \mathbb{N}_+ \cup \{\infty\}$ and initial state $x \in X$ by

$$J^\kappa_N(x \mid \ell) = J_N(x, u_\kappa(\cdot; x) \mid \ell) = \sum_{n=0}^{N-1} \ell(x_\kappa(n; x), u_\kappa(n; x)),$$

and for a time-varying feedback map $\kappa : \mathbb{N}_0 \times X \to U$ by

$$J^\kappa_N(n, x \mid \ell) = \sum_{k=0}^{N-1} \ell(x_\kappa(n+k; n, x), u_\kappa(n+k; n, x)).$$

For $n = 0$, define for brevity $J^\kappa_N(x \mid \ell) = J^\kappa_N(0, x \mid \ell)$.

B. Basic assumptions

In the remainder, we fix a constraint set $\mathbb{Z}$ with the corresponding induced constraint sets, and a stage cost $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$.

Assumption II.1. There exists $\gamma_\ell \in \mathcal{K}_\infty$ such that for all $x_1, x_2 \in \mathbb{X}$ and all $u_1, u_2 \in \mathbb{U}$ we have

$$|\ell(x_1, u_1) - \ell(x_2, u_2)| \leq \gamma_\ell(\|x_1 - x_2\|_X + \|u_1 - u_2\|_U). \tag{4}$$

Furthermore, for every $\rho \in (0, 1)$, there exists $\gamma^\rho_\ell \in \mathcal{K}_\infty$ such that for all $r \in \mathbb{R}_{\geq 0}$ we have

$$\sum_{n=0}^{\infty} \gamma_\ell(\rho^n r) \leq \gamma^\rho_\ell(r). \tag{5}$$

Assumption II.1 is essentially [6, Assumption 3].

Assumption II.2. There exists $B_\ell \in \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{X}$, $u \in \mathbb{U}$, we have $\ell(x, u) \leq B_\ell$.

Next, we fix an equilibrium pair $(x_e, u_e) \in \mathbb{Z}$, i.e., $x_e = f(x_e, u_e)$, and define $\ell_e = \ell(x_e, u_e)$, and given $x \in X$, $u \in U$, set $\|x\|_e = \|x - x_e\|_X$, $\|u\|_e = \|u - u_e\|_U$.

Assumption II.3. The equilibrium (x_e, u_e) is in the interior of $\mathbb{Z}$, i.e., $(x_e, u_e) \in \mathring{\mathbb{Z}}$.

Assumption II.3 implies that there exists $R_{\mathbb{X},e} \in \mathbb{R}_{>0}$ such that $B_{R_{\mathbb{X},e}}(x_e) \subseteq \mathbb{X}$.

Assumption II.4. There exists a function $\lambda : \mathbb{X} \to \mathbb{R}$, bounded from below, and $\rho \in \mathcal{K}_\infty$, such that for all $(x, u) \in \mathbb{Z}$ with $f(x, u) \in \mathbb{X}$, we have

$$\rho(\|x\|_e) \leq \ell(x, u) - \ell_e + \lambda(x) - \lambda(f(x, u)). \tag{6}$$

We use Assumption II.4 to define the rotated stage cost

$$\ell_{\text{rot}}(x, u) = \ell(x, u) - \ell_e + \lambda(x) - \lambda(f(x, u)) \tag{7}$$

for all $(x, u) \in \mathbb{Z}$ with $f(x, u) \in \mathbb{X}$. Following the usual terminology in the EMPC literature, we call Assumption II.4 strict dissipativity.

Assumption II.5. There exists $B_\lambda \in \mathbb{R}_{\geq 0}$, such that $|\lambda(x)| \leq B_\lambda$ for all $x \in \mathbb{X}$ with λ from Assumption II.4.

The next assumption describes local incremental stabilizability in the form of [6, Assumption 1].

Assumption II.6. There exists $V_\delta : X \times \mathbb{Z} \to \mathbb{R}_{\geq 0}$, $\kappa_\delta : X \times \mathbb{Z} \to U$, and constants $\underline{C}_\delta, \bar{C}_\delta, C_\kappa, C^{\max}_\delta \in \mathbb{R}_{>0}$, $\rho_\delta \in (0, 1)$, such that for all $(x, u) \in \mathbb{Z}$ and all $\tilde{x} \in X$ with $V_\delta(\tilde{x}, x, u) \leq C^{\max}_\delta$ we have

$$\underline{C}_\delta \|\tilde{x} - x\|^2_X \leq V_\delta(\tilde{x}, x, u) \leq \bar{C}_\delta \|\tilde{x} - x\|^2_X \tag{8}$$
$$\|\kappa_\delta(\tilde{x}, x, u) - u\|^2_U \leq C_\kappa V_\delta(\tilde{x}, x, u) \tag{9}$$
$$\forall u_+ \in \mathbb{U}(x_+) : \; V_\delta(\tilde{x}_+, x_+, u_+) \leq \rho^2_\delta V_\delta(\tilde{x}, x, u) \tag{10}$$

where $x_+ = f(x, u)$ and $\tilde{x}_+ = f(\tilde{x}, \kappa_\delta(\tilde{x}, x, u))$.

Next, we introduce a finite time approximate controllability assumption, essentially [9, Assumption 3.4] with an additional requirement on the neighbourhood of x_e.

Assumption II.7. There exists $N_e \in \mathbb{N}_+$ and some $R_e \in \mathbb{R}_{>0}$ with $R_e \leq \min\{\sqrt{\bar{C}^{-1}_\delta C^{\max}_\delta}, R_{\mathbb{X},e}\}$ such that for all $x \in \mathbb{X}$ there exists $u_x \in \mathbb{U}^{N_e}(x)$ with $\|x(N_e; x, u_x)\|_e \leq R_e$.

Note that in Assumption II.7, $R_e \leq R_{\mathbb{X},e}$ implies $u_x \in \mathbb{U}^{N_e}_{\mathbb{X}}(x)$.

III. Analysis results

We now analyse EMPC schemes without terminal constraints using Assumption II.6 as the central ingredient. However, for this we need an incremental variant of controlled positive invariance (CPI), introduced next. We discuss this property and alternative approaches at the end of the present work.

A. An incremental CPI assumption

The following assumption formalizes the CPI condition that is used later on.

Assumption III.1. Consider the situation of Assumption II.6. If $\tilde{x} \in \mathbb{X}$, then $\kappa_\delta(\tilde{x}, x, u) \in \mathbb{U}(\tilde{x})$, and if $f(x, u) \in \mathbb{X}$, then $f(\tilde{x}, \kappa_\delta(\tilde{x}, x, u)) \in \mathbb{X}$.

We now collect some technical results needed in the sequel, based on common arguments from the RMPC literature, cf. [5], [6].

Lemma III.2. Let $x \in \mathbb{X}$ and $u \in \mathbb{U}^N(x)$, $N \in \mathbb{N}_+$. Under Assumptions II.6 and III.1, for all $\tilde{x} \in \mathbb{X}$ with

$\|\tilde{x} - x\|_X \leq \sqrt{\bar{C}_\delta^{-1} C_\delta^{\max}}$, the control sequence $\tilde{u} \in U^N$ defined recursively by

$$\begin{aligned} \tilde{x}(0) &= \tilde{x} \\ \tilde{u}(n) &= \kappa_\delta(\tilde{x}(n), x(n; x, u), u(n)), \; n = 0, \dots, N-1 \\ \tilde{x}(n+1) &= f(\tilde{x}(n), \tilde{u}(n)), \; n = 0, \dots, N-2, \end{aligned}$$

is well-defined, and $(\tilde{x}(n), \tilde{u}(n)) \in \mathbb{Z}$ for all $n = 0, \dots, N-1$. If $x(N; x, u) \in \mathbb{X}$ (i.e., $u \in \mathbb{U}_{\mathbb{X}}^N(x)$), then we also have $\tilde{x}(N) \in \mathbb{X}$. Furthermore,

$$V_\delta(\tilde{x}(n), x(n; x, u), u(n)) \leq \rho_\delta^{2n} V_\delta(\tilde{x}, x, u) \quad (11)$$

$$\|\tilde{x}(n) - x(n; x, u)\|_X \leq \rho_\delta^n \sqrt{\underline{C}_\delta^{-1} \bar{C}_\delta} \|\tilde{x} - x\|_X \quad (12)$$

$$\|\tilde{u}(n) - u(n)\|_U \leq \rho_\delta^n \sqrt{C_\kappa \bar{C}_\delta} \|\tilde{x} - x\|_X \quad (13)$$

for all $n = 0, \dots, N-1$. Finally, if additionally Assumption II.1 holds, then

$$\sum_{n=0}^{N-1} \ell(\tilde{x}(n), \tilde{u}(n)) \leq \sum_{n=0}^{N-1} \ell(x(n; x, u), u(n)) + \gamma_\delta(\|\tilde{x} - x\|_X), \quad (14)$$

with $\gamma_\delta(r) = \gamma_\ell^{\rho_\delta}\left((\sqrt{\underline{C}_\delta^{-1} \bar{C}_\delta} + \sqrt{C_\kappa \bar{C}_\delta}) r \right)$, where $\gamma_\ell^{\rho_\delta} \in \mathcal{K}_\infty$ is from Assumption II.1.

Proof: Standard induction, hence omitted. □

B. Performance bound

We now aim at a bound on the closed loop performance by adapting [1, Lemma 8.26] to the present situation. For this, we use a turnpike argument as developed in [3], combining cheap reachability with the assumption of strict dissipativity, cf. also [4].

Proposition III.3. Under Assumptions II.1, II.2, II.6, III.1, II.3 and II.7, there exists $N_{\text{CR}} \in \mathbb{N}_+$ and $C_{\text{CR}} \in \mathbb{R}_{>0}$ such that for all $N \geq N_{\text{CR}}$ and all $x \in \mathbb{X}$ we have

$$V_N(x \mid \ell) \leq N \ell_e + C_{\text{CR}}. \quad (15)$$

Proof: Set $N_{\text{CR}} = N_e$ from Assumption II.7 and let $x \in \mathbb{X}$, $N \geq N_{\text{CR}}$ be arbitrary. Let $u_x \in \mathbb{U}^{N_{\text{CR}}}(x)$ be the control sequence from Assumption II.7 and set $\tilde{x} = x(N_e; x, u_x)$. Apply now Lemma III.2 to $x = x_e$, $u = (u_e \cdots u_e) \in U^{N - N_{\text{CR}}}$ (note that $\tilde{x} \in \mathbb{X}$), which results in $\tilde{x}(\cdot), \tilde{u}(\cdot)$ with $(\tilde{x}(n), \tilde{u}(n)) \in \mathbb{Z}$ for $n = 0, \dots, N - N_{\text{CR}} - 1$. Define $\hat{u} = \begin{pmatrix} u_x & \tilde{u} \end{pmatrix}$, then we have

$$\begin{aligned} V_N(x \mid \ell) &\leq J_N(x, \hat{u} \mid \ell) \\ &= J_{N_{\text{CR}}}(x, u_x \mid \ell) + \sum_{n=0}^{N - N_{\text{CR}} - 1} \ell(\tilde{x}(n), \tilde{u}(n)) \\ &\leq N_{\text{CR}} B_\ell + \gamma_\delta(\|\tilde{x}\|_e) + (N - N_{\text{CR}}) \ell_e \\ &\leq N \ell_e + N_{\text{CR}} B_\ell + \max\{0, -N_{\text{CR}} \ell_e\} + \gamma_\delta(R_e), \end{aligned}$$

where we used $\hat{u} \in \mathbb{U}_{\mathbb{X}}^N(x)$ in the first inequality, and Assumption II.2 and Lemma III.2 in the second inequality. □

Remark III.4. Proposition III-B and its proof show that $\mathbb{X}$ is viable, i.e., for all $x \in \mathbb{X}$ and all $N \in \mathbb{N}_+$, $\mathbb{U}^N(x) \neq \emptyset$, which also implies $\mathbb{U}_{\mathbb{X}}^N(x) \neq \emptyset$ and trivially $V_N(x \mid \ell) < \infty$. This also shows that Assumption III.1 together with Assumption II.6 and II.7 is rather strong.

For convenience, we recall the following result on (steady state) turnpikes, see e.g. [1, Proposition 8.15].

Lemma III.5. Under Assumption II.4 and II.5, for all $x \in \mathbb{X}$, $u \in \mathbb{U}_{\mathbb{X}}^N(x)$ for some $N \in \mathbb{N}_+$, and all $\delta \in \mathbb{R}_{\geq 0}$, if $J_N(x, u \mid \ell) \leq N \ell_e + \delta$, then $\sharp \mathcal{Q}((x, u), x_e, \sigma_\delta(P)) \geq N - P$ for all $P = 1, \dots, N-1$, where for all $\epsilon \in \mathbb{R}_{\geq 0}$ we define

$$\mathcal{Q}((x, u), x_e, \epsilon) = \{n \in \{0, \dots, N-1\} \mid \|x(n; x, u)\|_e \leq \epsilon\}$$

and $\sigma_\delta(P) = \rho^{-1}\left(\frac{2B_\lambda + \delta}{P}\right)$.

Here is now the adapted version of [1, Lemma 8.27].

Lemma III.6. Under Assumptions II.1, II.2, II.3, II.6, III.1, II.7, II.4 and II.5, there exists $\underline{N} \in \mathbb{N}_+$ and $\nu \in \mathcal{L}_{\mathbb{N}_+}$ such that for all $N \geq \underline{N}$ and $x \in \mathbb{X}$ we have

$$V_N(x \mid \ell) \leq V_{N-1}(x \mid \ell) + \ell_e + \nu(N-1). \quad (16)$$

Proof: Define

$$N_{\text{TP}} = \left\lceil \frac{2B_\lambda + C_{\text{CR}} + 1}{\rho\left(\frac{\sqrt{\underline{C}_\delta^{-1} C_\delta^{\max}}}{1 + \sqrt{\underline{C}_\delta^{-1} \bar{C}_\delta})^{-1}} \right)} \right\rceil \quad (17)$$

and set $\underline{N} = \max\{N_{\text{CR}}, N_{\text{TP}} + 2\}$. Let now $x \in \mathbb{X}$, $N \geq \underline{N}$ and $0 < \epsilon < 1$ be arbitrary. Choose $u_\epsilon \in \mathbb{U}_{\mathbb{X}}^{N-1}(x)$ with $J_{N-1}(x, u_\epsilon \mid \ell) \leq V_{N-1}(x \mid \ell) + \epsilon$, which exists since $\mathbb{U}_{\mathbb{X}}^{N-1}(x) \neq \emptyset$ (cf. Proposition III-B and Remark III.4) and $V_{N-1}(x \mid \ell) > -\infty$. To show the latter, let $u \in \mathbb{U}_{\mathbb{X}}^{N-1}(x)$ be arbitrary, then summing up the inequality from Assumption II.4 over $n = 0, \dots, N-2$ results in

$$\begin{aligned} \sum_{n=0}^{N-2} \ell(x(n; x, u), u(n)) &\geq (N-1)\ell_e + \lambda(x(N; x, u)) - \lambda(x) \\ &\geq (N-1)\ell_e - 2B_\lambda > -\infty. \end{aligned}$$

Since the right hand side in the preceding inequality chain is independent of u, we get that $V_{N-1}(x \mid \ell)$ is bounded from below. In the following, for readability define also $x_\epsilon(n) = x(n; x, u_\epsilon)$. Next, use Lemma III.5 with $N-1$, $P = (N-1)-1$ and $\delta = C_{\text{CR}} + 1$ to get $\sharp \mathcal{Q}((x, u_\epsilon), x_e, \sigma_\delta(P)) \geq (N-1) - (N-2) \geq 1$, which implies that there exists $k_x \in \{0, \dots, (N-1)-1\}$ with $\|x_\epsilon(k_x)\|_e \leq \rho^{-1}\left(\frac{2B_\lambda + C_{\text{CR}} + 1}{N-2}\right) \leq \rho^{-1}\left(\frac{2B_\lambda + C_{\text{CR}} + 1}{N_{\text{TP}}}\right)$, which implies that $\|x_\epsilon(k_x)\|_e \leq \sqrt{\bar{C}_\delta^{-1} C_\delta^{\max}}$. The latter fact allows us to use Lemma III.2 to define $u_+ = \kappa_\delta(x_\epsilon(k_x), x_e, u_e)$ and $\tilde{x} = f(x_\epsilon(k_x), u_+)$ and conclude that $(x_\epsilon(k_x), u_+) \in \mathbb{Z}$ and $\tilde{x} \in \mathbb{X}$. Furthermore, we also get from Lemma III.2 that

$$\begin{aligned} \ell(x_\epsilon(k_x), u_+) &\leq \ell_e + \gamma_\ell(\|x_\epsilon(k_x)\|_e + \|u_+\|_e) \\ &\leq \ell_e + \gamma_\ell\left((1 + \sqrt{C_\kappa \bar{C}_\delta}) \rho^{-1}\left(\frac{2B_\lambda + C_{\text{CR}} + 1}{N-2} \right) \right) \\ &= \ell_e + \nu_1(N-1) \end{aligned}$$

Additionally,

$$\begin{aligned}V_\delta(\tilde{x}, x_\epsilon(k_x), u_\epsilon(k_x)) &\leq \bar{C}_\delta \|\tilde{x} - x_\epsilon(k_x)\|_X^2 \\ &\leq \bar{C}_\delta \left(\|\tilde{x}\|_e + \|x_\epsilon(k_x)\|_e\right)^2 \\ &\leq \bar{C}_\delta \left(1 + \rho_\delta \sqrt{\underline{C}_\delta^{-1} \bar{C}_\delta}\right)^2 \|x_\epsilon(k_x)\|_e^2,\end{aligned}$$

and the choice of N_{TP} implies then $V_\delta(\tilde{x}, x_\epsilon(k_x), u_\epsilon(k_x)) \leq C_\delta^{\max}$. We can now use Lemma III.2 again (note that $u_\epsilon(\cdot + k_x) \in \mathbb{U}^{N-1-k_x}(x_\epsilon(k_x))$) to define

$$\begin{aligned}\tilde{x}(0) &= \tilde{x} \\ \tilde{u}(n) &= \kappa_\delta(\tilde{x}(n), x_\epsilon(k_x + n), u_\epsilon(k_x + n) \\ \tilde{x}(n+1) &= f(\tilde{x}(n), \tilde{u}(n)),\ n = 0, \dots, (N-1) - k_x - 1\end{aligned}$$

and ensure that $(\tilde{x}(n), \tilde{u}(n)) \in \mathbb{Z}$ for $n = 0, \dots, (N-1) - k_x - 1$ and $\tilde{x}((N-1) - k_x) \in \mathbb{X}$ (here we used that $u_\epsilon \in \mathbb{U}_{\mathbb{X}}^{N-1}(x)$ instead of just $u_\epsilon \in \mathbb{U}^{N-1}(x)$). Observe that

$$\begin{aligned}&\|\tilde{x}(n) - x_\epsilon(k_x + n)\|_X + \|\tilde{u}(n) - u_\epsilon(k_x + n)\|_U \\ &\leq \rho_\delta^n \sqrt{(\underline{C}_\delta^{-1} + C_\kappa) V_\delta(\tilde{x}, x_\epsilon(k_x), u_\epsilon(k_x))} \\ &\leq \rho_\delta^n \sqrt{(\underline{C}_\delta^{-1} + C_\kappa)\bar{C}_\delta} \left(1 + \rho_\delta \sqrt{\underline{C}_\delta^{-1} \bar{C}_\delta}\right) \|x_\epsilon(k_x)\|_e,\end{aligned}$$

so we also have $J_{(N-1)-k_x}(\tilde{x}, \tilde{u} \mid \ell) \leq J_{(N-1)-k_x}(x_\epsilon(k_x), u_\epsilon(\cdot + k_x) \mid \ell) + \nu_2(N_1)$ with

$$\nu_2(N_1) = \gamma_\ell^{\rho_\delta} \left(\sqrt{(\underline{C}_\delta^{-1} + C_\kappa)\bar{C}_\delta} \left(1 + \rho_\delta \sqrt{\underline{C}_\delta^{-1} \bar{C}_\delta}\right) \rho^{-1} \left(\frac{2B_\lambda + C_{\text{CR}} + 1}{N-2}\right)\right)$$

Finally, define

$$\hat{u} = (u_\epsilon(0) \cdots u_\epsilon(k_x - 1)\, u_+\, \tilde{u}(0) \cdots \tilde{u}((N-1) - k_x - 1))$$

and note that by construction $\hat{u} \in \mathbb{U}_{\mathbb{X}}^N(x)$. We now have

$$\begin{aligned}V_N(x \mid \ell) &\leq J_N(x, \hat{u} \mid \ell) \\ &\leq J_{k_x - 1}(x, u_\epsilon \mid \ell) + \ell_e + \nu_1(N-1) \\ &\quad + J_{(N-1)-k_x-1}(x_\epsilon(k_x), u_\epsilon(\cdot + k_x) \mid \ell) + \nu_2(N-1) \\ &= J_{N-1}(x, u_\epsilon \mid \ell) + \ell_e + \nu(N-1) \\ &\leq V_{N-1}(x \mid \ell) + \ell_e + \nu(N-1) + \epsilon,\end{aligned}$$

with $\nu(N-1) = \nu_1(N-1) + \nu_2(N-1)$. Since $0 < \epsilon < 1$ was arbitrary and $\nu \in \mathcal{L}_{\mathbb{N}_+}$ is independent of ϵ, N and x, the claim follows. ☐

We can now state and prove the following bound on average closed loop performance, which is essentially [1, Theorem 8.27] adapted to the present situation. Motivated by results like [1, Theorem 4.16] and the approach from [10], we work with approximate minimizers.

Proposition III.7. Let $\mu_N : \mathbb{N}_0 \times \mathbb{X} \to U$ and $(\epsilon_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{R}_{\geq 0}$ such that for all $n \in \mathbb{N}_0$ and $x \in \mathbb{X}$ we have $\mu_N(n, x) = u_n(0)$ for some $u_n \in \mathbb{U}_{\mathbb{X}}^N(x)$ with $J_N(x, u_n \mid \ell) \leq V_N(x \mid \ell) + \epsilon_n$. Under Assumptions II.1, II.2, II.3, II.4, II.5, II.7, II.6, III.1, there exists $\underline{N} \in \mathbb{N}$ and $\nu \in \mathcal{L}_{\mathbb{N}_+}$ such that for all $N \geq \underline{N}$ and all $x \in \mathbb{X}$ we have $(x_{\mu_N}(n; x), u_{\mu_N}(n; x)) \in \mathbb{Z}$ for all $n \in \mathbb{N}_0$ and

$$\begin{aligned}&\limsup_{M \to \infty} \frac{1}{M} \sum_{m=0}^{M-1} \ell(x_{\mu_N}(m; x), u_{\mu_N}(m; x)) \\ &\leq \ell_e + \nu(N) + \limsup_{M \to \infty} \frac{1}{M} \sum_{m=0}^{M-1} \epsilon_m \qquad (18)\end{aligned}$$

Proof: Since $\mathbb{X}$ is viable, cf. Proposition III-B and Remark III.4, and $f(x, u(0)) \in \mathbb{X}$ for all $x \in \mathbb{X}$ and $u \in \mathbb{U}_{\mathbb{X}}^N(x)$, the first claim is clear.

Let now $N \geq \underline{N}$, $x \in \mathbb{X}$ arbitrary and define $x(m) = x_{\mu_N}(m; x)$ and $u(n) = x_{\mu_N}(m; x)$, and denote by $u_m(\cdot \mid x) \in \mathbb{U}_{\mathbb{X}}^N(x)$ the control sequence used for the construction of μ_N. We then have

$$\begin{aligned}\ell(x(m), u(m)) &= J_N(x(m), u_m(0 \mid x(m)) \mid \ell) \\ &\quad - J_{N-1}(x(m+1), u_m(\cdot + 1 \mid x(m)) \mid \ell) \\ &\leq V_N(x(m) \mid \ell) + \epsilon_m - V_{N-1}(x(m+1) \mid \ell) \\ &\leq V_N(x(m) \mid \ell) - V_N(x(m+1) \mid \ell) + \ell_e \\ &\quad + \nu(N-1) + \epsilon_m,\end{aligned}$$

where we used the ϵ_m-approximate optimality of $u_m(\cdot \mid x(m))$ and $u_m(\cdot + 1 \mid x(m)) \in \mathbb{U}_{\mathbb{X}}^{N-1}(x(m+1))$ in the first inequality, followed by Lemma III.6. We then get

$$\begin{aligned}\frac{1}{M} \sum_{m=0}^{M-1} \ell(x(m), u(m)) &\leq \frac{1}{M}(V_N(x \mid \ell) \\ &\quad - v_N(x(M) \mid \ell)) + \ell_e + \nu(N-1) + \frac{1}{M} \sum_{m=0}^{M-1} \epsilon_m\end{aligned}$$

and since $V_N(\cdot \mid \ell) \geq N\ell_e - B_\lambda$, the result follows. ☐

C. Practical asymptotic stability

Finally, we turn to stability of the closed-loop system. As is well-known, when using EMPC without terminal conditions, one cannot expect asymptotic stability, but rather practical asymptotic stability [9]. In the following, we rederive standard practical asymptotic stability results, but using local incremental stabilizability as the starting point. We start with a continuity result for the finite-horizon value function.

Lemma III.8. Under Assumptions II.6, III.1, II.1, there exists $\gamma_V \in \mathcal{K}_\infty$ such that for all $N \in \mathbb{N}_+$ and for all $x_1, x_2 \in \mathbb{X}$ with $\mathbb{U}_{\mathbb{X}}^N(x_1), \mathbb{U}_{\mathbb{X}}^N(x_2) \neq \emptyset$, $V_N(x_1 \mid \ell), V_N(x_2 \mid \ell) \in \mathbb{R}$ and $\|x_1 - x_2\|_X \leq \sqrt{\bar{C}_\delta^{-1} C_\delta^{\max}}$ we have

$$|V_N(x_1 \mid \ell) - V_N(x_2 \mid \ell)| \leq \gamma_V(\|x_1 - x_2\|_X) \qquad (19)$$

Proof: Let $N \in \mathbb{N}_+$ and $x_1, x_2 \in \mathbb{X}$ fulfilling the conditions of the lemma and let $\epsilon > 0$ be arbitrary. Choose $u_1 \in \mathbb{U}_{\mathbb{X}}^N(x_1)$ with $J_N(x, u_1 \mid \ell) \leq V_N(x_1 \mid \ell) + \epsilon$. Such a control sequence exists since $V_N(x_1 \mid \ell) > -\infty$ and $\mathbb{U}_{\mathbb{X}}^N(x_1) \neq \emptyset$ by assumption. Define $x_1(n) = x(n; x_1, u_1)$. Since $\|x_1 - x_2\|_X \leq \sqrt{\bar{C}_\delta^{-1} C_\delta^{\max}}$ and $x_2 \in \mathbb{X}$, we can use

Lemma III.2 to define

$$\tilde{x} = x_2$$
$$\tilde{u}(n) = \kappa_\delta(\tilde{x}(n), x_1(n), u_1(n))$$
$$\tilde{x}(n+1) = f(\tilde{x}(n), \tilde{u}(n)),\ n = 0, \dots, N-1$$

and get $\tilde{u} \in \mathbb{U}_{\mathbb{X}}^N(x_2)$. Now,

$$\begin{aligned} V_N(x_2 \mid \ell) &\leq J_N(x_2, \tilde{u} \mid \ell) \\ &\leq J_N(x_1, u_1 \mid \ell) + \gamma_\delta(\|x_1 - x_2\|_X) \\ &\leq V_N(x_1 \mid \ell) + \gamma_\delta(\|x_1 - x_2\|_X) + \epsilon, \end{aligned}$$

where we used Lemma III.2 again in the second inequality. Interchanging the roles of x_1 and x_2 shows that

$$|V_N(x_1 \mid \ell) - V_N(x_2 \mid \ell)| \leq \gamma_\delta(\|x_1 - x_2\|_X) + \epsilon,$$

and since $\epsilon > 0$ was arbitrary, the claim is established with $\gamma_V = \gamma_\delta$. □

Under some additional assumptions the previous result can be simplified.

Corollary III.9. Under Assumptions II.6, III.1, II.1, II.4, II.5 and II.7, there exists $\gamma_V \in \mathcal{K}_\infty$ such that for all $x_1, x_2 \in \mathbb{X}$ with $\|x_1 - x_2\|_X \leq \sqrt{\bar{C}_\delta^{-1} C_\delta^{\max}}$ we have

$$|V_N(x_1 \mid \ell) - V_N(x_2 \mid \ell)| \leq \gamma_V(\|x_1 - x_2\|_X) \quad (20)$$

Proof: Proposition III-B (cf. also Remark III.4) and its proof ensure that $\mathbb{U}_{\mathbb{X}}^N(x_i) \neq \emptyset$ and $V_N(x_i \mid \ell) < \infty$, $i = 1, 2$. Furthermore, from the proof of Lemma III.6 we get that $V_N(x_i \mid \ell) > -\infty$, so all the conditions of Lemma III.8 hold for all $x_1, x_2 \in \mathbb{X}$ with $\|x_1 - x_2\|_X \leq \sqrt{\bar{C}_\delta^{-1} C_\delta^{\max}}$. □

Let $s : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ be some function. We say that system (1) is dissipative w.r.t. supply rate s under the constraints $\mathbb{Z}$ if there exists a function $S : \mathbb{X} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{X}$, $N \in \mathbb{N}_+$, $u \in \mathbb{U}_{\mathbb{X}}^N(x)$ we have

$$S(x(N; x, u)) \leq S(x) + \sum_{n=0}^{N-1} s(x(n; x, u), u(n)). \quad (21)$$

Every such function S is called a storage function. Define now the available storage

$$S_a(x \mid s) = \sup_{\substack{N \in \mathbb{N} \\ u \in \mathbb{U}_{\mathbb{X}}^N(x)}} -\sum_{n=0}^{N-1} s(x(n; x, u), u(n)). \quad (22)$$

It is well-known that system (1) is dissipative w.r.t. supply rate s under constraints $\mathbb{Z}$ if and only if $S_a(x \mid \ell) < \infty$ for all $x \in \mathbb{X}$, and in this case $S_a(\cdot \mid s)$ is a valid storage function, cf. [11, Proposition 3.3]. It turns out that in our setting, we can easily show continuity of the available storage.

Proposition III.10. Let system (1) be dissipative w.r.t. storage function $s : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$. Assume that there exists $\gamma_s \in \mathcal{K}_\infty$ such that s fulfills Assumption II.1 (with s instead of ℓ) with γ_s instead of γ_ℓ. Under Assumptions II.6 and III.1, there exists then $\gamma_a \in \mathcal{K}_\infty$ such that for all $x_1, x_2 \in \mathbb{X}$ with $\|x_1 - x_2\|_X \leq \sqrt{\bar{C}_\delta^{-1} C_\delta^{\max}}$ we have

$$|S_a(x_1 \mid s) - S_a(x_2 \mid s)| \leq \gamma_a(\|x_1 - x_2\|_X). \quad (23)$$

Proof: Let $x_1, x_2 \in \mathbb{X}$ with $\|x_1 - x_2\|_X \leq \sqrt{\bar{C}_\delta^{-1} C_\delta^{\max}}$ and $\epsilon > 0$ be arbitrary. By construction, $S_a(\cdot \mid s) \geq 0$, and since the system is assumed to be dissipative, $S_a(x \mid s) < \infty$ for all $x \in \mathbb{X}$. Therefore, there exists $N_1 \in \mathbb{N}$ and $u_1 \in \mathbb{U}_{\mathbb{X}}^{N_1}(x_1)$ with $J_N(x, u_1 \mid -s) + \epsilon \geq S_a(x_1 \mid s)$. If $N_1 > 0$, then use Lemma III.2 with $\tilde{x}(0) = x_2$ and $x = x_1$, $u = u_1$ as well as s and γ_s instead of ℓ and γ_ℓ, resulting in $\tilde{u} \in \mathbb{U}_{\mathbb{X}}^{N_1}(x_2)$. Now,

$$\begin{aligned} S_a(x_1 \mid s) &\leq J_{N_1}(x_1, u_1 \mid -s) + \epsilon \\ &\leq J_{N_1}(x_2, \tilde{u} \mid -s) + \epsilon \\ &\quad + \gamma_s^{\rho_\delta}((\sqrt{\underline{C}_\delta^{-1} \bar{C}_\delta} + \sqrt{C_\kappa \bar{C}_\delta}) \|x_1 - x_2\|_X) \\ &\leq S_a(x_2 \mid s) + \gamma_a(\|x_1 - x_2\|_X) + \epsilon, \end{aligned}$$

where we used Lemma III.2 in the second inequality as well as the definition of $S_a(\cdot \mid s)$ and $\gamma_a(r) = \gamma_s^{\rho_\delta}((\sqrt{\underline{C}_\delta^{-1} \bar{C}_\delta} + \sqrt{C_\kappa \bar{C}_\delta}) r)$. If $N_1 = 0$, then we have trivially $S_a(x_1 \mid s) = 0 \leq S_a(x_2 \mid s) + \epsilon$. Interchanging the roles of x_1 and x_2 shows that

$$|S_a(x_1 \mid s) - S_a(x_2 \mid s)| \leq \gamma_a(\|x_1 - x_2\|_X) + \epsilon,$$

and since $\epsilon > 0$ was arbitrary, the claim follows. □

It is well-known that in the discrete-time case, dissipativity w.r.t. s is equivalent to the following, cf. [12, Section 4.6]. There exists a function $\lambda : \mathbb{X} \to \mathbb{R}$, bounded from below, such that for all $(x, u) \in \mathbb{Z}$ with $f(x, u) \in \mathbb{X}$ we have

$$\lambda(f(x, u)) \leq \lambda(x) + s(x, u). \quad (24)$$

We hence see that Assumption II.4 is equivalent to dissipativity of (1) w.r.t. the supply rate $s_\ell(x, u) = \ell(x, u) - \ell_e - \rho(\|x\|_e)$, and $\underline{\lambda}(x) = \lambda(x) + B$ is a corresponding storage function, where $B \in \mathbb{R}_{\geq 0}$ is any lower bound on λ. Since we can use any valid λ in Assumption II.4, we can use the available storage instead of λ. The next result then establishes (local) continuity. However, we first need an additional technical assumption.

Assumption III.11. In the situation of Assumption II.4, let $\underline{\rho} \in \mathcal{K}_\infty$ such that

$$\underline{\rho}(r) \leq \rho(r)\ \forall r \in \mathbb{R}_{\geq 0} \quad (25)$$
$$|\underline{\rho}(r) - \underline{\rho}(r')| \leq \gamma_\rho(|r - r'|)\ \forall r, r' \in \mathbb{R}_{\geq 0} \quad (26)$$
$$\forall \xi \in (0, 1) \exists \gamma_\rho^\xi \in \mathcal{K}_\infty : \sum_{n=0}^{\infty} \underline{\rho}(\xi^n r) \leq \gamma_\rho^\xi(r)\ \forall r \in \mathbb{R}_{\geq 0}, \quad (27)$$

where $\gamma_\rho \in \mathcal{K}_\infty$.

Lemma III.12. Under Assumptions II.4 and III.11, the system is dissipative w.r.t. supply rate $\tilde{s}_\ell(x, u) = \ell(x, u) - \ell_e - \underline{\rho}(\|x\|_e)$ and $\tilde{\lambda}(x) = S_a(x \mid \tilde{s}_\ell)$ is a storage function. In particular, Assumption II.4 is fulfilled with $\tilde{\lambda}$ and $\underline{\rho}$ instead of λ and ρ. If additionally Assumption II.6, III.1, II.1 hold, then there exists $\gamma_\lambda \in \mathcal{K}_\infty$ such that for all $x_1, x_2 \in \mathbb{X}$ with $\|x_1 - x_2\|_X \leq \sqrt{\bar{C}_\delta^{-1} C_\delta^{\max}}$ we have

$$|\tilde{\lambda}(x_1) - \tilde{\lambda}(x_2)| \leq \gamma_\lambda(\|x_1 - x_2\|_X). \quad (28)$$

Proof: The first part follows immediately from the definition of dissipativity and the discussion preceding this result. Observe that for all $(x_1,u_1),(x_2,u_2)\in\mathbb{Z}$ we have

$$
\begin{aligned}
|\tilde{s}_\ell(x_1,u_1)-\tilde{s}_\ell(x_2,u_2)| &\le |\ell(x_1,u_1)-\ell(x_2,u_2)| \\
&\quad + |\underline{\rho}(\|x_1\|_2)-\underline{\rho}(\|x_2\|_e)| \\
&\le \gamma_\ell(\|x_1-x_2\|_X+\|u_1-u_2\|_U)+\gamma_\rho(|\|x_1\|_e-\|x_2\|_e|) \\
&\le \gamma_\ell(\|x_1-x_2\|_X+\|u_1-u_2\|_U)+\gamma_\rho(\|x_1-x_2\|_X) \\
&\le (\gamma_\ell+\gamma_\rho)(\|x_1-x_2\|_X+\|u_1-u_2\|_U) \\
&= \gamma_s(\|x_1-x_2\|_X+\|u_1-u_2\|_U),
\end{aligned}
$$

where we used $|\|x_1\|_e-\|x_2\|_e|\le\|(x_1-x_e)-(x_2-x_e)\|_X=\|x_1-x_2\|_X$ in the third inequality. Furthermore, we have for all $\xi\in(0,1)$ and $r\in\mathbb{R}_{\ge 0}$

$$
\begin{aligned}
\sum_{n=0}^{\infty}\gamma_s(\xi^n r) &\le \sum_{n=0}^{\infty}\gamma_\ell(\xi^n r)+\gamma_\rho(\xi^n r) \\
&\le \gamma_\ell^\xi(r)+\gamma_\rho^\xi(r)=\gamma_s^\xi(r).
\end{aligned}
$$

The claim now follows from Proposition III.10. □

We now establish properties of the value function for the rotated stage cost ℓ_{rot}.

Lemma III.13. Under Assumptions II.4, II.5 and III.11, there exists $B_a\in\mathbb{R}_{\ge 0}$ such that $0\le S_a(x\mid\tilde{s}_\ell)\le B_a$ for all $x\in\mathbb{X}$.

Proof: Let λ be the function from Assumption II.4, then $\underline{\lambda}=\lambda+B$, where B is any lower bound on λ, is a storage function for s_ℓ. Observe now that any storage function w.r.t. the supply rate s_ℓ is also a storage function w.r.t. the supply rate $\tilde{s}_\ell$. Furthermore, since $|\lambda(x)|\le B_\lambda$ for all $x\in\mathbb{X}$ according to Assumption II.5, we have $|\underline{\lambda}(x)|\le B_\lambda+\max\{0,-B\}=B_a$. It is well known that $S_a(\cdot\mid\tilde{s}_\ell)$ is a lower bound for all storage functions w.r.t. the supply rate $\tilde{s}_\ell$, so we get that $S_a(x\mid\tilde{s}_\ell)\le\underline{\lambda}(x)\le B_a$ for all $x\in\mathbb{X}$, establishing the result. □

Since we need (local) continuity of λ, **from now on we assume that** $\lambda(x)=S_a(x\mid\tilde{s}_\ell)$, so that the preceding results are applicable. We can use this strategy since in the definition of dissipativity, only the existence of a storage function is required, so we can simply choose a convenient storage function.

Next, we establish bounds of the finite-horizon value function for the rotated stage cost.

Lemma III.14. Under Assumptions II.6, III.1, II.1, II.2, II.4, III.11, II.7, there exists $\underline{\tilde{N}}\in\mathbb{N}_+$, $\underline{\alpha}_{\tilde{V}},\bar{\alpha}_{\tilde{V}}\in\mathcal{K}_\infty$ such that for all $N\ge\underline{\tilde{N}}$ and $x\in\mathbb{X}$ we have

$$
\underline{\alpha}_{\tilde{V}}(\|x\|_e)\le V_N(x\mid\ell_{\text{rot}})\le\bar{\alpha}_{\tilde{V}}(\|x\|_e). \tag{29}
$$

Proof: We start with the upper bound. The proof is similar to the one of [9, Theorem 3.3], but adapted to the present situation. Define $\underline{\tilde{N}}=N_{\text{CR}}+\lceil 1/2\log(\rho_\delta)\log(\bar{C}_\delta^{-1}\underline{C}_\delta)\rceil$ and let $x\in\mathbb{X}$ be arbitrary.

Case $\|x\|_e\le\sqrt{\bar{C}_\delta^{-1}C_\delta^{\max}}$: In this case we can apply Lemma III.2 for $\tilde{x}(0)=x$ and $x=x_e$, $u\equiv u_e$, resulting in $(\tilde{x}(n),\tilde{u}(n))\in\mathbb{Z}$ for $n=0,\dots,N-1$ and $\tilde{x}(N)\in\mathbb{X}$. We then have

$$
\begin{aligned}
V_N(x\mid\ell_{\text{rot}}) &\le J_N(x,\tilde{u}\mid\ell_{\text{rot}}) \\
&= J_N(x,\tilde{u}\mid\ell)-N\ell_e+\lambda(x)-\lambda(\tilde{x}(N)) \\
&\le \gamma_\delta(\|x\|_e)+|\lambda(x)-\lambda(x_e)|+|\lambda(\tilde{x}(N))-\lambda(x_e)| \\
&\le \gamma_\delta(\|x\|_e)+\gamma_\lambda(\|x\|_e)+\gamma_\lambda(\sqrt{\underline{C}_\delta^{-1}\bar{C}_\delta\rho_\delta^N}\|x\|_e) \\
&= \tilde{\alpha}(\|x\|_e),
\end{aligned}
$$

where we used $\tilde{u}\in\mathbb{U}_{\mathbb{X}}^N(x)$ in the first inequality, Lemma III.2 in the second inequality, and then again Lemma III.2 as well as Lemma III.12. Note that since the choice of N implies that $\|\tilde{x}(N)\|_e\le\sqrt{\bar{C}_\delta^{-1}C_\delta^{\max}}$, hence Lemma III.2 has been applicable.

Case $\|x\|_e>\sqrt{\bar{C}_\delta^{-1}C_\delta^{\max}}$: Choose $u_x\in\mathbb{U}_{\mathbb{X}}^{N_e}(x)$ from Assumption II.7 and set $\tilde{x}=x(N_e;x,u_x)\in\mathbb{X}$. Note that

$$
\begin{aligned}
\ell_{\text{rot}}(x,u) &= \ell(x,u)-\ell_e+\lambda(x)-\lambda(f(x,u)) \\
&\le B_\ell-\ell_e+2B_a=\tilde{B}_\ell,
\end{aligned}
$$

where B_a is the bound from Lemma III.13. This shows that $J_{N_e}(x,u_x\mid\ell_{\text{rot}})\le N_e\tilde{B}_\ell$. Next, use Lemma III.2 for $\tilde{x}(0)=\tilde{x}$ and $x=x_e$, $u\equiv u_e$ to get $\tilde{u}\in\mathbb{U}_{\mathbb{X}}^{N-N_e}(\tilde{x})$. Similarly to the first case, we get $J_{N-N_e}(\tilde{x},\tilde{u}\mid\ell_{\text{rot}})\le\tilde{\alpha}(\|\tilde{x}\|_e\le\tilde{\alpha}(R_e)$, where we used that $\|\tilde{x}(N-N_e)\|_e\le\sqrt{\bar{C}_\delta^{-1}C_\delta^{\max}}$ by choice of N. Define now $\hat{u}=(u_x\tilde{u})$, then the preceding developments show that $\hat{u}\in\mathbb{U}_{\mathbb{X}}^N(x)$ and hence

$$
\begin{aligned}
V_N(x\mid\ell_{\text{rot}}) &\le J_N(x,\hat{u}\mid\ell_{\text{rot}}) \\
&= J_{N_e}(x,u_x\mid\ell_{\text{rot}})+J_{N-N_e}(\tilde{x},\tilde{u}\mid\ell_{\text{rot}}) \\
&\le N_e\tilde{B}_\ell+\tilde{\alpha}(R_e)=\tilde{C}.
\end{aligned}
$$

Summarizing, $V(x\mid\ell_{\text{rot}})\le\tilde{\alpha}(\|x\|_e)$ for $\|x\|_e\le\sqrt{\bar{C}_\delta^{-1}C_\delta^{\max}}$, $V(x\mid\ell_{\text{rot}})\le\tilde{C}$ for $\|x\|_e>\sqrt{\bar{C}_\delta^{-1}C_\delta^{\max}}$, so the upper bound follows with $\bar{\alpha}_{\tilde{V}}(r)=\max\{1,\tilde{C}\tilde{\alpha}(R_e)^{-1}\}\tilde{\alpha}(\|x\|_e)$.

We now turn to the lower bound. Let $N\ge\underline{\tilde{N}}$ and $x\in\mathbb{X}$ be arbitrary. Proposition III-B shows under the present assumptions, mutatis mutandis, that $\mathbb{U}_{\mathbb{X}}^N(x)\ne\emptyset$. Let $u\in\mathbb{U}_{\mathbb{X}}^N(x)$, then we get

$$
V_N(x\mid\ell_{\text{rot}})\ge J_N(x,u\mid\ell_{\text{rot}})\ge\ell_{\text{rot}}(x,u(0))\ge\underline{\rho}(\|x\|_e),
$$

where we used $u\in\mathbb{U}_{\mathbb{X}}^N(x)$ in the first inequality and then $\ell_{\text{rot}}(x,u)\ge\underline{\rho}(\|x\|_e\ge 0$. □

Finally, we have the following continuity result for finite-horizon value function for the rotated stage cost.

Lemma III.15. Under Assumptions II.1, II.6, III.1, II.4, III.11, there exists $\underline{\tilde{N}}_\gamma\in\mathbb{N}_+$ and $\gamma_{\tilde{V}}\in\mathcal{K}_\infty$ such that for all $N\ge\underline{\tilde{N}}_\gamma$ and $x_1,x_2\in\mathbb{X}$ with $\|x_1-x_2\|_X\le\sqrt{\bar{C}_\delta^{-1}C_\delta^{\max}}$ and $\mathbb{U}_{\mathbb{X}}^N(x_1),\mathbb{U}_{\mathbb{X}}^N(x_2)\ne\emptyset$, we have

$$
|V_N(x_1\mid\ell_{\text{rot}})-V_N(x_2\mid\ell_{\text{rot}})|\le\gamma_{\tilde{V}}(\|x_1-x_2\|_X). \tag{30}
$$

Proof: Define $\underline{\tilde{N}}_\gamma=\lceil 1/2\log(\rho_\delta)\log(\bar{C}_\delta^{-1}\underline{C}_\delta)\rceil$ and let $N\ge\underline{\tilde{N}}_\gamma$ and $x_1,x_2\in\mathbb{X}$ with $\|x_1-x_2\|_X\le\sqrt{\bar{C}_\delta^{-1}C_\delta^{\max}}$

and $\mathbb{U}^N_{\mathbb{X}}(x_1), \mathbb{U}^N_{\mathbb{X}}(x_2) \neq \emptyset$, and $\epsilon > 0$ be arbitrary. Observe that Lemma III.14 is applicable, hence $0 \leq V_N(x_i \mid \ell_{\text{rot}}) < \infty$, $i = 1, 2$. Choose now $u_2 \in \mathbb{U}^N_{\mathbb{X}}(x_2)$ with $J_N(x_2, u_2 \mid \ell_{\text{rot}}) \leq V_N(x_2 \mid \ell_{\text{rot}}) + \epsilon$ and define $x_2(n) = x(n; x_2, u_2)$. We can now use Lemma III.2 to construct

$$\begin{aligned}\tilde{x}(0) &= x_1\\ \tilde{u}(n) &= \kappa_\delta(\tilde{x}(n), x_2(n), u_2(n))\\ \tilde{x}(n+1) &= f(\tilde{x}(n), \tilde{u}(n)),\ n = 0, \dots, N-1\end{aligned}$$

with $\tilde{u} \in \mathbb{U}^N_{\mathbb{X}}(x_1)$. Now,

$$\begin{aligned}&V_N(x_1 \mid \ell_{\text{rot}}) \leq J_N(x_1, \tilde{u} \mid \ell_{\text{rot}})\\ &= J_N(x_1, \tilde{u} \mid \ell) - N\ell_e + \lambda(x_1) - \lambda(\tilde{x}(N))\\ &\leq J_N(x_2, u_2 \mid \ell) - N\ell_e + \lambda(x_2) - \lambda(x_2(N))\\ &\quad + |\lambda(x_1) - \lambda(x_2)| + |\lambda(x_2(N)) - \lambda(\tilde{x}(N))|\\ &\quad + \gamma_\delta(\|x_1 - x_2\|_X)\\ &\leq J_N(x_2, u_2 \mid \ell_{\text{rot}}) + \gamma_\lambda(\|x_1 - x_2\|_X)\\ &\quad + \gamma_\lambda(\sqrt{\underline{C}_\delta^{-1}\bar{C}_\delta}\rho_\delta^N\|x_1 - x_2\|_X) + \gamma_\delta(\|x_1 - x_2\|_X)\\ &\leq V_N(x_2 \mid \ell_{\text{rot}}) + \epsilon + \gamma_{\tilde{V}}(\|x_1 - x_2\|_X),\end{aligned}$$

where we used Lemma III.2 in the second inequality, Lemma III.12 in the second and third inequality (note that due to the choice of $\underline{\tilde{N}}_\gamma$ it is applicable), and finally the choice of u_2 and $\gamma_{\tilde{V}}(r) = \gamma_\lambda(r) + \gamma_\lambda(\sqrt{\underline{C}_\delta^{-1}\bar{C}_\delta}\rho_\delta^N r) + \gamma_\delta(r)$. Interchanging the roles of x_1 and x_2 shows then

$$|V_N(x_1 \mid \ell_{\text{rot}}) - V_N(x_2 \mid \ell_{\text{rot}})| \leq \gamma_{\tilde{V}}(\|x_1 - x_2\|_X) + \epsilon,$$

and since $\epsilon > 0$ was arbitrary, the result follows. □

After the preceding auxiliary results, an inspection of the proof of [1, Proposition 8.32] reveals that this result applies to the present situation and hence we get practical asymptotic stability. Since this is a standard result that immediately applies to our situation, we omit the details.

IV. Conclusion

We have been able to rederive from the assumption of local incremental stabilizability most of the essential analysis results for EMPC without terminal conditions. Additionally, we also provided some novel continuity results for the storage function of dissipative systems in the context of EMPC. By building on local incremental stabilizability, we connected the analysis of EMPC schemes to the theory of RMPC, where this notion is playing an increasing important role. Furthermore, this is particularly interesting since incremental notions have received considerable attention in the control community lately [13], [14]. A potential limitation of the present work is the incremental CPI condition in Assumption III.1. While CPI assumptions are standard in nonlinear NMPC [1], it is unclear whether this assumption can be a significant limitation in the incremental context. Therefore, ongoing work is concerned with relaxing the incremental CPI condition, for which we are exploring artificial tightening approaches, along the lines of [5]. Interestingly, this brings the analysis of EMPC schemes without terminal conditions and RMPC even closer together.

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