Abstract—This paper develops a compositional framework for formal safety verification of an interconnected network comprised of a countably infinite number of discrete-time nonlinear subsystems with unknown mathematical models. Our proposed scheme involved subdividing the infinite network problem into individual subsystems, wherein the safety concept is modelled through a robust optimization program (ROP) via a notion of local-barrier certificates (L-BC). To address the difficulties associated with solving the ROP directly, primarily due to the absence of a mathematical model, we gather finite data from subsystem trajectories and leverage them to provide a scenario optimization program (SOP). We proceed with solving the resulted SOP and construct a local-barrier certificate for each unknown subsystem with a guarantee of correctness. Finally, in accordance with some small-gain conditions, we construct a global-barrier certificate (G-BC) derived from individual local certificates of subsystems, thus guaranteeing the safety of the infinite network within infinite time horizons. The practicality of our compositional findings becomes evident through a vehicle platooning scenario, characterized by a countably infinite number of vehicles with a single leader and an unlimited number of followers.

I. INTRODUCTION

Motivations and State of the Art. Due to the rapid advancements in data science and the pervasive integration of large-scale networks into various facets of modern life, it is imperative to prioritize safety concerns in order to guarantee safe interactions and minimize potential risks when utilizing data across these networks. When dealing with complex networks comprising numerous subsystems, it is often more pragmatic to represent a vast yet finite network as effectively infinite. For instance, in the road traffic control, precisely counting the number of vehicles on the road poses a challenge due to the seemingly endless, interconnected nature of the network. Treating these systems as finite networks would result in unrealistic models that fail to capture the true complexity of the real-world situation [1].

Formal verification primarily seeks to ascertain whether a given dynamical system complies with a specified set of desired specifications. However, the analysis of systems with continuous state spaces presents a significant challenge, as closed-form solutions are often unavailable, leading to considerable computational complexities. This complexity arises from the need to handle infinite sets of states and actions, which is especially crucial in safety-critical applications. The prevailing research on formal verification and controller synthesis over complex dynamical systems predominantly employs finite abstractions, as a discretization-based technique [2], [3]. Specifically, finite abstractions provide a means to represent continuous-space control systems in a more abstract fashion, by associating discrete states and inputs with aggregated continuous ones from the original system (see e.g., [4], [5], [6], [7], [8], [9]). Nevertheless, as the system’s dimension increases, the computational complexities grow exponentially, referred to as curse of dimensionality, making construction methods impractical. To address this challenge, a potential solution has emerged in the form of compositional techniques that construct abstractions of large-scale networks based on those of smaller subsystems (see e.g., [10], [11], [12], [13], [14], [15], [16]).

In recent years, there has been growing interest among researchers in exploring a discretization-free approach for analyzing complex systems, which entails the utilization of (control) barrier certificates, initially introduced in [17], [18]. The adoption of barrier certificates has increasingly emerged as a prominent technique for verifying and synthesizing controllers across a diverse array of complex systems (see e.g., [19], [20], [21]). Compositional techniques have also been employed to construct barrier certificates for interconnected systems, building upon barrier certificates of smaller subsystems (see e.g., [22], [23], [24], [25], [26], [27]).

The previously discussed compositional techniques, primarily intended for finite networks utilizing both abstraction and barrier methodologies, encountered limitations when applied to networks consisting of an infinite number of subsystems. While certain efforts have been made to tackle stability analysis in the context of infinite networks (e.g., [28], [29], [30], [31]), or construction of finite abstractions for infinite networks (e.g., [32], [33], [34], [35]), there has not been any work addressing the compositional construction of safety barrier certificates within the domain of infinite networks of subsystems without resorting to discretization. As another primary challenge, all the aforementioned compositional techniques require precise knowledge of the system’s model. Particularly, obtaining closed-form mathematical models for large-scale systems is often challenging or impractical due to their complexity. As a result, model-based techniques may not be applicable for analyzing these complex systems. To address this concern, data-driven analysis has emerged by

Compositional Safety Verification of Infinite Networks: A Data-Driven Approach

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offering two approaches: indirect and direct methods. In indirect data-driven methods, the literature presents solutions to verification and synthesis challenges by approximating models through identification techniques [36]. Nevertheless, obtaining a precise model for complex systems continues to pose challenges as it demands substantial resources. In response, direct data-driven approaches have emerged, enabling analysis directly from system trajectories without the need for system identification [37].

**Original Contributions.** The primary contribution of this work is the development of a compositional data-driven framework for ensuring the safety of an interconnected network, comprised of a countably infinite number of subsystems, each with unknown mathematical models. The sole required information is knowledge of Lipschitz constants of subsystems, for which we offer an algorithm utilizing data to estimate them, accompanied by an asymptotic guarantee during the estimation process. In our proposed setting, we begin by formulating the original safety problem of each subsystem into a robust optimization program (ROP). However, the ROP is not tractable due to the lack of knowledge about the subsystem’s dynamics. To overcome this limitation, we leverage a given data set collected from subsystems and formulate a scenario optimization program (SOP) that aligns with the original ROP. By solving the associated SOP, we design local-barrier certificates for each subsystem with a guarantee of correctness. To ensure the safety of the entire infinite network, we employ small-gain conditions and compose a global-barrier certificate from individual local certificates of subsystems. A visual representation of our data-driven compositional method is presented in Fig. 1. Proofs of all statements are omitted due to space limitations.

**Related Literature on Data-Driven Techniques.** Several studies have delved into the formal analysis of unknown dynamical systems by employing direct data-driven methods. Notable findings encompass the development of control laws to stabilize nonlinear polynomial-type models for verifying and synthesizing controllers using barrier certificates for unknown dynamical systems [41], [42]. The methods discussed in the earlier literature are all primarily focused on obtaining a precise model for complex systems. In this subsection, we initiate by introducing discrete-time nonlinear subsystems. We then interconnect countably infinite number of these subsystems to establish an infinite network.

**Definition 2.1:** A discrete-time nonlinear subsystem, denoted as $\Psi_i$, $i \in \mathbb{N}$, can be described as a tuple

$$\Psi_i = (X_i, W_i, f_i),$$

where

- $X_i \subseteq \mathbb{R}^{n_i}$ and $W_i \subseteq \mathbb{R}^{p_i}$ denote the state and input sets of the subsystem;
- $f_i : X_i \times W_i \rightarrow X_i$ denotes the transition map characterizing the subsystem’s evolution, and it is assumed to be unknown in our context.

The discrete-time subsystem $\Psi_i$ is defined by a difference equation represented as:

$$x_i(k + 1) = f_i(x_i(k), w_i(k)), \quad k \in \mathbb{N}_0.$$  \hfill (1)

The internal input structure of $\Psi_i$, $i \in \mathbb{N}$, is defined by

$$w_i = (w_{ij})_{j \in \mathbb{N}_i} \in W_i := \prod_{j \in \mathbb{N}_i} W_{ij},$$  \hfill (2)
where \( w_{ij} \in W_{ij} \), and \( N_i \) is a finite subset of \( \mathbb{N} \) comprising the index of \( \Psi_j, j \in N_i \), that affect \( \Psi_i, i \notin N_i, \forall i \in \mathbb{N} \). The primary role of \( w_i \) is for the sake of interconnection to build an interconnected network, as opposed to control inputs, which are not considered in this work.

In the following definition, we present a formal description of infinite networks, which are composed of individual subsystems.

**Definition 2.2:** Consider discrete-time subsystems \( \Psi_i = (X_i, W_i, f_i), i \in \mathbb{N} \), with their internal input structure defined in (2). The infinite network can be expressed as the tuple \( \Psi = (X, f) \), where
\[
X = \{ x = (x_i)_{i \in \mathbb{N}}: x_i \in X_i, \| x \| = \sup_{i \in \mathbb{N}} \{ |x_i| \} < \infty \}, \]
\[
f(x) = (f_i(x_i, w_{ij}))_{i \in \mathbb{N}}.
\]
The infinite network, denoted as \( \Psi = \mathcal{N}(\Psi_i)_{i \in \mathbb{N}} \), operates according to
\[
\Psi: x(k + 1) = f(x(k)), \quad k \in \mathbb{N}_0,
\]
wherein the interconnection of subsystems is illustrated through the following constraints:
\[
\forall i \in \mathbb{N}, \forall j \in N_i: \quad w_{ij} = x_j, \quad X_j \subseteq W_{ij}.
\]
We refer to the sequence \( x_{x_0} : \mathbb{N} \to X \) that satisfies (3) for any initial state \( x_0 \in X \) as the state trajectory of \( \Psi \) starting from an initial state \( x_0 \).

In the following section, we introduce the concepts of local-barrier certificates (L-BC) for subsystems with internal signals and global-barrier certificates (G-BC) for interconnected networks without internal signals.

### III. Local and Global Barrier Certificates

**Definition 3.1:** Given a subsystem \( \Psi_i = (X_i, W_i, f_i) \) defined in Definition 2.1, with \( X_0, X_u \subseteq X_i \) denoting, respectively, initial and unsafe sets of \( \Psi_i \), a function \( B_i : X_i \to \mathbb{R}_{\geq 0} \) is called a local-barrier certificate (L-BC) for \( \Psi_i \) if there exist \( \delta_i, \sigma_i, \phi_i \in \mathbb{R}_{>0}, \xi_i \in (0, 1) \), and \( \rho_{w_i} \in \mathbb{R}_{\geq 0} \), such that
\[
B_i(x_i) \geq \delta_i |x_i|^2, \quad \forall x_i \in X_i, \quad (5a)
\]
\[
B_i(x_i) \leq \sigma_i, \quad \forall x_i \in X_u, \quad (5b)
\]
\[
B_i(x_i) \geq \phi_i, \quad \forall x_i \in X_w, \quad (5c)
\]
and \( \forall x_i \in X_i, \forall w_i \in W_i \), one has
\[
B_i((x_i)(k + 1)) \leq \max \{ \xi_i B_i(x_i(k), \rho_{w_i}|w_i(k)|^2) \}. \quad (5d)
\]

We now introduce a complementary definition of global-barrier certificates for interconnected networks without internal inputs, which is subsequently employed to enforce safety specifications across an infinite network.

**Definition 3.2:** Consider an infinite network \( \Psi = (X, f) \) in Definition 2.2, with \( X_0, X_u \subseteq X \) denoting the initial and unsafe sets of \( \Psi \), respectively. A function \( B : X \to \mathbb{R}_{\geq 0} \) is called a global-barrier certificate (G-BC) for \( \Psi \) if there exist \( \sigma, \phi \in \mathbb{R}_{>0}, \xi \in (0, 1) \) and \( \rho \geq \sigma \), such that
\[
B(x) \leq \sigma, \quad \forall x \in X_0, \quad (6a)
\]
\[
B(x) \geq \phi, \quad \forall x \in X_u, \quad (6b)
\]
\[
B(x(k + 1)) \leq \xi B(x(k)), \quad \forall x \in X. \quad (6c)
\]

**Remark 3.3:** It is important to highlight that the additional condition (5a) in L-BC plays a pivotal role in facilitating compositional techniques in Section V. In addition, while condition \( \phi > \sigma \) in G-BC is vital in ensuring a safety certificate for an infinite network, as demonstrated in Theorem 3.4, the L-BC of subsystems does not impose such a requirement. In fact, L-BC are solely utilized for constructing G-BC for infinite networks without ensuring subsystem safety.

The subsequent theorem, borrowed from [17], provides a guarantee that the state trajectories of interconnected networks will never enter an unsafe region.

**Theorem 3.4:** Consider an infinite network \( \Psi = (X, f) \), as defined in Definition 2.2, and assuming that \( B \) is a G-BC for \( \Psi \), as specified in Definition 3.2. Then for all \( x_0 \in X_0 \) and \( k \in \mathbb{N} \), the state trajectory \( x_{x_0} \) remains outside of the unsafe region \( X_u \), i.e., \( x_{x_0} \notin X_u \) within an infinite time horizon.

The computational complexity of finding G-BC for interconnected networks is notably high, primarily due to the system's dimensionality. This challenge was a primary motivation for introducing the concept of L-BC for individual subsystems. Subsequently, in Section V, we propose a compositional approach for constructing a G-BC for an infinite network based on L-BC of individual subsystems.

To formally ensure the safety of the infinite network in (3) via Theorem 3.4, one needs precise knowledge of the mapping \( f_i \) for each subsystem to verify condition (5d), which is not accessible in our current context. To overcome this challenge, we present our data-driven approach in the next section, wherein we construct L-BC based on finite data sets obtained from trajectories of subsystems.

### IV. Data-Driven Construction of L-BC

In our data-driven setting, we consider the structure of L-BC as \( B_i(q_i, x_i) = \sum_{j=1}^{\xi_i} (q_i^j)(p_i^j(x_i)) \), where \( p_i^j \) denote user-defined (possibly nonlinear) basis functions, and \( q_i = [q_i^1, \ldots, q_i^\xi_i] \in \mathbb{R}^\xi_i \) represent unknown coefficients. To satisfy conditions (5a)-(5d), we approach the problem by transforming it into the subsequent robust optimization program (ROP):
\[
\min_{\Lambda_i; \rho_{w_i}} \eta_i
\]
\[
\text{s.t.} \quad -B_i(q_i, x_i) + \delta_i(x_i^\top x_i) \leq \eta_i, \quad \forall x_i \in X_i, \quad (7a)
\]
\[
B_i(q_i, x_i) - \sigma_i \leq \eta_i, \quad \forall x_i \in X_0, \quad (7b)
\]
\[
-B_i(q_i, x_i) + \phi_i \leq \eta_i, \quad \forall x_i \in X_u, \quad (7c)
\]
\[
B_i(q_i, f(x_i, w_i)) - \xi_i B_i(q_i, x_i)
\]
\[
- \rho_{w_i} (w_i^\top w_i) \leq \eta_i, \quad \forall x_i \in X_i, \forall w_i \in W_i, \quad (7d)
\]
\[
\Lambda_i = [\delta_i; \xi_i; \sigma_i; \phi_i; \rho_{w_i}; q_i^1; \ldots, q_i^\xi_i],
\]
\[
\delta_i, \xi_i, \sigma_i, \phi_i \in \mathbb{R}_{>0}, \rho_{w_i} \in \mathbb{R}_{\geq 0}, q_i^j \in \mathbb{R}, \xi_i \in (0, 1),
\]
where \( \eta_i^* \) in (7d) refers to the dimension of \( W_i \). If \( \eta_R^* \leq 0 \), with \( \eta_R^* \) being the optimal value of ROP, solving the ROP indicates that conditions (5a)-(5d) are fulfilled.

**Remark 4.1:** Within the constraints specified in (5a) and (5d), we encounter infinity norms associated with variables
\( x_i \) and \( w_i \), respectively. To render them computationally suitable for use in the ROP, we convert these infinity norms into Euclidean norms by incorporating their corresponding weight factors. Likewise, we have reformulated the max-form condition (5d) into a summation form represented by (7d). This reformulation is achieved by recovering the variables \( \xi_i \) and \( \rho_{w_i} \), based on \( \xi_i \) and \( \rho_{w_i} \), as follows, for any \( 0 < \pi_i < 1 \):

\[
\xi_i = 1 - (1 - \pi_i)(1 - \xi_i), \quad \rho_{w_i} = \frac{\rho_{w_i}}{(1 - \xi_i)\pi_i}.
\]

The ROP stated in (7) poses a main challenge due to its dependence on the precise mapping \( f_i(x_i, w_i) \), which is not available within the context of our problem. To address this difficulty, we propose our data-driven solution by introducing a scenario optimization program for the ROP in (7). To do so, we leverage a dataset of samples denoted as \( (\hat{x}_i, \hat{w}_i)^{S_i} \) within \( X_i \times W_i \), with \( S_i \in \mathbb{N} \), as follows:

\[
(\hat{x}_i, \hat{w}_i), f(\hat{x}_i, \hat{w}_i), \quad \forall s \in \{1, \ldots, S_i\}.
\]

We now define a ball of radius \( \alpha_i \) around each sample \( (\hat{x}_i, \hat{w}_i) \) as \( X_i \times W_i \), such that \( X_i \times W_i \subseteq \bigcup_{i=1}^{S_i} (X_i^i \times W_i^i) \) and

\[
\| (x_i, w_i) - (\hat{x}_i, \hat{w}_i) \| \leq \alpha_i, \quad \forall (x_i, w_i) \in X_i \times W_i.
\]

Now instead of addressing the ROP as presented in (7), we focus on solving the following scenario optimization program (SOP), \( \forall s \in \{1, \ldots, S_i\} \):

\[
\min_{\eta_i} \eta_i \\
\text{s.t.} \quad -B_i(q_i, \hat{x}_i) + \delta_i(\hat{x}_i^\top \hat{x}_i) \leq \eta_i, \quad \forall \hat{x}_i \in X_i, \tag{9a}
\]

\[
B_i(q_i, \hat{x}_i) - \sigma_i \leq \eta_i, \quad \forall \hat{x}_i \in X_0, \tag{9b}
\]

\[
-B_i(q_i, \hat{x}_i) + \phi_i \leq \eta_i, \quad \forall \hat{x}_i \in \bar{X}_u_i, \tag{9c}
\]

\[
B_i(q_i, \hat{x}_i, f(\hat{x}_i, \hat{w}_i)) - \hat{\zeta}_i(B_i(q_i, \hat{x}_i)) - \rho_{w_i}(\hat{w}_i^\top \hat{w}_i) \leq \eta_i, \quad \forall (\hat{x}_i, \hat{w}_i) \in X_i \times W_i, \tag{9d}
\]

\[
\eta_i \geq \max \{ \eta_i^1, \eta_i^2, \eta_i^3 \}.
\]

We represent the optimal value of SOP as \( \eta_i^* \).

**Remark 4.2:** Due to a mild bilinearity between unknown variables \( \xi_i \in (0, 1) \) and \( q_i \) in condition (9d), we constrain \( \xi_i \) to the discrete set \( \xi_i \in \{ 0, \xi_i^1, \ldots, \xi_i^l \} \) with a cardinality of \( l \). This allows us to tackle the bilinearity by solving the SOP for a specific \( \xi_i \) while designing \( q_i \).

**A. Data-Driven L-BC Construction with Guarantee**

Here, our primary objective is to solve the proposed SOP in (9) and construct L-BC for unknown discrete-time subsystems \( \Psi_i \) with a guarantee of correctness. To attain this objective, we commence by raising the following assumption.

**Assumption 4.3:** Suppose \( B_i(q_i, x_i) \) and \( \delta_i(x_i^\top x_i) - B_i(q_i, x_i) \) are Lipschitz continuous with respect to \( x_i \) with, respectively, Lipschitz constants \( \mathcal{L}_i\mathcal{L}_i \) and \( \mathcal{L}_2\mathcal{L}_2 \), for any \( i \in \mathbb{N} \). Moreover, \( B_i(q_i, f(x_i, w_i)) - \xi_i B_i(q_i, x_i) - \rho_{w_i}(\frac{w_i^\top w_i}{p_i}) \) is Lipschitz continuous with respect to \( (x_i, w_i) \) with Lipschitz constant \( \mathcal{L}_3\mathcal{L}_3 \), for any \( i \in \mathbb{N} \).

Under Assumption 4.3, and inspired by [41], the following theorem outlines our data-driven approach for constructing L-BC for unknown discrete-time subsystems \( \Psi_i \) with a guarantee of correctness.

**Theorem 4.4:** Consider subsystems \( \Psi_i = (X_i, W_i, f_i) \) defined in Definition 2.1. Assume Assumption 4.3 is satisfied. Let SOP (9) be solved with \( S_i \) sampled data, as in (8), and an optimal value \( \eta_i^* \) and a solution \( \xi_i^* = [\delta_i; \xi_i^1; \sigma_i; \phi_i; \rho_{w_i}; q_i^1; \ldots; q_i^3]^\tau \). If

\[
\eta_i^* + \mathcal{L}_i\mathcal{L}_1\sigma_i \leq 0 \tag{10}
\]

with \( \mathcal{L}_i = \max \{ \mathcal{L}_1\mathcal{L}_1, \mathcal{L}_2\mathcal{L}_2, \mathcal{L}_3\mathcal{L}_3 \} \), then \( B_i \) resulting from solving the SOP in (9) is an L-BC for \( \Psi \) with a guarantee of correctness.

**Remark 4.5:** Note that the ball’s radius \( \alpha_i \) is crucial in satisfying the data-driven condition (10), providing correctness guarantee over the construction of L-BC based on data. To potentially reduce the required number of samples, one may initially gather data using a larger value of \( \alpha_i \) to solve the SOP in (9). If condition (10) is not met with the chosen (potentially large) \( \alpha_i \), it becomes necessary to opt for a smaller \( \alpha_i \) and re-solve the SOP. When working with real data, it is feasible to consider a sufficiently large \( \alpha_i \) (worst-case scenario) and ensure satisfaction of condition (10).

To verify condition (10) in Theorem 4.4, the computation of \( \mathcal{L}_i \) is required. To achieve this, we utilize the fundamental results of [46] and present Algorithm 1 for estimating the corresponding Lipschitz constants using a finite set of data for each subsystem. While the algorithm is dedicated to estimating \( \mathcal{L}_i \), by following the similar steps, one can estimate \( \mathcal{L}_i \), \( \mathcal{L}_2 \), using a finite set of data, where \( g(\hat{x}_i^\tau) = B_i^*(q_i, \hat{x}_i^\tau) \) and \( g(\hat{x}_i^\tau) = \delta_i^*(\hat{x}_i^\top \hat{x}_i^\tau) - B_i^*(q_i, \hat{x}_i^\tau) \) in Step 3, respectively. Under Algorithm 1, the convergence of the estimated values \( \mathcal{L}_1\mathcal{L}_1, \mathcal{L}_2\mathcal{L}_2, \mathcal{L}_3\mathcal{L}_3 \) to their actual values is guaranteed in the limit, as supported by the following lemma [46].

**Lemma 4.6:** The estimated values \( \mathcal{L}_1\mathcal{L}_1, \mathcal{L}_2\mathcal{L}_2, \mathcal{L}_3\mathcal{L}_3 \) in Algorithm 1 converge to their actual values if and only if \( \beta \) approaches zero while \( \kappa \) and \( \kappa \) tend to infinity.

**Remark 4.7:** Given the necessity of determining unknown coefficients \( q_i \) to estimate the Lipschitz constants \( \mathcal{L}_1\mathcal{L}_1, \mathcal{L}_2\mathcal{L}_2, \mathcal{L}_3\mathcal{L}_3 \) in Algorithm 1, it is crucial to initially solve the proposed SOP outlined in (9). To avoid the need for subsequent verification of condition (10), one may initially assume a certain range for unknown coefficients \( q_i \) and estimate the Lipschitz constants \( \mathcal{L}_1\mathcal{L}_1, \mathcal{L}_2\mathcal{L}_2, \mathcal{L}_3\mathcal{L}_3 \) before solving the SOP. Consequently, it is essential to enforce these established ranges during the solution of the SOP.

In the forthcoming section, we introduce a compositional approach based on small-gain reasoning to construct a G-BC for an infinite network, building upon L-BC of individual subsystems, constructed from data in Theorem 4.4.

**V. COMPOSITIONAL CONSTRUCTION OF G-BC**

In this section, we propose a compositional framework that enables the construction of G-BC for an infinite network \( \Psi \) by leveraging L-BC of individual subsystems \( \Psi_i, i \in \mathbb{N} \).
Algorithm 1 Estimation of Lipschitz constant $L^3_i$ using data

Inputs: $L$-BC $B^*_{\Psi_i}$, $\bar{\xi}^*_i$, $\bar{\rho}^*_w$

1: Choose $\bar{\kappa}, \tilde{\kappa} \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}_{>0}$
2: Select $\bar{\kappa}$ sampled pairs $((\hat{x}^i, \hat{w}^i), (\hat{x}'^i, \hat{w}'^i))$ from $X_i \times W_i$ such that $\|\hat{x}^i - \hat{x}'^i\|_\infty \leq \beta$
3: Compute the slope $\Delta^i_{\hat{r}}$ as
   $$\Delta^i_{\hat{r}} = \frac{g(\hat{x}^i, \hat{w}^i) - g(\hat{x}'^i, \hat{w}'^i)}{\|\hat{x}'^i - \hat{x}^i\|_\infty}, \forall \hat{r} \in \{1, \ldots, \bar{\kappa}\}$$
4: Compute the maximum slope as $q_i = \max\{\Delta^i_{\hat{r}}\}$
5: Repeat Steps 2-4 $\bar{\kappa}$ times and acquire $q^i_1, \ldots, q^i_{\bar{\kappa}}$
6: Through the utilization of the Reverse Weibull distribution [46] on $q^i_1, \ldots, q^i_{\bar{\kappa}}$, which yields location, scale, and shape parameters, select the location parameter as an estimation of $L^3_i$

Output: Lipschitz constant $L^3_i$

Considering subsystems $\Psi_i$ as in Definition 2.1, assume the existence of L-BC $B_i$ as defined in Definition 3.1 with constants $\delta_i, \sigma_i, \phi_i \in \mathbb{R}_{>0}, \xi_i \in (0, 1)$, and $\rho_w \in \mathbb{R}_{>0}$. We now define, $\forall i, j \in \mathbb{N}$,

$$\xi_{ij} := \begin{cases} \xi_i, & \text{if } i = j, \\ \frac{\rho_w}{\delta_i}, & \text{if } j \in N_i, \\ 0, & \text{if } i \neq j, j \notin N_i. \end{cases}$$

Correspondingly, we define $\zeta : l_+^\infty \rightarrow l_+^\infty$ as

$$\zeta(s) = (\sup_{j \in \mathbb{N}} \{\xi_{ij}s_j\})_{i \in \mathbb{N}}, \quad s \in l_+^\infty. \quad (11)$$

We also assume that there exist constants $\bar{\xi}, \tilde{\delta} \in \mathbb{R}_{>0}$, and $\rho_w \in \mathbb{R}_{>0}$, such that $\xi_i \leq \bar{\xi}, \rho_w \leq \rho_w, \delta_i \geq \tilde{\delta}$, for all $i \in \mathbb{N}$. This assumption ensures the well-posedness of $\zeta$ in (11).

To establish the primary compositionality findings of the paper, we present the following small-gain assumption, inspired from [29].

**Assumption 5.1:** Consider operator $\zeta$ as defined in (11). Assume that $\sup_{j \in \mathbb{N}} \{\xi_{ij}s_j\} > 0, \forall s_j > 0, \forall i, j \in \mathbb{N}$, $\zeta$ is continuous on $l_+^\infty$, $\lim_{n \rightarrow +\infty} \zeta^n(s) = 0$, and there exist positive constants $c_1$ and $c_2$ such that for all $i, j \in \mathbb{N}$, the operator $\zeta_{ij}(s) := \zeta(s) + c_1s_jc_i, s \in l_+^\infty$ fulfills the following condition:

$$\zeta_{ij}(s) \not\geq (1 - c_2)s, \quad s \in l_+^\infty \setminus \{0\}. \quad (12)$$

The small gain condition (12) implies the existence of a vector $\mu := (\mu_i)_{i \in \mathbb{N}} \in l_+^\infty$ with $\mu_i \in \mathbb{R}_{>0}, i \in \mathbb{N}$, and $\epsilon \in (0, 1)$, such that [29, Lemma 4.5]

$$\zeta(\mu) \leq (1 - \epsilon)\mu. \quad (13)$$

As a result, according to (11) and since $\epsilon \in (0, 1)$, one has, $\forall i \in \mathbb{N}$:

$$\sup_{j \in \mathbb{N}} \{\xi_{ij}\mu_j\} \leq (1 - \epsilon)\mu_i < \mu_i.$$ 

By applying $\frac{1}{\mu_i}$ to both sides, we have

$$\frac{1}{\mu_i} \sup_{j \in \mathbb{N}} \{\xi_{ij}\mu_j\} = \sup_{j \in \mathbb{N}} \left(\frac{\xi_{ij}\mu_j}{\mu_i}\right) < 1. \quad (14)$$

Since inequality (13) holds for all $i \in \mathbb{N}$, it can be generalized as

$$\sup_{i, j \in \mathbb{N}} \left\{\frac{\xi_{ij}\mu_j}{\mu_i}\right\} < 1. \quad (15)$$

In the upcoming theorem, under Assumption 5.1, we construct a G-BC for the infinite network $\Psi$ using L-BC of $\Psi_i, i \in \mathbb{N}$, constructed from data.

**Theorem 5.2:** Consider an infinite network $\Psi = \mathcal{N}(\Psi_i)_{i \in \mathbb{N}}$, which arises from infinitely many subsystems $\Psi_i$. Assume that each individual subsystem $\Psi_i$ possesses an L-BC $B_i$, constructed from data according to Theorem 4.4 with a guarantee of correctness. If Assumption 5.1 is satisfied, and

$$\sup_{i} \left\{\frac{\phi_i}{\mu_i}\right\} > \sup_{i} \left\{\frac{\sigma_i}{\mu_i}\right\}, \quad (16)$$

then the function $B(x)$ defined as

$$B(x) := \sup_{i} \left\{\frac{1}{\mu_i}\chi_i(x_i)\right\}. \quad (17)$$

is a G-BC for the infinite network $\Psi = \mathcal{N}(\Psi_i)_{i \in \mathbb{N}}$, with $\sigma := \sup_{i} \left\{\frac{\sigma_i}{\mu_i}\right\}, \phi := \sup_{i} \left\{\frac{\phi_i}{\mu_i}\right\}$, and $\xi = \sup_{i, j} \left\{\frac{\xi_{ij}\mu_j}{\mu_i}\right\}$. 

**Remark 5.3:** Assuming that $\xi_{ij} \leq 1$ for any $i, j \in \mathbb{N}$, inequality (16) can be satisfied by setting $\mu_i = 1$ for all $i \in \mathbb{N}$. Consequently, inequality (16) is simplified as $B(x) := \sup_{i} \left\{\{B_i(x_i)\}\right\}$, and as a result, the small-gain condition (12), equivalently inequality (14), is automatically fulfilled.

VI. CASE STUDY

In this section, we showcase the efficacy of our proposed results by applying them into a vehicle platoon consisting of an infinitely countable number of vehicles, as depicted in Fig. 2. The dynamics of the interconnected network can be described as [47]

$$\Psi : x(k + 1) = Ax(k) + u(k),$$

where $u(k)$ is a previously designed and deployed controller and $A$ is a block matrix featuring diagonal elements represented by $A_i$, and off-diagonal blocks denoted as $A_{i(i-1)} = A_w, i \geq 2$, as:

$$\hat{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad A_w = \begin{bmatrix} 0 & \tau \\ 0 & 0 \end{bmatrix},$$

with the interconnection strength characterized by $\tau = 0.005$. Additionally, all non-diagonal blocks are specified as zero matrices of appropriate dimensions. Moreover, $x(k) = (x_i(k))_{i \in \mathbb{N}}$ and $u(k) = (u_i(k))_{i \in \mathbb{N}}$. We now proceed with presenting a description of each individual vehicle $\Psi_i$ as $\Psi_i : (k + 1) = \hat{A}x(k) + u_i(k) + A_wu_i(k)$.
It can be readily ascertained that $\Psi = \mathcal{N}(\Psi_i)_{i \in \mathbb{N}}$, where $w_i(k) = [0; w_{i(i-1)}(k)]$, with $w_{i(i-1)}(k) = [0; \ldots C. Belta, B. Yordanov, and E. A. Gol, Formal methods for discrete-time dynamical systems . Springer, 2017, vol. 89.

We assume that the well-posedness of $\Psi = \mathcal{N}(\Psi_i)_{i \in \mathbb{N}}$ holds true by validating the condition of $\|f(x)\| < \infty$, as in Definition 2.2. By deploying the controller in (17), and defining $C = \max(\{A, |A_w|\})$, where $A$ is the new matrix of the system after deploying the controller, we have:

$$
\|f(x)\| = \sup_{i \in \mathbb{N}} \{f_i(x_i, u_i)\} = \sup_{i \in \mathbb{N}} \{\dot{x}_i + A_i x_i + A_{wi} w_i\}
\leq |A| \sup_{i \in \mathbb{N}} \{|x_i|\} + |A_w| \sup_{i \in \mathbb{N}} \{|w_i|\}
\leq C (\sup_{i \in \mathbb{N}} \{|x_i|\} + \sup_{i \in \mathbb{N}} \{|x_i|\}) = C (\|x\| + \|x\|) < \infty.
$$

Consequently, one can assert that $\Psi = \mathcal{N}(\Psi_i)_{i \in \mathbb{N}}$ is well-defined. It is worth noting that even though matrices $A$ and $A_w$ are unknown, which is the case here, the resulting well-posedness conclusion remains valid.

The regions of interest for each vehicle are defined as follows: $X_i \in [0,1] \times [-0.3, 0.7]$, $X_0 \in [0.25, 0.75] \times [-0.05, 0.45]$, and $X_{ui} \in [0,1] \times [-0.3, 0.15] \cup [0,1] \times [0.55, 0.7]$. Our primary objective is to construct an L-BC, using collected data from each vehicle with unknown dynamics, by solving the SOP (9) for each $\Psi_i$. Subsequently, under the proposed small-gain condition, we aim to construct a G-BC based on individual L-BC while ensuring that the trajectory of the infinite network remains safe in infinite time horizons. To do so, by considering $\alpha_i = 0.005$ and $\xi_i^* = 0.95$, we solve the SOP in (9) and compute coefficients of L-BC, along with other decision variables in the SOP, as follows:

$$
B_i(q_i, x_i) = -0.0843d_i^4 - 0.14d_i^3 v_i + 0.14d_i^4 + 0.1228d_i^2 v_i^2 + 0.14d_i^2 v_i - 0.0514d_i + 0.14d_i v_i^3 - 0.14d_i v_i - 0.125d_i v_i - 0.0123d_i - 0.14v_i^4 + 0.14v_i^3 + 0.14v_i^2 - 0.1073v_i + 0.0506
\delta_i^* = 0.05, \alpha_i^* = 0.05, \phi_i^* = 0.06, \bar{\rho}_{wi} = 2 \times 10^{-8}, \eta_{i,g}^* = -0.005.
$$

Consequently, according to Remark 4.1, values of $\xi_{i}$ and $\rho_{wi}$ outlined in (5d) are computed as $\xi_i = 0.975$ and $\rho_{wi} = 8 \times 10^{-7}$, by selecting $\pi_i = 0.5$. We then apply Algorithm 1 and compute $L_i^1 = 0.1648$, $L_i^2 = 0.1682$ and $L_i^3 = 0.1598$.

We now proceed with showing the small-gain condition as an essential requirement for the compositional results. Given that $\delta_i^* > \rho_{wi}$, it becomes evident that $\xi_{ij} \leq 1$, for any $i, j \in \mathbb{N}$. According to Remark 5.3, with the choice of $\mu_i = 1$ for each $i \in \mathbb{N}$, it can be concluded that condition (12), and consequently (14), are satisfied without imposing any constraints on the number of vehicles. As a result, $B_i(x) := \sup_{i \in \mathbb{N}} \{B_i^i(x_i)\}$ arises as a G-BC for the infinite network $\Psi$ with $\sigma = 0.058$, $\phi = 0.064$, and $\xi = 0.975$. By applying Theorem 3.4, we can guarantee that all trajectories of the interconnected network originating from $X_0$ will remain within the safe domain throughout an infinite time horizon. Closed-loop state trajectories of a representative vehicle under the previously designed and deployed controller are depicted in Fig. 3.

VII. CONCLUSION

In this paper, we offered a framework for formally verifying the safety of an interconnected network that consisted of a countably infinite array of discrete-time subsystems, each with unknown mathematical models. Our approach involved breaking down the overall problem into subsystem levels, where we modeled the safety concept for each subsystem using a robust optimization program (ROP) based on the notion of local-barrier certificates. We then collected finite data from subsystem trajectories and created a scenario optimization program (SOP). We solved the resulting SOP and constructed a local-barrier certificate for each unknown subsystem, while providing a guarantee of correctness. Finally, under some small-gain conditions, we constructed a global-barrier certificate from local certificates of subsystems to ensure the safety of the infinite network.

REFERENCES
