Minimal water consumption for a crop fertirrigation model

M.G. Dadjo 1,3, A. Rapaport 1, J. Harmand 2, R. Ushirobira 3 and D. Efimov 3

Abstract — In this work, we consider a simplified model of crop fertirrigation as a non-autonomous controlled system, with soil moisture, nitrogen content, and biomass as state variables and the delivered water flow rate as input. We study the problem of minimizing the total water quantity delivered during the agricultural season under the constraints that the crops are not suffering from water or nitrogen stress at any time. We establish sufficient conditions for the feasibility of the problem and depict several control strategies depending on the initial nitrogen content. In particular, we show that this problem can exhibit an infinity of singular trajectories of the same cost.

I. INTRODUCTION

In many regions of the world, and especially in the arid ones, agricultural production faces water scarcity, leading to difficulties in satisfying food population needs. Crops irrigation with treated wastewater, instead of fresh water, is a solution for preserving water resources. This practice increasingly attracts decision-makers attention, but some essential technical and sanitary issues (such as time-varying fluctuations, pipes clogging, and undesirable nano-particles or pathogens) must be addressed.

The underlying idea is to preserve, at the water treatment step, the nutrients that are beneficial for crop growth, notably nitrogen, so that no additional nutrients, which may spread and contaminate soils, have to be brought. In this regard, irrigation with reused water amounts to fertirrigation (i.e., irrigation with nutrients supplied in water). In the present work, we propose to address the problem of optimizing crop production via fertirrigation. Although many works about optimization with classical irrigation can be found in the literature [17], [15], [14], [10], [2], and several commercial software are available on the market [4], [13], the optimization of fertirrigation has been comparatively much less considered, apart some recent works [11], [9], [7].

With classical irrigation that covers the crop’s water needs during the whole agricultural season, the soil’s nitrogen concentration can be diluted to the point that it no longer meets the crop demands if the initial nitrogen content is insufficient. This often causes farmers to supply nitrogen at the beginning of the season with the risk of spreading unnecessary quantities of nitrogen in soils. Bringing nitrogen with water appears to be a safe solution, provided that its concentration is appropriate and does not end up with too much nitrogen in the soil at the end of the season.

Here also, one faces a dilemma between nitrogen dilution and water supply, which we will find in the optimization problem. In the former works [11], [9], [7], numerical solutions have been proposed but there was not yet mathematical analysis of optimal control under constraints, as we do here.

The outline of this work is as follows. The considered model of the crop is introduced in Section II. The constrained optimization problem is formulated in Section III, along with the analysis of conditions of the existence of solutions. The structure of the optimal solutions is investigated in Section IV. Some numeric illustrations and discussions are given in Section V.

II. THE FERTIRRIGATION MODEL

Following existing literature [16], [12] and former works [8], [2], [7], we consider a simplified crop model in greenhouse suitable for decision-making, where we explicitly consider a nitrogen compartment. The dynamics includes soil water balance, soil nitrogen balance, and biomass production as follows:

\[ \dot{S}(t) = \frac{K_S}{S(t) - b} \left( -\varphi(t) K_S(S(t)) - (1 - \varphi(t)) K_R(S(t)) + k_2 u(t) \right) \]  

\[ \dot{N}(t) = -k_3 \varphi(t) K_S(S(t)) f \left( \frac{N(t)}{S(t)} \right) + k_4 C_N^u u(t) \]

\[ \dot{B}(t) = k_5 \varphi(t) K_S(S(t)) f \left( \frac{N(t)}{S(t)} \right) \]

where \( S \) denotes the soil humidity level (between 0 and 1), \( N \) and \( B \) (in \( \mathbb{R}_+ \)) are the nitrogen content and the biomass per unit of soil surface, and \( k_i, i = 1, \ldots, 5 \) are positive constants. The time function \( \varphi \) is the crop radiation interception efficiency related to the canopy cover, which is supposed to be \( C^1 \) and increasing during the season period \([0, T]\) with values in \([0, 1]\), where times 0 and \( T \) correspond to seeding and harvesting dates. The terms \( \varphi(t) K_S(S(t)) \) and \( (1 - \varphi(t)) K_R(S(t)) \) represent the crop transpiration and soil evaporation at time \( t \) depending on the water content \( S \). The functions \( K_S, K_R, \) and \( f \) are piecewise continuous functions whose classical forms can be found in the agronomic literature:

\[ K_S(S) = \begin{cases} 
0, & S \in [0, S_w] \\
\frac{S - S_w}{S - S^*}, & S \in [S_w, S^*] \\
1, & S \in [S^*, 1]
\end{cases} \]

\[ K_R(S) = \begin{cases} 
0, & S \in [0, S_h] \\
\frac{S - S_h}{1 - S^*}, & S \in [S_h, 1]
\end{cases} \]
where $0 < S_h < S_w < S^* < 1$,
\[ f(C_N) = \begin{cases} \frac{C_N}{\eta_C}, & C_N \in [0, \eta_C] \\ 1, & C_N > \eta_C > 0 \end{cases} \]
where numbers $S_h$, $S_w$, $S^*$, $\eta_C$ describe thresholds corresponding to change of modes in the growth. In particular, the crop suffers from hydric stress when the value of the function $K_2$ is not maximal (that is, when $S < S^*$), and from nitrogen stress when the value of the function $f$ is not maximal (that is when $N < \eta_C S$). Parameters $C_N^0$ (concentration of nitrogen in the delivered water) and $u_{\max}$ (maximum water flow) are the operating characteristics of the fertirrigation system. The control $u$ is the flow rate of the delivered water. Given an initial value $S(0) = S_0$, a function $u$ is admissible for $S_0$ if it is Lebesgue measurable on $[0, T]$ with values in $[0, u_{\max}]$ and the corresponding solution $S$ remains in $[0, 1]$, which amounts to imposing the constraint on the control
\[ S = 1 \Rightarrow u \leq \frac{1}{k_2}. \]

From the agronomic viewpoint, this means that when the soil is saturated, i.e., $S = 1$, the irrigation has to be regulated to avoid water loss.

The authors have already considered this model to estimate in real-time simultaneously the model parameters and the nitrogen content of the soil ($N$), while measuring the humidity ($S$) and the produced biomass ($B$), for any persistently exciting control $u(\cdot)$ [5].

### III. THE OPTIMIZATION PROBLEM

From (3), one can see that the biomass production at time $T$ is maximal if one has
\[ K_2(S(t)) f \left( \frac{N(t)}{S(t)} \right) = 1, \quad t \in [0, T] \]
that is, when there is no water or nitrogen stress during the whole season. That amounts to having trajectories lying in the subset defined by the two constraints
\[ E := \{ (S, N) \in [0, 1] \times \mathbb{R}_+, S \geq S^* \geq N \geq \eta_C S \}. \]

We shall look for irrigation strategies to minimize the total quantity of delivered water while ensuring maximal biomass production. This can be formulated via the following constrained optimization problem:

**Problem $\mathcal{P}$:** For $(S_0, N_0) \in E$ and $U(S_0)$ the set of admissible controls $u$ for $S_0$, we seek
\[
\inf_{u \in U(S_0)} \int_0^T u(\tau) d\tau,
\]
subject to the state constraint
\[ (S_0^u, S_0^w, N_0^u, S_0^w, N_0^w) \in E, \quad t \in [0, T] \]
where $(S_0^u, S_0^w, N_0^u, S_0^w, N_0^w)$ denote the solution of (1)-(2) with $S(0) = S_0$, $N(0) = N_0$.

We give here sufficient conditions for the problem $\mathcal{P}$ to be feasible. We have chosen the viability approach [1] which consists in showing the the domain $E$. Another (equivalent) approach based on projected dynamical systems [3] could have been used.

**Definition 3.1:** The domain $E$ is viable if for any initial condition $(t_0, S_0, N_0) \in [0, T] \times E$, there exists an admissible control $u \in U(S_0)$ such that the solution of (1)-(2) with $S(t_0) = S_0$, $N(t_0) = N_0$ verifies $(S(t), N(t)) \in E$ for any $t \in [t_0, T]$.

For convenience, we define the numbers
\[ C_1 = \eta_C k_1 - k_3, \quad C_2 = k_4 C_N^{\text{in}} - \eta_C k_1 k_2. \]

**Proposition 3.2:** The domain $E$ is viable in the sense of Definition 3.1 if the condition
\[ C_N^0 \geq C_N^{\text{in}} := \max \left\{ \frac{k_2}{k_4} \max \{ \eta_C k_1 (1 - K_R(S^*)), k_3 \} \right\} \]
and $u_{\max} \geq u_{\max}^{\text{max}}$ with
\[ u_{\max}^{\text{max}} := \left\{ \begin{array}{ll} \max \left\{ \frac{1}{k_2}, \frac{-C_1}{C_2} \right\} & \text{if } C_1 < 0 \text{ and } C_2 > 0, \\ \frac{1}{k_2} & \text{otherwise.} \end{array} \right. \]

are fulfilled.

**Proof:** At $S = S^*$, one has from equation (1)
\[ \dot{S} = k_1 (-\varphi - (1 - \varphi)) K_R(S^*) + k_2 u. \]

A necessary condition to have $\dot{S}(t) \geq 0$ is that $u(t) \in [0, u_{\max}]$ satisfies
\[ u(t) \geq \frac{\varphi(t) + (1 - \varphi(t)) K_R(S^*)}{k_2}. \]

for any possible $t \in [0, T]$, which implies the condition
\[ u_{\max}^{\text{max}} \geq \max_{t \in [0, T]} \frac{\varphi(t) + (1 - \varphi(t)) K_R(S^*)}{k_2} = \frac{1}{k_2}. \]

When $N = \eta_C S$, one has from equations (1)-(2)
\[ \dot{N}(t) - \eta_C \dot{S}(t) = k_1 (\varphi(t) + (1 - \varphi(t)) K_R(S(t))) - k_4 \varphi(t) + (k_4 C_N^{\text{in}} - \eta_C k_1 k_2) u(t). \]

Then, the condition ensuring $\dot{N} - \eta_C \dot{S} \geq 0$ is that for any $(t, S) \in [0, T] \times [S^*, 1]$, there exists $u$ with $u(t) \in [0, u_{\max}]$ such that
\[ (k_1 \eta_C - k_3) \varphi(t) + k_1 \eta_C (1 - \varphi(t)) K_R(S(t)) + (k_4 C_N^{\text{in}} - \eta_C k_1 k_2) u(t) \geq 0 \]
which is implied by
\[ \min \left\{ k_1 \eta_C - k_3, k_1 \eta_C K_R(S^*) \right\} + (k_4 C_N^{\text{in}} - \eta_C k_1 k_2) u(t) \geq 0, \]
that is,
\[ \min \{ C_1, k_1 \eta_C K_R(S^*) \} + C_2 u(t) \geq 0. \]

At $S = 1$ one should have also $\dot{S} \leq 0$, that is $u \leq \frac{1}{k_2}$, along with
\[ \min \{ C_1, k_1 \eta_C \} + C_2 u(t) \geq 0. \]
Let us distinguish cases depending on the signs of $C_1, C_2$.

i. If $C_1 < 0$ and $C_2 \leq 0$, clearly condition (7) cannot be fulfilled for a non-negative $u$.

ii. If $C_1 < 0$ and $C_2 > 0$, one has to have

$$k_2 C_1 + C_2 \geq 0$$

for condition (8) to be fulfilled with $u \leq \frac{1}{k_2}$. Condition (7) implies that $u_{\text{max}}$ satisfies

$$u_{\text{max}} \geq \frac{-C_1}{C_2} > 0$$

and combining with condition (6)

$$u_{\text{max}} \geq \max \left\{ \frac{1}{k_2}, \frac{-C_1}{C_2} \right\}$$

This last condition is compatible with condition (6) if

$$k_2 \min \left( C_1, k_1\eta C K_R(S^*) \right) + C_2 \geq 0$$

So condition (8) with $u \leq \frac{1}{k_2}$ is necessarily satisfied. Finally, $u_{\text{max}}$ has simply to satisfy condition (6).

Note that conditions for the set $E$ to be viable in case ii with (9), or case iii, or case iv with (10), are satisfied when the single condition (10) is satisfied. Equivalently, this condition is fulfilled if and only if (4) is verified. Finally, the upper bound of $u_{\text{max}}$ has to fulfill condition (5) depending on the case ii, iii or iv.

IV. ANALYSIS OF THE OPTIMAL SOLUTIONS

We assume the following conditions

$$C_{\text{in}}^n \geq C_{\text{in}}^r, \quad u_{\text{max}} \geq u_{\text{max}}$$

are fulfilled, so that the set $E$ is viable, accordingly to Proposition 3.2, and Problem $\mathcal{P}$ is feasible for any $(S_0, N_0)$ in $E$. For simplicity, we shall consider initial conditions with $S_0 = 1$ (and thus $N_0 \geq \eta_C$) only, which are often met in practice (assuming the soil humidity $S$ to be maximal at the beginning of the agronomic season). Let us begin by highlighting the next property:

Lemma 4.1: If there exists an admissible control such that the solution remains in $E$ and verifies $N(T) = \eta_C S^*$, then it is optimal.

Proof: From equation (2), solutions in the set $E$ satisfy

$$N(t) = N_0 - k_3 \int_0^t \varphi(\tau) d\tau + k_4 C_{\text{in}}^n \int_0^t u(\tau) d\tau, \quad t \in [0, T].$$

Therefore, minimizing $\int_0^T u(\tau) d\tau$ amounts to minimizing $N(T)$ among all solutions in $E$. In particular, if there exists an admissible control such that the solution remains in $E$ and verifies $N(T) = \eta_C S^*$, which is the smallest value of $N$ in the set $E$, then this control is necessarily optimal.

The optimal control problem $\mathcal{P}$ has been solved numerically using the software Bocop [18]. We have observed that the optimal solution always saturates the constraints (in the sense that along an optimal solution there exists a time $t$ such that one has $S(t) = S^*$ or $N(t) = \eta_C S(t)$), possibly at the final time only, provided that the time horizon $T$ is sufficiently large. This has guided us to distinguish several optimal strategies depending on which constraint is saturated. Sketches of proofs of optimality of all these strategies are given in the report [6].

A. The S-strategy

Definition 4.2: We will call $S$-strategy the following control input:

$$u_S(t, S) := \begin{cases} 0, & S(t) > S^*, \\ u_{\text{sing}}^S(t), & S(t) = S^* \end{cases}$$

where $u_{\text{sing}}^S(t) := \frac{1}{k_2} (\varphi(t) + (1 - \varphi(t)) K_R(S^*))$

Note that under condition (5), this control takes values in $[0, u_{\text{max}}]$ and is thus admissible. This strategy consists of no irrigation until the humidity level reaches the threshold $S^*$ (if possible), and then maintaining the humidity level constant at the value $S^*$ up to the final time. Let $t_S$ be the hitting time of the set $\{S = S^*\}$:

$$t_S := \sup \{ t \in [0, T] \mid S(t) > S^* \} = S_0^0(t_S), \quad S_S := S_0^0(t_S).$$

From this time instant, using the control $u_{\text{sing}}^S$, one obtains for $t \in [t_S, T]$

$$\dot{N} - \eta_C S = \dot{N} = \left( C_1 + \frac{C_2}{k_2} \right) (\varphi + (1 - \varphi) K_R(S^*))$$

where

$$C_1 + \frac{C_2}{k_2} = \frac{k}{k_2} C_{\text{in}}^n - k_3$$

is non-negative under the condition $C_{\text{in}}^n \geq C_{\text{in}}^r$. Therefore, one has the property

$$\dot{N}(t) - \eta_C S(t) = \dot{N}(t) \geq 0, \quad t \in [t_S, T]$$

with the $S$-strategy, which is feasible if the nitrogen constraint $N \geq \eta_C S$ is not violated before time $t_S$. That amounts to have

$$N_0 \geq N_0^s := \max \left\{ \eta_C, \eta_C S^* + k_3 \int_0^{t_S} \varphi(t) dt \right\}.$$

Remark 4.3: From equations (1)-(2), one gets

$$\dot{N} - \eta_C S = C_1 \varphi + (1 - \varphi) k_1 \eta_C K_R(S) + C_2 u.$$
B. The NS-strategy

We generalize the former S-strategy when condition (12) is not fulfilled with \( N_0 < N^*_0 \) (i.e., when \( S > S^* \)).

Definition 4.4: We will call NS-strategy the following control input:

\[
u_{NS}(t, S, N) = \begin{cases} 
0, & S > \frac{N}{\eta C}, \\
\max \{0, u_N^{\text{sing}}(t, S)\}, & S = \frac{N}{\eta C} > S^*, \\
u^{\text{sing}}(t), & S = S^*
\end{cases}
\]

with \( u_N^{\text{sing}}(t, S) := \frac{C_1 \varphi(t) + k_1 \eta C K_R(S(t))(1 - \varphi(t))}{-C_2} \)

and \( u_S^{\text{sing}} \) is as before.

Note that when (12) is not satisfied, one has \( C_1 < 0 \) and condition (11) gives \( C_2 \geq -k_2 C_1 > 0 \). Then for any \( (t, S) \in [0, T] \times [0, 1] \), one has \( u^{\text{sing}}(t, S) \leq \frac{C_1}{C_2} \), which is upper bounded by \( u_{\text{max}} \) with condition (5). The control \( u_{NS} \) is thus admissible.

This strategy consists of maintaining the humidity level equal to \( N_0 \) and continuously irrigating from the beginning until time \( t_c \), when one stops the irrigation up to the final time \( T \). Let us define

\[
N_0^\dagger := \eta C S^*_S + k_3 \int_0^T \varphi(t) dt - k_4 C_N^n \bar{u} T_c
\]

where \( L_c := \inf \{t \in [0, T] | S(t, S_S^0) < 1\} \) (\( S^*_T, S^*_S(t) \) with \( t < T \) denotes the solution in backward time from \( S(T) = S^* \)). Note that \( t_c \) is well defined as the solution \( S = 0 \) with the control \( u = 0 \) is decreasing in forward time and \( S_{0,1}^0(t) \leq S^*_0,t(t) = S^*_S \). When \( N_0^\dagger \geq \eta C \), this implies that for any \( N_0 \in [\eta C, N_0^\dagger] \), there exists a unique \( t_c^* \in [L_c, T] \) such that the solution with control \( u_N(\cdot, t_c^*) \) verifies exactly \( N(T) = \eta C S(T) \geq \eta C S^* \). The commutation time \( t_c = t_c^* \) is such that the nitrogen constraint is saturated only at the final time, and such a solution is optimal [6].

Remark 4.7: The solution for \( N_0 = N_0^\dagger \) with the control \( u_N \) and commutation time \( L_c \) reaches the point \( (S^*_S, \eta C S^*_S) \) in \( E \) at the final time \( T \).

D. The singular strategies

Note that when one has \( N_0^\dagger < N_0^\dagger \), initial conditions with \( N_0 \in (N_0^\dagger, N_0^\dagger) \) are not covered by the former cases. One has the following property.

Proposition 4.8: Assume \( N_0^\dagger < N_0^\dagger \), where \( N_0^\dagger \) and \( N_0^\dagger \) are defined in subsections IV-C and IV-B. Let \( u^\dagger \) and \( u^\dagger \) be the optimal open-loop controls given by the N- and NS-strategies for the initial conditions \( N_0^\dagger \) and \( N_0^\dagger \), respectively. Then for any \( N_0 \in (N_0^\dagger, N_0^\dagger) \), the control

\[
u(t) = \lambda u^\dagger(t) + (1 - \lambda) u^\dagger(t), \quad \lambda = \frac{N_0^\dagger - N_0}{N_0^\dagger - N_0^\dagger}, \quad t \in [0, T]
\]

is optimal.

Proof: Let \((S^\dagger, N^\dagger) \) and \((S^\dagger, N^\dagger) \) be the solutions given by the N- and NS-strategies for the initial condition \( N_0^\dagger \) and \( N_0^\dagger \) respectively. From equations (1), (2) one has

\[
\begin{align*}
\Lambda S^\dagger(t) + (1 - \lambda) N^\dagger(t) &= -k_1 \varphi(t) - k_1 (\lambda K_R(S^\dagger(t)) + (1 - \lambda) K_R(S^\dagger(t))) + k_2 (\lambda u^\dagger(t) + (1 - \lambda) u^\dagger(t)), \\
\Lambda N^\dagger(t) + (1 - \lambda) N^\dagger(t) &= -k_2 \varphi(t) + k_2 C_N^{\dagger} \left( \lambda u^\dagger(t) + (1 - \lambda) u^\dagger(t) \right)
\end{align*}
\]

Note that one has \( \Lambda K_R(S^\dagger) + (1 - \lambda) K_R(S^\dagger) = K_R(\lambda S^\dagger + (1 - \lambda) S^\dagger) \) as \( K_R \) is an affine function in the domain \( E \). Therefore

\[
S(t) = \lambda S^\dagger(t) + (1 - \lambda) S^\dagger(t), \quad N(t) = \lambda N^\dagger(t) + (1 - \lambda) N^\dagger(t)
\]

defined for any \( t \in [0, T] \) is the solution of (1), (2) for the control \( u \) and the initial condition (1), \( N_0 \). As solutions \((S^\dagger, N^\dagger) \) and \((S^\dagger, N^\dagger) \) remain in the convex set \( E \), we deduce that the solution \((S, N) \) also, and the control \( u \) is admissible. One has \( N^\dagger(T) = N^\dagger(T) = \eta C S^* \) (see Remarks 4.5 and 4.7). So we get \( N(T) = \eta C S^* \). According to Lemma 4.1, we conclude that \( u \) is optimal. ■

Such a control is singular in the sense that it is not composed of arcs with extreme values of the control set (0 or \( u_{\text{max}} \)), nor arcs on the boundary of the constraint set \( E \).

Remark 4.9: Indeed, it can be shown that for \( N_0 \in (N_0^\dagger, N_0^\dagger) \), there exists an infinity of singular controls such that the trajectory stays in \( E \) and hits the corner point \((S^*_S, \eta C S^*_S) \) of \( E \) at the final time (with Bocop, we obtain an
optimal control different to (4.8)). According to the remark to Lemma 4.1, all these controls are optimal.

V. NUMERICAL ILLUSTRATIONS

The former analysis has revealed the sub-domains on which the various strategies are optimal. This section gives an illustration with realist values of the parameters. We have considered a concave function

$$\varphi(t) = \frac{t(1 + \alpha)}{t + \alpha}, \quad \alpha > 0$$

and chosen a plausible set of values of model parameters inspired by the literature. Then, we have computed the lower bounds (4), (5) on the operating parameters $C_{in}^N$, $u_{max}$ for the set $E$ to be viable: $C_{in}^N = 1.4$, $u_{max} = 1$ and chosen the following values of operating parameters $C_{in}^N = 1.45$, $u_{max} = 2$, which guarantee the set $E$ to be viable (Proposition 3.2). Then, we have determined numerically the thresholds defined in Sections IV-A, IV-C, IV-C. Examples of trajectories and their controls using different strategies presented previously are given in the figures below:

<table>
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<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$k_4$</th>
<th>$S^*$</th>
<th>$S_w$</th>
<th>$S_h$</th>
<th>$T$</th>
<th>$\alpha$</th>
<th>$\eta_C$</th>
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<th>$N_0^\dagger$</th>
<th>$N_0^\flat$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.547681</td>
<td>0.579387</td>
<td>0.641166</td>
</tr>
</tbody>
</table>

Optimal solution for $N_0 = \eta_C$

Optimal solution for $N_0 = N^\dagger_0$

Optimal solution for $N_0 = N^\flat_0$

Optimal solution for $N_0 \in (N^\dagger_0, N^\flat_0)$

Optimal solution for $N_0 \in (N^\sharp_0, N^\dagger_0)$ found by Bocop

Optimal solution for $N_0 \in (N^\sharp_0, N^\flat_0)$ with the control given by Proposition 4.8

Optimal solution for $N_0 = N^\flat_0$
Let us make some comments.

- When the initial quantity of nitrogen is high ($N_0 > N_0^\dagger$), it is not surprising that the nitrogen constraint is not saturated (the S-strategy coincides with the optimal one already found for the model with no nitrogen stress [2]).

- For a lower value of initial nitrogen $N_0 \in (N_0^\dagger, N_0^\sharp)$, the lack of irrigation conducts the system to face the nitrogen stress before the hydric one, and the NS-strategy is optimal.

- A surprising feature occurs when keeping the system at the edge of the nitrogen stress does not allow to reach the humidity threshold (the time horizon been reached before) for initial nitrogen content $N_0 < N_0^\sharp$. The NS-strategy is not optimal and the N-strategy is fundamentally different: it requires anticipating future needs since the beginning.

- Another non-intuitive feature occurs for $N_0 \in (N_0^\dagger, N_0^\star)$ for which the "corner" point $(S^\star, \eta_C-S^\star)$ is reached exactly at the final time. The optimal control has to irrigate all the time with particular profiles (that are non-unique), avoiding the boundaries of $E$, unlike the other strategies. Note that the numerical software provides a continuous control while we proposed another optimal one that is discontinuous. In practice, the continuous one might be easier to apply.

This study reveals that the minimal residual nitrogen content in soil (i.e., at the final time) is obtained when the initial nitrogen content belongs to a particular interval of values. Therefore, having a low initial nitrogen content in the soil could paradoxically lead to more significant residual contents. This is explained by the larger water consumption needed to maintain maximal production.

VI. CONCLUSION

Considering the problem of minimal irrigation under maximal biomass production for a simplified crop model, we have given conditions for the optimal control with constraints to be feasible and depicted the different structures of the optimal strategy depending on the initial nitrogen content. These strategies can be implemented as simple feedback measuring online: the humidity level $S$ if the limit of water stress is met before the harvesting time, or the nitrogen concentration $N/S$ if the limit of the nitrogen stress is met before the harvesting time. In the other cases, the optimal control is determined as an open-loop control that can be computed using only the model’s parameters. We have also exhibited a particular interval of values of the initial nitrogen content that conducts to the smallest residual nitrogen in the soil, which could serve as a theoretical target for the practitioners.

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