2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Compositional Safety Verification of Infinite Networks: A Data-Driven Approach

Ali Aminzadeh, Abdalla Swikir, Sami Haddadin, and Abolfazl Lavaei

Abstract— **This paper develops a compositional framework for formal safety verification of an interconnected network comprised of a *countably infinite* number of discrete-time nonlinear subsystems with *unknown mathematical models*. Our proposed scheme involved subdividing the infinite network problem into individual subsystems, wherein the safety concept is modelled through a robust optimization program (ROP) via a notion of *local-barrier certificates* (L-BC). To address the difficulties associated with solving the ROP directly, primarily due to the absence of a mathematical model, we gather finite data from subsystem trajectories and leverage them to provide a scenario optimization program (SOP). We proceed with solving the resulted SOP and construct a local-barrier certificate for each unknown subsystem with a guarantee of correctness. Finally, in accordance with some *small-gain conditions*, we construct a *global-barrier certificate* (G-BC) derived from individual *local* certificates of subsystems, thus guaranteeing the safety of the infinite network within infinite time horizons. The practicality of our compositional findings becomes evident through a vehicle platooning scenario, characterized by a *countably infinite number* of vehicles with a single leader and an *unlimited number* of followers.**

I. Introduction

Motivations and State of the Art. Due to the rapid advancements in data science and the pervasive integration of large-scale networks into various facets of modern life, it is imperative to prioritize safety concerns in order to guarantee safe interactions and minimize potential risks when utilizing data across these networks. When dealing with complex networks comprising numerous subsystems, it is often more pragmatic to represent a vast yet finite network as effectively infinite. For instance, in the road traffic control, precisely counting the number of vehicles on the road poses a challenge due to the seemingly endless, interconnected nature of the network. Treating these systems as finite networks would result in unrealistic models that fail to capture the true complexity of the real-world situation [1].

Formal verification primarily seeks to ascertain whether a given dynamical system complies with a specified set of desired specifications. However, the analysis of systems with *continuous state spaces* presents a significant challenge, as closed-form solutions are often unavailable, leading to considerable computational complexities. This complexity arises from the need to handle infinite sets of states and actions, which is especially crucial in safety-critical applications. The prevailing research on formal verification and controller synthesis over complex dynamical systems predominantly employs finite abstractions, as a discretization-based technique [2], [3]. Specifically, finite abstractions provide a means to represent continuous-space control systems in a more abstract fashion, by associating discrete states and inputs with aggregated continuous ones from the original system (see *e.g.,* [4], [5], [6], [7], [8], [9]). Nevertheless, as the system's dimension increases, the computational complexities grow exponentially, referred to as *curse of dimensionality*, making construction methods impractical. To address this challenge, a potential solution has emerged in the form of *compositional techniques* that construct abstractions of large-scale networks based on those of smaller subsystems (see *e.g.,* [10], [11], [12], [13], [14], [15], [16]).

In recent years, there has been growing interest among researchers in exploring a *discretization-free* approach for analyzing complex systems, which entails the utilization of (control) barrier certificates, initially introduced in [17], [18]. The adoption of barrier certificates has increasingly emerged as a prominent technique for verifying and synthesizing controllers across a diverse array of complex systems (see *e.g.*, [19], [20], [21]). Compositional techniques have also been employed to construct barrier certificates for interconnected systems, building upon barrier certificates of smaller subsystems (see *e.g.*, [22], [23], [24], [25], [26], [27]).

The previously discussed compositional techniques, primarily intended for *finite networks* utilizing both abstraction and barrier methodologies, encountered limitations when applied to networks consisting of an *infinite number* of subsystems. While certain efforts have been made to tackle *stability analysis* in the context of infinite networks (*e.g.,* [28], [29], [30], [31]), or construction of *finite abstractions* for infinite networks (*e.g.,* [32], [33], [34], [35]), there has not been any work addressing the compositional construction of *safety barrier certificates* within the domain of *infinite networks* of subsystems without resorting to discretization. As another primary challenge, all the aforementioned compositional techniques require precise knowledge of the system's model. Particularly, obtaining closed-form mathematical models for large-scale systems is often challenging or impractical due to their complexity. As a result, model-based techniques may not be applicable for analyzing these complex systems. To address this concern, data-driven analysis has emerged by

This work was partially funded by the European Union's Horizon 2020 program (grant agreement No. 101070596 - euROBIN) and European Union's Horizon 2020 research and innovation programme as part of the project Darko under grant no. 101017274.

A. Aminzadeh is with the K. N. Toosi University of Technology, Iran. A. Swikir and S. Haddadin are with the Munich Institute of Robotics and Machine Intelligence, Technical University of Munich, Germany. A. Swikir is also with the Department of Electrical and Electronic Engineering, Omar Al-Mukhtar University (OMU), Albaida, Libya. A. Lavaei is with the School of Computing, Newcastle University, United Kingdom. Email: aliaminzadeh@email.kntu.ac.ir, {abdalla.swikir,haddadin}@tum.de, abolfazl.lavaei@newcastle.ac.uk.

offering two approaches: *indirect and direct* methods. In *indirect* data-driven methods, the literature presents solutions to verification and synthesis challenges by approximating models through identification techniques [36]. Nevertheless, obtaining a precise model for complex systems continues to pose challenges as it demands substantial resources. In response, *direct* data-driven approaches have emerged, enabling analysis directly from system trajectories without the need for system identification [37].

Original Contributions. The primary contribution of this work is the development of a *compositional data-driven* framework for ensuring the safety of an interconnected network, comprised of a *countably infinite* number of subsystems, each with *unknown mathematical models*. The sole required information is knowledge of Lipschitz constants of subsystems, for which we offer an algorithm utilizing data to estimate them, accompanied by an asymptotic guarantee during the estimation process. In our proposed setting, we begin by formulating the original safety problem of each subsystem into a robust optimization program (ROP). However, the ROP is not tractable due to the lack of knowledge about the subsystem's dynamics. To overcome this limitation, we leverage a given data set collected from subsystems and formulate a scenario optimization program (SOP) that aligns with the original ROP. By solving the associated SOP, we design *local-barrier* certificates for each subsystem with a guarantee of correctness. To ensure the safety of the entire infinite network, we employ *small-gain conditions* and compose a *global-barrier* certificate from individual local certificates of subsystems. A visual representation of our data-driven compositional method is presented in Fig. 1. Proofs of all statements are omitted due to space limitations.

Related Literature on Data-Driven Techniques. Several studies have delved into the formal analysis of unknown dynamical systems by employing direct data-driven methods. Notable findings encompass the development of data-driven control laws to stabilize nonlinear polynomial-type models [38], stability verification in unknown switched (linear) systems through data [39], [40], and data-driven techniques for verifying and synthesizing controllers using barrier certificates for unknown dynamical systems [41], [42]. The methods discussed in the earlier literature are all primarily designed for monolithic systems and are not well-suited for high-dimensional underlying systems, roughly defined as those with more than three dimensions. To tackle this limitation, some recent studies have focused on compositional data-driven approaches, as exemplified by work [43], [44], [45]. However, it is important to note that these approaches are not applicable to infinite networks and are specifically tailored for large-scale systems with a *finite* number of subsystems.

II. Discrete-Time Nonlinear Systems

A. Notation and Preliminaries

We employ the symbols $\mathbb{R}, \mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{N}_0$, and $\mathbb{N}$, to represent the sets of real numbers, positive real numbers, non-negative real numbers, non-negative integers, and positive integers, respectively. We use the notation $\Lambda = [\lambda_1; \dots; \lambda_n]$

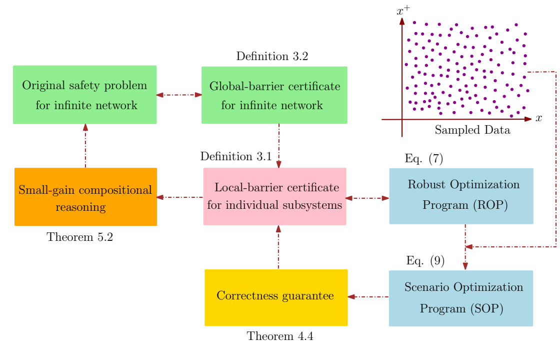

Fig. 1. Visual representation of the structure and contributions of the paper.

to represent a vector of n decision variables in our optimization problems. A vector μ consisting of infinitely many components μ_i is denoted by $\mu := (\mu_i)_{i\in\mathbb{N}}$. For any $c \in \mathbb{R}$, $|c|$ represents the absolute value of c, and for any $x \in \mathbb{R}^n$, the infinity and Euclidean norms of x are denoted by $|x|$ and $\|x\|_2$, respectively. Moreover, for any $m \times n$ matrix $C = (c_{ij})_{1\leq i\leq m, 1\leq j\leq n}$, we define $|C| = \max_{1\leq i\leq m} \sum_{j=1}^{n} |c_{ij}|$ as infinity norm of C. We use the notation l^∞ to refer to the Banach space that includes all infinite uniformly bounded sequences $s := (s_i) \in l^\infty, i \in \mathbb{N}$, where s_i represents the i-th element of a sequence $s \in l^\infty$. Furthermore, l^∞_+ denotes the positive cone within l^∞, encompassing all vectors $s \in l^\infty$ for which $s_i \geq 0, i \in \mathbb{N}$ holds true. In the context of l^∞, when comparing two sequences s and s', we establish that $s \leq s'$ holds true if each element in sequence s is less than or equal to its corresponding element in sequence s'. The standard unit vectors in l^∞, denoted as e_i, depict sequences that consist primarily of zeros with a solitary "1" positioned at index i, while all other entries are "0". Given $\zeta : l^\infty_+ \to l^\infty_+$ being an operator, for all $n \in \mathbb{N}$, $\zeta^n(\cdot)$ represents the result of applying ζ repeatedly n times.

B. Infinite Networks

In this subsection, we initiate by introducing discrete-time nonlinear subsystems. We then interconnect countably infinite number of these subsystems to establish an infinite network.

Definition 2.1: A discrete-time nonlinear subsystem, denoted as $\Psi_i, i \in \mathbb{N}$, can be described as a tuple

$$\Psi_i = (X_i, W_i, f_i),$$

where

- $X_i \subseteq \mathbb{R}^{n_i}$ and $W_i \subseteq \mathbb{R}^{p_i}$ denote the state and input sets of the subsystem;
- $f_i : X_i \times W_i \to X_i$ denotes the transition map characterizing the subsystem's evolution, and it is *assumed to be unknown* in our context.

The discrete-time subsystem Ψ_i is defined by a difference equation represented as:

$$\Psi_i : x_i(k+1) = f_i(x_i(k), w_i(k)), \quad k \in \mathbb{N}_0. \tag{1}$$

The internal input structure of $\Psi_i, i \in \mathbb{N}$, is defined by

$$w_i = (w_{ij})_{j\in \boldsymbol{N}_i} \in W_i := \prod_{j\in \boldsymbol{N}_i} W_{ij}, \tag{2}$$

where $w_{ij} \in W_{ij}$, and $\boldsymbol{N}_i$ is a finite subset of $\mathbb{N}$ comprising the index of Ψ_j, $j \in \boldsymbol{N}_i$, that affect Ψ_i, $i \notin \boldsymbol{N}_i, \forall i \in \mathbb{N}$. The primary role of w_i is for the sake of interconnection to build an interconnected network, as opposed to control inputs, which are not considered in this work.

In the following definition, we present a formal description of infinite networks, which are composed of individual subsystems.

Definition 2.2: Consider discrete-time subsystems $\Psi_i = (X_i, W_i, f_i), i \in \mathbb{N}$, with their internal input structure defined in (2). The infinite network can be expressed as the tuple $\Psi = (X, f)$, where
$X = \{x = (x_i)_{i\in\mathbb{N}} \colon x_i \in X_i, \|x\| := \sup_{i\in\mathbb{N}}\{|x_i|\} < \infty\}$,
$f(x) = (f_i(x_i, w_i))_{i\in\mathbb{N}}$.
The infinite network, denoted as $\Psi = \mathcal{N}(\Psi_i)_{i\in\mathbb{N}}$, operates according to

$$\Psi \colon x(k+1) = f(x(k)), \quad k \in \mathbb{N}_0, \tag{3}$$

wherein the interconnection of subsystems is illustrated through the following constraints:

$$\forall i \in \mathbb{N}, \forall j \in \boldsymbol{N}_i \colon \quad w_{ij} = x_j, \; X_j \subseteq W_{ij}. \tag{4}$$

We refer to the sequence $x_{x_0} : \mathbb{N} \to X$ that satisfies (3) for any initial state $x_0 \in X$ as the *state trajectory* of Ψ starting from an initial state x_0.

In the following section, we introduce the concepts of local-barrier certificates (L-BC) for subsystems with internal signals and global-barrier certificates (G-BC) for interconnected networks without internal signals.

III. Local and Global Barrier Certificates

Definition 3.1: Given a subsystem $\Psi_i = (X_i, W_i, f_i)$ defined in Definition 2.1, with $X_{0_i}, X_{u_i} \subseteq X_i$ denoting, respectively, initial and unsafe sets of Ψ_i, a function $\mathbb{B}_i : X_i \to \mathbb{R}_{\geq 0}$ is called a local-barrier certificate (L-BC) for Ψ_i if there exist $\delta_i, \sigma_i, \phi_i \in \mathbb{R}_{>0}$, $\xi_i \in (0, 1)$, and $\rho_{w_i} \in \mathbb{R}_{\geq 0}$, such that

$$\mathbb{B}_i(x_i) \geq \delta_i |x_i|^2, \qquad \forall x_i \in X_i, \tag{5a}$$
$$\mathbb{B}_i(x_i) \leq \sigma_i, \qquad \forall x_i \in X_{0_i}, \tag{5b}$$
$$\mathbb{B}_i(x_i) \geq \phi_i, \qquad \forall x_i \in X_{u_i}, \tag{5c}$$

and $\forall x_i \in X_i$, $\forall w_i \in W_i$, one has

$$\mathbb{B}_i(x_i(k+1)) \leq \max\Big\{\xi_i \mathbb{B}_i(x_i(k)), \rho_{w_i}|w_i(k)|^2\Big\}. \tag{5d}$$

We now introduce a complementary definition of *global-barrier certificates* for interconnected networks without internal inputs, which is subsequently employed to enforce safety specifications across an infinite network.

Definition 3.2: Consider an infinite network $\Psi = (X, f)$ in Definition 2.2, with $X_0, X_u \subseteq X$ denoting the initial and unsafe sets of Ψ, respectively. A function $\mathbb{B} \colon X \to \mathbb{R}_{\geq 0}$ is called a global-barrier certificate (G-BC) for Ψ if there exist $\sigma, \phi \in \mathbb{R}_{>0}$, $\xi \in (0, 1)$ and $\phi > \sigma$, such that

$$\mathbb{B}(x) \leq \sigma, \qquad \forall x \in X_0, \tag{6a}$$
$$\mathbb{B}(x) \geq \phi, \qquad \forall x \in X_u, \tag{6b}$$
$$\mathbb{B}(x(k+1)) \leq \xi \mathbb{B}(x(k)), \qquad \forall x \in X. \tag{6c}$$

Remark 3.3: It is important to highlight that the additional condition (5a) in L-BC plays a pivotal role in facilitating compositional techniques in Section V. In addition, while condition $\phi > \sigma$ in G-BC is vital in ensuring a safety certificate for an infinite network, as demonstrated in Theorem 3.4, the L-BC of subsystems does not impose such a requirement. In fact, L-BC are solely utilized for constructing G-BC for infinite networks without ensuring subsystem safety.

The subsequent theorem, borrowed from [17], provides a guarantee that the state trajectories of interconnected networks will never enter an unsafe region.

Theorem 3.4: Consider an infinite network $\Psi = (X, f)$, as defined in Definition 2.2, and assuming that $\mathbb{B}$ is a G-BC for Ψ, as specified in Definition 3.2. Then for all $x_0 \in X_0$ and $k \in \mathbb{N}$, the state trajectory x_{x_0} remains outside of the unsafe region X_u, *i.e.,* $x_{x_0} \notin X_u$, within an infinite time horizon.

The computational complexity of finding G-BC for interconnected networks is notably high, primarily due to the system's dimensionality. This challenge was a primary motivation for introducing the concept of L-BC for individual subsystems. Subsequently, in Section V, we propose a compositional approach for constructing a G-BC for an infinite network based on L-BC of individual subsystems.

To formally ensure the safety of the infinite network in (3) via Theorem 3.4, one needs precise knowledge of the mapping f_i for each subsystem to verify condition (5d), which is not accessible in our current context. To overcome this challenge, we present our data-driven approach in the next section, wherein we construct L-BC based on finite data sets obtained from trajectories of subsystems.

IV. Data-Driven Construction of L-BC

In our data-driven setting, we consider the structure of L-BC as $\mathbb{B}_i(q_i, x_i) = \sum_{j=1}^{z_i}(q_i^j \bar{p}_i^j(x_i))$, where $\bar{p}_i^j$ denote user-defined (possibly nonlinear) basis functions, and $q_i = [q_i^1; ...; q_i^{z_i}] \in \mathbb{R}^{z_i}$ represent unknown coefficients. To satisfy conditions (5a)-(5d), we approach the problem by transforming it into the subsequent robust optimization program (ROP):

$$\min_{[\Lambda_i; \eta_i]} \; \eta_i$$
$$\textbf{s.t.} \quad -\mathbb{B}_i(q_i, x_i) + \delta_i(x_i^\top x_i) \leq \eta_i, \qquad \forall x_i \in X_i, \tag{7a}$$
$$\mathbb{B}_i(q_i, x_i) - \sigma_i \leq \eta_i, \qquad \forall x_i \in X_{0_i}, \tag{7b}$$
$$-\mathbb{B}_i(q_i, x_i) + \phi_i \leq \eta_i, \qquad \forall x_i \in X_{u_i}, \tag{7c}$$
$$\mathbb{B}_i(q_i, f(x_i, w_i)) - \bar{\xi}_i \mathbb{B}_i(q_i, x_i) - \bar{\rho}_{w_i}\Big(\frac{w_i^\top w_i}{p_i}\Big) \leq \eta_i, \qquad \forall x_i \in X_i, \forall w_i \in W_i, \tag{7d}$$
$$\Lambda_i = [\delta_i; \bar{\xi}_i; \sigma_i; \phi_i; \bar{\rho}_{w_i}; q_i^1; \ldots; q_i^{z_i}],$$
$$\delta_i, \bar{\xi}_i, \sigma_i, \phi_i \in \mathbb{R}_{>0}, \bar{\rho}_{w_i} \in \mathbb{R}_{\geq 0}, q_i^{z_i} \in \mathbb{R}, \bar{\xi}_i \in (0, 1),$$

where p_i in (7d) refers to the dimension of W_i. If $\eta_{i_R}^* \leq 0$, with $\eta_{i_R}^*$ being the optimal value of ROP, solving the ROP indicates that conditions (5a)-(5d) are fulfilled.

Remark 4.1: Within the constraints specified in (5a) and (5d), we encounter infinity norms associated with variables

x_i and w_i, respectively. To render them computationally suitable for use in the ROP, we convert these infinity norms into Euclidean norms by incorporating their corresponding weight factors. Likewise, we have reformulated the max-form condition (5d) into a summation form represented by (7d). This reformulation is achieved by recovering the variables ξ_i and ρ_{w_i} based on $\bar{\xi}_i$ and $\bar{\rho}_{w_i}$, as follows, for any $0 < \pi_i < 1$:

$$\xi_i = 1 - (1-\pi_i)(1-\bar{\xi}_i),\ \ \rho_{w_i} = \frac{\bar{\rho}_{w_i}}{(1-\bar{\xi}_i)\pi_i}.$$

The ROP stated in (7) poses a main challenge due to its dependence on the precise mapping $f_i(x_i, w_i)$, which is not available within the context of our problem. To address this difficulty, we propose our data-driven solution by introducing a scenario optimization program for the ROP in (7). To do so, we leverage a dataset of samples denoted as $(\hat{x}_i^s, \hat{w}_i^s)_{s=1}^{S_i}$ within $X_i \times W_i$, with $S_i \in \mathbb{N}$, as follows:

$$((\hat{x}_i^s, \hat{w}_i^s), f(\hat{x}_i^s, \hat{w}_i^s)), \quad \forall s \in \{1, ..., S_i\}. \tag{8}$$

We now define a ball of radius α_i around each sample $(\hat{x}_i^s, \hat{w}_i^s)$ as $\mathsf{X}_i^s \times \mathsf{W}_i^s$, such that $X_i \times W_i \subseteq \cup_{s=1}^{S_i} (\mathsf{X}_i^s \times \mathsf{W}_i^s)$ and

$$\|(x_i, w_i) - (\hat{x}_i^s, \hat{w}_i^s)\|_2 \leq \alpha_i, \quad \forall (x_i, w_i) \in X_i \times W_i.$$

Now instead of addressing the ROP as presented in (7), we focus on solving the following scenario optimization program (SOP), $\forall s \in \{1, ..., S_i\}$:

$$\min_{[\Lambda_i;\eta_i]} \ \eta_i$$

$$\begin{aligned}
\mathbf{s.t.}\ \ & -\mathbb{B}_i(q_i, \hat{x}_i^s) + \delta_i(\hat{x}_i^{s\top}\hat{x}_i^s) \leq \eta_i, && \forall \hat{x}_i^s \in X_i, && \text{(9a)}\\
& \mathbb{B}_i(q_i, \hat{x}_i^s) - \sigma_i \leq \eta_i, && \forall \hat{x}_i^s \in X_{0_i}, && \text{(9b)}\\
& -\mathbb{B}_i(q_i, \hat{x}_i^s) + \phi_i \leq \eta_i, && \forall \hat{x}_i^s \in X_{u_i}, && \text{(9c)}\\
& \mathbb{B}_i(q_i, f(\hat{x}_i^s, \hat{w}_i^s)) - \bar{\xi}_i(\mathbb{B}_i(q_i, \hat{x}_i^s)) && && \\
& \quad - \bar{\rho}_{w_i}\Big(\frac{\hat{w}_i^{s\top}\hat{w}_i^s}{p_i}\Big) \leq \eta_i, && \forall (\hat{x}_i^s, \hat{w}_i^s) \in X_i \times W_i, && \\
& && && \text{(9d)}\\
& \Lambda_i = [\delta_i; \bar{\xi}_i; \sigma_i; \phi_i; \bar{\rho}_{w_i}; q_i^1; \ldots; q_i^{z_i}], && && \\
& \delta_i, \bar{\xi}_i, \sigma_i, \phi_i \in \mathbb{R}_{>0}, \bar{\rho}_{w_i} \in \mathbb{R}_{\geq 0}, q_i^{z_i} \in \mathbb{R}, \bar{\xi}_i \in (0,1). && &&
\end{aligned}$$

We represent the optimal value of SOP as $\eta_{i_S}^*$.

Remark 4.2: Due to a mild bilinearity between unknown variables $\bar{\xi}_i \in (0,1)$ and q_i in condition (9d), we constrain $\bar{\xi}_i$ to the discrete set $\bar{\xi}_i \in \{\bar{\xi}_i^1, \ldots, \bar{\xi}_i^l\}$ with a cardinality of l. This allows us to tackle the bilinearity by solving the SOP for a specific $\bar{\xi}_i$ while designing q_i.

A. Data-Driven L-BC Construction with Guarantee

Here, our primary objective is to solve the proposed SOP in (9) and construct L-BC for unknown discrete-time subsystems Ψ_i with a guarantee of correctness. To attain this objective, we commence by raising the following assumption.

Assumption 4.3: Suppose $\mathbb{B}_i(q_i, x_i)$ and $\delta_i(x_i^\top x_i) - \mathbb{B}_i(q_i, x_i)$ are Lipschitz continuous with respect to x_i with, respectively, Lipschitz constants $\mathscr{L}_i^1$ and $\mathscr{L}_i^2$, for any $i \in \mathbb{N}$. Moreover, $\mathbb{B}_i(q_i, f(x_i, w_i)) - \bar{\xi}_i \mathbb{B}_i(q_i, x_i) - \bar{\rho}_{w_i}(\frac{w_i^\top w_i}{p_i})$ is Lipschitz continuous with respect to (x_i, w_i) with Lipschitz constant $\mathscr{L}_i^3$, for any $i \in \mathbb{N}$.

Under Assumption 4.3, and inspired by [41], the following theorem outlines our data-driven approach for constructing L-BC for unknown discrete-time subsystems Ψ_i with a guarantee of correctness.

Theorem 4.4: Consider subsystems $\Psi_i = (X_i, W_i, f_i)$ defined in Definition 2.1. Assume Assumption 4.3 is satisfied. Let SOP (9) be solved with S_i sampled data, as in (8), and an optimal value $\eta_{i_S}^*$ and a solution $\Lambda_i^* = [\delta_i^*; \bar{\xi}_i^*; \sigma_i^*; \phi_i^*; \bar{\rho}_{w_i}^*; q_i^{1*}; \ldots; q_i^{z_i *}]$. If

$$\eta_{i_S}^* + \mathscr{L}_i \alpha_i \leq 0 \tag{10}$$

with $\mathscr{L}_i = \max\{\mathscr{L}_i^1, \mathscr{L}_i^2, \mathscr{L}_i^3\}$, then $\mathbb{B}_i$ resulting from solving the SOP in (9) is an L-BC for Ψ_i with a guarantee of correctness.

Remark 4.5: Note that the ball's radius α_i is crucial in satisfying the data-driven condition (10), providing correctness guarantee over the construction of L-BC based on data. To potentially reduce the required number of samples, one may initially gather data using a larger value of α_i to solve the SOP in (9). If condition (10) is not met with the chosen (potentially large) α_i, it becomes necessary to opt for a smaller α_i and re-solve the SOP. When working with real data, it is feasible to consider a sufficiently large α_i (worst-case scenario) and ensure satisfaction of condition (10).

To verify condition (10) in Theorem 4.4, the computation of $\mathscr{L}_i$ is required. To achieve this, we utilize the fundamental results of [46] and present Algorithm 1 for estimating the corresponding Lipschitz constants using a finite set of data for each subsystem. While the algorithm is dedicated to estimating $\mathscr{L}_i^3$, by following the similar steps, one can estimate $\mathscr{L}_i^1, \mathscr{L}_i^2$ using a finite set of data, where $g(\hat{x}_i^r) = \mathbb{B}_i^*(q_i, \hat{x}_i^r)$ and $g(\hat{x}_i^r) = \delta_i^*(\hat{x}_i^{r\top}\hat{x}_i^r) - \mathbb{B}_i^*(q_i, \hat{x}_i^r)$ in Step 3, respectively. Under Algorithm 1, the convergence of the estimated values $\mathscr{L}_i^1$-$\mathscr{L}_i^3$ to their actual values is guaranteed in the limit, as supported by the following lemma [46].

Lemma 4.6: The estimated values $\mathscr{L}_i^1$-$\mathscr{L}_i^3$ in Algorithm 1 converge to their actual values if and only if β approaches zero while $\bar{\kappa}$ and $\tilde{\kappa}$ tend to infinity.

Remark 4.7: Given the necessity of determining unknown coefficients q_i to estimate the Lipschitz constants $\mathscr{L}_i^1$-$\mathscr{L}_i^3$ in Algorithm 1, it is crucial to initially solve the proposed SOP outlined in (9). To avoid the need for subsequent verification of condition (10), one may initially assume a certain range for unknown coefficients q_i and estimate the Lipschitz constants $\mathscr{L}_i^1$-$\mathscr{L}_i^3$ before solving the SOP. Consequently, it is essential to enforce these established ranges during the solution of the SOP.

In the forthcoming section, we introduce a *compositional approach* based on *small-gain reasoning* to construct a G-BC for an infinite network, building upon L-BC of individual subsystems, constructed from data in Theorem 4.4.

V. Compositional Construction of G-BC

In this section, we propose a compositional framework that enables the construction of G-BC for an infinite network Ψ by leveraging L-BC of individual subsystems $\Psi_i, i \in \mathbb{N}$.

Algorithm 1 Estimation of Lipschitz constant $\mathscr{L}_i^3$ using data

Inputs: L-BC $\mathbb{B}_i^*$, $\bar{\xi}_i^*$, $\bar{\rho}_{w_i}^*$

1: Choose $\bar{\kappa}, \tilde{\kappa} \in \mathbb{N}$ and $\beta \in \mathbb{R}_{>0}$

2: Select $\bar{\kappa}$ sampled pairs $((\hat{x}_i^r, \hat{w}_i^r), (\hat{x}_i'^r, \hat{w}_i'^r))$ from $X_i \times W_i$ such that $\|(\hat{x}_i^r, \hat{w}_i^r) - (\hat{x}_i'^r, \hat{w}_i'^r)\|_2 \leq \beta$

3: Compute the slope Δ_i^r as $\Delta_i^r = \frac{\|g(\hat{x}_i^r, \hat{w}_i^r) - g(\hat{x}_i'^r, \hat{w}_i'^r)\|_2}{\|(\hat{x}_i^r, \hat{w}_i^r) - (\hat{x}_i'^r, \hat{w}_i'^r)\|_2}$, $\forall r \in \{1, ..., \bar{\kappa}\}$ with $g(\hat{x}_i^r, \hat{w}_i^r) = \mathbb{B}_i^*(q_i, f(\hat{x}_i^r, \hat{w}_i^r)) - \bar{\xi}_i^*(\mathbb{B}_i(q_i, \hat{x}_i^r)) - \bar{\rho}_{w_i}^*(\frac{\hat{w}_i^{r\top} \hat{w}_i^r}{p_i})$, ($g(\hat{x}_i'^r, \hat{w}_i'^r)$ is computed similarly)

4: Compute the maximum slope as $\varrho_i = \max\{\Delta_i^1, \ldots, \Delta_i^{\bar{\kappa}}\}$

5: Repeat Steps 2-4 $\tilde{\kappa}$ times and acquire $\varrho_1^i, ..., \varrho_i^{\tilde{\kappa}}$

6: Through the utilization of the *Reverse Weibull distribution* [46] on $\varrho_1^i, \ldots, \varrho_{\tilde{\kappa}}^i$, which yields location, scale, and shape parameters, select the *location parameter* as an estimation of $\mathscr{L}_i^3$

Output: Lipschitz constant $\mathscr{L}_i^3$

Considering subsystems Ψ_i as in Definition 2.1, assume the existence of L-BC $\mathbb{B}_i$ as defined in Definition 3.1 with constants $\delta_i, \sigma_i, \phi_i \in \mathbb{R}_{>0}, \xi_i \in (0, 1)$, and $\rho_{w_i} \in \mathbb{R}_{\geq 0}$. We now define, $\forall i, j \in \mathbb{N}$,

$$\xi_{ij} := \begin{cases} \xi_i, & \text{if } i = j, \\ \frac{\rho_{w_i}}{\delta_j}, & \text{if } j \in \boldsymbol{N}_i, \\ 0 & \text{if } i \neq j, j \notin \boldsymbol{N}_i. \end{cases}$$

Correspondingly, we define $\zeta : l_+^\infty \to l_+^\infty$ as

$$\zeta(s) = (\sup_{j \in \mathbb{N}}\{\xi_{ij} s_j\})_{i \in \mathbb{N}}, \quad s \in l_+^\infty. \tag{11}$$

We also assume that there exist constants $\tilde{\xi}, \tilde{\delta} \in \mathbb{R}_{>0}$, and $\tilde{\rho}_w \in \mathbb{R}_{\geq 0}$, such that $\xi_i \leq \tilde{\xi}, \rho_{w_i} \leq \tilde{\rho}_w, \delta_i \geq \tilde{\delta}$, for all $i \in \mathbb{N}$. This assumption ensures the well-posedness of ζ in (11).

To establish the primary compositionality findings of the paper, we present the following *small-gain assumption*, inspired from [29].

Assumption 5.1: Consider operator ζ as defined in (11). Assume that $\sup_{j \in \mathbb{N}}\{\xi_{ij} s_j\} > 0, \forall s_j > 0, \forall i, j \in \mathbb{N}, \zeta$ is continuous on l_+^∞, $\lim_{n \to +\infty} \zeta^n(s) = 0$, and there exist positive constants c_1 and c_2 such that for all $i, j \in \mathbb{N}$, the operator $\zeta_{i,j}(s) := \zeta(s) + c_1 s_j e_i, s \in l_+^\infty$ fulfills the following condition:

$$\zeta_{i,j}(s) \ngeq (1 - c_2)s, \quad s \in l_+^\infty \setminus \{0\}. \tag{12}$$

The small gain condition (12) implies the existence of a vector $\mu := (\mu_i)_{i \in \mathbb{N}} \in l_+^\infty$ with $\mu_i \in \mathbb{R}_{>0}, i \in \mathbb{N}$, and $\epsilon \in (0, 1)$, such that [29, Lemma 4.5]

$$\zeta(\mu) \leq (1 - \epsilon)\mu.$$

As a result, according to (11) and since $\epsilon \in (0, 1)$, one has, $\forall i \in \mathbb{N}$:

$$\sup_{j \in \mathbb{N}}\{\xi_{ij} \mu_j\} \leq (1 - \epsilon)\mu_i < \mu_i.$$

By applying $\frac{1}{\mu_i}$ to both sides, we have

$$\frac{1}{\mu_i}(\sup_{j \in \mathbb{N}}\{\xi_{ij} \mu_j\}) = \sup_{j \in \mathbb{N}}\{\frac{\xi_{ij} \mu_j}{\mu_i}\} < 1. \tag{13}$$

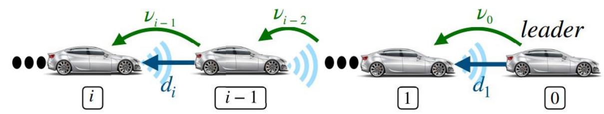

Fig. 2. A platoon comprising an *infinitely countable* number of vehicles.

Since inequality (13) holds for all $i \in \mathbb{N}$, it can be generalized as

$$\sup_{i, j \in \mathbb{N}}\{\frac{\xi_{ij} \mu_j}{\mu_i}\} < 1. \tag{14}$$

In the upcoming theorem, under Assumption 5.1, we construct a G-BC for the infinite network Ψ using L-BC of $\Psi_i, i \in \mathbb{N}$, constructed from data.

Theorem 5.2: Consider an infinite network $\Psi = \mathcal{N}(\Psi_i)_{i \in \mathbb{N}}$, which arises from infinitely many subsystems Ψ_i. Assume that each individual subsystem Ψ_i possesses an L-BC $\mathbb{B}_i$, constructed from data according to Theorem 4.4 with a guarantee of correctness. If Assumption 5.1 is satisfied, and

$$\sup_i \Big\{\frac{\phi_i}{\mu_i}\Big\} > \sup_i \Big\{\frac{\sigma_i}{\mu_i}\Big\}, \tag{15}$$

then the function $\mathbb{B}(x)$ defined as

$$\mathbb{B}(x) := \sup_i \Big\{\frac{1}{\mu_i} \mathbb{B}_i(x_i)\Big\}, \tag{16}$$

is a G-BC for the infinite network $\Psi = \mathcal{N}(\Psi_i)_{i \in \mathbb{N}}$, with $\sigma := \sup_i \Big\{\frac{\sigma_i}{\mu_i}\Big\}$, $\phi := \sup_i \Big\{\frac{\phi_i}{\mu_i}\Big\}$, and $\xi = \sup_{i,j} \Big\{\frac{\xi_{ij} \mu_j}{\mu_i}\Big\}$.

Remark 5.3: Assuming that $\xi_{ij} \leq 1$ for any $i, j \in \mathbb{N}$, inequality (14) can be satisfied by setting $\mu_i = 1$ for all $i \in \mathbb{N}$. Consequently, inequality (16) is simplified as $\mathbb{B}(x) := \sup_i \big\{\mathbb{B}_i(x_i)\big\}$, and as a result, the small-gain condition (12), equivalently inequality (14), is *automatically fulfilled*.

VI. Case Study

In this section, we showcase the efficacy of our proposed results by applying them into a vehicle platoon consisting of an *infinitely countable* number of vehicles, as depicted in Fig. 2. The dynamics of the interconnected network can be described as [47]

$$\Psi : x(k + 1) = Ax(k) + u(k),$$

where $u(k)$ is a previously designed and deployed controller and A is a block matrix featuring diagonal elements represented by $\hat{A}$, and off-diagonal blocks denoted as $A_{i(i-1)} = A_w$, $i \geq 2$, as:

$$\hat{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad A_w = \begin{bmatrix} 0 & \tau \\ 0 & 0 \end{bmatrix},$$

with the interconnection strength characterized by $\tau = 0.005$. Additionally, all non-diagonal blocks are specified as zero matrices of appropriate dimensions. Moreover, $x(k) = (x_i(k))_{i \in \mathbb{N}}$ and $u(k) = (u_i(k))_{i \in \mathbb{N}}$. We now proceed with presenting a description of each individual vehicle Ψ_i as

$$\Psi_i : (k + 1) = \hat{A}x_i(k) + u_i(k) + A_w w_i(k).$$

It can be readily ascertained that $\Psi = \mathcal{N}(\Psi_i)_{i\in\mathbb{N}}$, where $w_i(k) = [0; w_{i(i-1)}(k)]$, with $w_{i(i-1)}(k) = [0;1]^\top x_{i-1}(k)$, and setting $w_{1,0}(k) = 0$. Each vehicle's state is expressed as $x_i = [d_i; v_i]$, $i \in \mathbb{N}$, where d_i represents the relative distance between vehicle i and its preceding vehicle $i-1$, with the 0-th vehicle representing the leader, while v_i signifies the velocity of vehicle i in relation to the leader's reference frame. The main objective in vehicle platoon is to adjust each vehicle's speed to uphold a safe distance from its preceding vehicle [47]. We assume a controller is previously designed and deployed to each vehicle as

$$u_i = \begin{bmatrix} -0.4d_i + 0.2v_i + 0.3 \\ 0.4d_i - 0.9v_i + 0.05 \end{bmatrix}. \tag{17}$$

We assume that the model of each vehicle is unknown. Let us ensure that the well-posedness of $\Psi = \mathcal{N}(\Psi_i)_{i\in\mathbb{N}}$ holds true by validating the condition of $\|f(x)\| < \infty$, as in Definition 2.2. By deploying the controller in (17), and defining $C = \max\{|\bar{A}|, |A_w|\}$, where $\bar{A}$ is the new matrix of the system after deploying the controller, we have:

$$\begin{aligned}\|f(x)\| &= \sup_{i\in\mathbb{N}}\{|f_i(x_i, w_i)|\} = \sup_{i\in\mathbb{N}}\{|\bar{A}x_i + A_w w_i|\} \\ &\leq |\bar{A}|\sup_{i\in\mathbb{N}}\{|x_i|\} + |A_w|\sup_{i\in\mathbb{N}}\{|w_i|\} \\ &\leq C(\sup_{i\in\mathbb{N}}\{|x_i|\} + \sup_{i\in\mathbb{N}}\{|x_i|\}) = C(\|x\| + \|x\|) < \infty.\end{aligned}$$

Consequently, one can assert that $\Psi = \mathcal{N}(\Psi_i)_{i\in\mathbb{N}}$ is well-defined. It is worth noting that even though matrices $\bar{A}$ and A_w are unknown, which is the case here, the resulting well-posedness conclusion remains valid.

The regions of interest for each vehicle are defined as follows: $X_i \in [0,1] \times [-0.3, 0.7]$, $X_{0_i} \in [0.25, 0.75] \times [-0.05, 0.45]$, and $X_{u_i} \in [0,1] \times [-0.3, -0.15] \cup [0,1] \times [0.55, 0.7]$. Our primary objective is to construct an L-BC, using collected data from each vehicle with unknown dynamics, by solving the SOP (9) for each Ψ_i. Subsequently, under the proposed small-gain condition, we aim to construct a G-BC based on individual L-BC while ensuring that the trajectory of the infinite network remains safe in infinite time horizons. To do so, by considering $\alpha_i = 0.005$ and $\bar{\xi}_i^* = 0.95$, we solve the SOP in (9) and compute coefficients of L-BC, along with other decision variables in the SOP, as follows:

$$\begin{aligned}\mathbb{B}_i(q_i, x_i) = &-0.0843d_i^4 - 0.14d_i^3 v_i + 0.14d_i^3 + 0.1228d_i^2 v_i^2 \\ &+ 0.14d_i^2 v_i - 0.0541d_i^2 + 0.14d_i v_i^3 - 0.14d_i v_i^2 \\ &- 0.1254d_i v_i - 0.0123d_i - 0.14v_i^4 + 0.14v_i^3 \\ &+ 0.14v_i^2 - 0.1073v_i + 0.0506\end{aligned}$$

$$\delta_i^* = 0.05, \sigma_i^* = 0.05, \phi_i^* = 0.06, \bar{\rho}_{w_i}^* = 2\times 10^{-8}, \eta_{i_S}^* = -0.005.$$

Consequently, according to Remark 4.1, values of ξ_i and ρ_{w_i} outlined in (5d) are computed as $\xi_i^* = 0.975$ and $\rho_{w_i}^* = 8 \times 10^{-7}$, by selecting $\pi_i = 0.5$. We then apply Algorithm 1 and compute $\mathscr{L}_i^1 = 0.1648$, $\mathscr{L}_i^2 = 0.1682$ and $\mathscr{L}_i^3 = 0.1598$. Given that $\eta_{i_S}^* + \mathscr{L}_i \alpha_i = -0.0049 \leq 0$, as stipulated by Theorem 4.4, we can assert that the data-driven L-BC is valid for unknown vehicle Ψ_i with the guarantee of correctness.

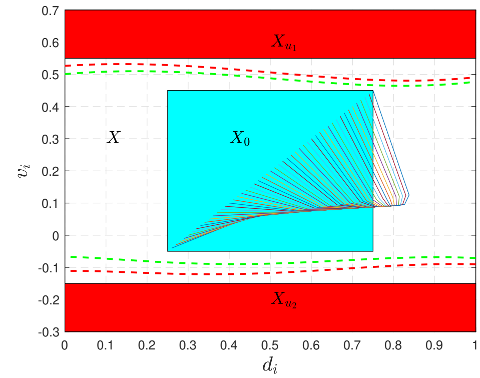

Fig. 3. Closed-loop states trajectories of a representative vehicle, with previously designed and deployed controller, are depicted for 30 initial conditions in range $[0.25, 0.75] \times [-0.05, 0.45]$. Green and red dashed lines are initial and unsafe level sets, respectively.

We now proceed with showing the small-gain condition as an essential requirement for the compositional results. Given that $\delta_i^* > \rho_{w_i}^*$, it becomes evident that $\xi_{ij} \leq 1$, for any $i, j \in \mathbb{N}$. According to Remark 5.3, with the choice of $\mu_i = 1$ for each $i \in \mathbb{N}$, it can be concluded that condition (12), and consequently (14), are satisfied without imposing any constraints on the number of vehicles. As a result, $\mathbb{B}(x) := \sup_i \{(\mathbb{B}_i(x_i))\}$ arises as a G-BC for the infinite network Ψ with $\sigma = 0.058$, $\phi = 0.064$, and $\xi = 0.975$. By applying Theorem 3.4, we can guarantee that all trajectories of the interconnected network originating from X_0 will remain within the safe domain throughout an infinite time horizon. Closed-loop state trajectories of a representative vehicle under the previously designed and deployed controller are depicted in Fig. 3.

VII. Conclusion

In this paper, we offered a framework for formally verifying the safety of an interconnected network that consisted of a *countably infinite* array of discrete-time subsystems, each with *unknown mathematical models*. Our approach involved breaking down the overall problem into subsystem levels, where we modeled the safety concept for each subsystem using a robust optimization program (ROP) based on the notion of *local-barrier certificates*. We then collected finite data from subsystem trajectories and created a scenario optimization program (SOP). We solved the resulting SOP and constructed a local-barrier certificate for each unknown subsystem, while providing a guarantee of correctness. Finally, under some *small-gain conditions*, we constructed a *global-barrier certificate* from *local* certificates of subsystems to ensure the safety of the infinite network.

References

[1] C. Enyioha, M. A. Rahimian, G. J. Pappas, and A. Jadbabaie, "Controllability and fraction of leaders in infinite networks," in *53rd IEEE Conference on Decision and Control*, 2014, pp. 1359–1364.

[2] P. Tabuada, *Verification and control of hybrid systems: a symbolic approach*. Springer Science & Business Media, 2009.

[3] C. Belta, B. Yordanov, and E. A. Gol, *Formal methods for discrete-time dynamical systems*. Springer, 2017, vol. 89.

[4] A. Abate, M. Prandini, J. Lygeros, and S. Sastry, "Probabilistic reachability and safety for controlled discrete time stochastic hybrid systems," *Automatica*, vol. 44, no. 11, pp. 2724–2734, 2008.

[5] A. A. Julius and G. J. Pappas, "Approximations of stochastic hybrid systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 6, pp. 1193–1203, 2009.

[6] A. Girard, G. Pola, and P. Tabuada, "Approximately bisimilar symbolic models for incrementally stable switched systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 1, pp. 116–126, 2009.

[7] M. Zamani, G. Pola, M. Mazo, and P. Tabuada, "Symbolic models for nonlinear control systems without stability assumptions," *IEEE Transactions on Automatic Control*, vol. 57, no. 7, pp. 1804–1809, 2011.

[8] A. Girard, G. Gössler, and S. Mouelhi, "Safety controller synthesis for incrementally stable switched systems using multiscale symbolic models," *IEEE Transactions on Automatic Control*, vol. 61, no. 6, pp. 1537–1549, 2015.

[9] S. Coogan and M. Arcak, "Efficient finite abstraction of mixed monotone systems," in *Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control*, 2015, pp. 58–67.

[10] Y. Tazaki and J. Imura, "Bisimilar finite abstractions of interconnected systems," in *International Workshop on Hybrid Systems: Computation and Control*, 2008, pp. 514–527.

[11] G. Pola, P. Pepe, and M. D. Di Benedetto, "Symbolic models for networks of control systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3663–3668, 2016.

[12] A. Swikir, A. Girard, and M. Zamani, "From dissipativity theory to compositional synthesis of symbolic models," in *2018 Indian Control Conference (ICC)*, 2018, pp. 30–35.

[13] M. Zamani, M. Mazo, M. Khaled, and A. Abate, "Symbolic abstractions of networked control systems," *IEEE Transactions on Control of Network Systems*, vol. 5, no. 4, pp. 1622–1634, 2017.

[14] A. Swikir and M. Zamani, "Compositional synthesis of finite abstractions for networks of systems: A small-gain approach," *Automatica*, vol. 107, pp. 551–561, 2019.

[15] A. Lavaei and M. Zamani, "From dissipativity theory to compositional synthesis of large-scale stochastic switched systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 9, pp. 4422–4437, 2022.

[16] A. Nejati and M. Zamani, "Compositional construction of finite MDPs for continuous-time stochastic systems: A dissipativity approach," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 1962–1967, 2020.

[17] S. Prajna and A. Jadbabaie, "Safety verification of hybrid systems using barrier certificates," in *International Workshop on Hybrid Systems: Computation and Control*, 2004, pp. 477–492.

[18] S. Prajna, A. Jadbabaie, and G. J. Pappas, "A framework for worst-case and stochastic safety verification using barrier certificates," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1415–1428, 2007.

[19] U. Borrmann, L. Wang, A. D. Ames, and M. Egerstedt, "Control barrier certificates for safe swarm behavior," *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 68–73, 2015.

[20] P. Wieland and F. Allgöwer, "Constructive safety using control barrier functions," *IFAC Proceedings Volumes*, vol. 40, no. 12, pp. 462–467, 2007.

[21] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in *18th European control conference (ECC)*, 2019, pp. 3420–3431.

[22] P. Jagtap, A. Swikir, and M. Zamani, "Compositional construction of control barrier functions for interconnected control systems," in *Proceedings of the 23rd International Conference on Hybrid Systems: Computation and Control*, 2020, pp. 1–11.

[23] Z. Lyu, X. Xu, and Y. Hong, "Small-gain theorem for safety verification of interconnected systems," *Automatica*, vol. 139, pp. 1–6, 2022.

[24] A. Lavaei and E. Frazzoli, "Compositional controller synthesis for interconnected stochastic systems with markovian switching," in *American Control Conference (ACC)*, 2022, pp. 4838–4843.

[25] M. Anand, A. Lavaei, and M. Zamani, "From small-gain theory to compositional construction of barrier certificates for large-scale stochastic systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 10, pp. 5638–5645, 2022.

[26] A. Nejati and M. Zamani, "From dissipativity theory to compositional construction of control barrier certificates," *Schloss-Dagstuhl-Leibniz Zentrum für Informatik*, 2022.

[27] A. Lavaei and E. Frazzoli, "Scalable synthesis of safety barrier certificates for networks of stochastic switched systems," *IEEE Transactions on Automatic Control*, 2024.

[28] A. Mironchenko, C. Kawan, and J. Glück, "Nonlinear small-gain theorems for input-to-state stability of infinite interconnections," *Mathematics of Control, Signals, and Systems*, vol. 33, no. 4, pp. 573–615, 2021.

[29] S. Dashkovskiy, A. Mironchenko, J. Schmid, and F. Wirth, "Stability of infinitely many interconnected systems," *IFAC-PapersOnLine*, vol. 52, no. 16, pp. 550–555, 2019.

[30] C. Kawan, A. Mironchenko, A. Swikir, N. Noroozi, and M. Zamani, "A Lyapunov-based small-gain theorem for infinite networks," *IEEE Transactions on Automatic Control*, vol. 66, no. 12, pp. 5830–5844, 2020.

[31] S. Dashkovskiy and S. Pavlichkov, "Stability conditions for infinite networks of nonlinear systems and their application for stabilization," *Automatica*, vol. 112, pp. 1–12, 2020.

[32] G. Pola, P. Pepe, M. D. Di Benedetto, and P. Tabuada, "Symbolic models for nonlinear time-delay systems using approximate bisimulations," *Systems & Control Letters*, vol. 59, no. 6, pp. 365–373, 2010.

[33] A. Girard, "Approximately bisimilar abstractions of incrementally stable finite or infinite dimensional systems," in *53rd IEEE conference on decision and control*, 2014, pp. 824–829.

[34] A. Swikir, N. Noroozi, and M. Zamani, "Compositional synthesis of symbolic models for infinite networks," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 1868–1873, 2020.

[35] S. Liu, N. Noroozi, and M. Zamani, "Symbolic models for infinite networks of control systems: A compositional approach," *Nonlinear Analysis: Hybrid Systems*, vol. 43, p. 101097, 2021.

[36] Z. Hou and Z. Wang, "From model-based control to data-driven control: Survey, classification and perspective," *Information Sciences*, vol. 235, pp. 3–35, 2013.

[37] F. Dörfler, J. Coulson, and I. Markovsky, "Bridging direct and indirect data-driven control formulations via regularizations and relaxations," *IEEE Transactions on Automatic Control*, vol. 68, no. 2, pp. 883–897, 2022.

[38] M. Guo, C. De Persis, and P. Tesi, "Learning control for polynomial systems using sum of squares relaxations," in *59th IEEE conference on decision and control (CDC)*, 2020, pp. 2436–2441.

[39] J. Kenanian, A. Balkan, R. M. Jungers, and P. Tabuada, "Data driven stability analysis of black-box switched linear systems," *Automatica*, vol. 109, pp. 1–9, 2019.

[40] C. Fan and S. Mitra, "Data-driven safety verification of complex cyber-physical systems," *Design Automation of Cyber-Physical Systems*, pp. 107–142, 2019.

[41] A. Nejati and M. Zamani, "Data-driven synthesis of safety controllers via multiple control barrier certificates," *IEEE Control Systems Letters*, vol. 7, pp. 2497–2502, 2023.

[42] A. Nejati, A. Lavaei, P. Jagtap, S. Soudjani, and M. Zamani, "Formal verification of unknown discrete-and continuous-time systems: A data-driven approach," *IEEE Transactions on Automatic Control*, vol. 68, no. 5, pp. 3011–3024, 2023.

[43] N. Noroozi, A. Salamati, and M. Zamani, "Data-driven safety verification of discrete-time networks: a compositional approach," *IEEE Control Systems Letters*, vol. 6, pp. 2210–2215, 2021.

[44] A. Lavaei, L. Di Lillo, A. Censi, and E. Frazzoli, "Formal estimation of collision risks for autonomous vehicles: A compositional data-driven approach," *IEEE Transactions on Control of Network Systems*, vol. 10, no. 1, pp. 407–418, 2022.

[45] A. Lavaei, S. Soudjani, and E. Frazzoli, "A compositional dissipativity approach for data-driven safety verification of large-scale dynamical systems," *IEEE Transactions on Automatic Control*, vol. 68, no. 12, pp. 7240–7253, 2023.

[46] G. Wood and B. Zhang, "Estimation of the Lipschitz constant of a function," *Journal of Global Optimization*, vol. 8, pp. 91–103, 1996.

[47] S. Sadraddini, S. Sivaranjani, V. Gupta, and C. Belta, "Provably safe cruise control of vehicular platoons," *IEEE Control Systems Letters*, vol. 1, no. 2, pp. 262–267, 2017.