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Distributed Average Consensus with Beep Communication

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***Abstract*— Motivated by the hostile working environments that lack a robust communication infrastructure, such as in the case of precision agriculture settings, we propose a novel bandwidth-saving average consensus procedure that exploits the beep communication model. Specifically, we allow the agents to alternatively perform traditional average consensus steps and steps where the agents only inform their neighbors about the fact that their state has increased or decreased with respect to the previous time step. All the information is transmitted among the agents via beeps, which represent a weak communications model with bandwidth preservation. We theoretically characterized the practical convergence property of the proposed algorithm towards the network average, i.e., the consensus error can be made arbitrarily small by acting on the parameters of the protocol. Additionally, we also numerically demonstrate that, for a proper choice of such parameters, the protocol exhibits an interesting trade-off between convergence rate and achievable accuracy.**

I. Introduction

In the last decades, multi-agent systems (MAS) [1] have received great attention due to their capability to foster agents' cooperation, as in the case of consensus [2], [3], which represents the cornerstone of applications in various fields [4], [5]. Specifically, in the context of precision agriculture, multi-agent systems are exploited to improve all the aspects of agricultural production, from crop performance to environmental quality [6]–[8]. However, despite the potential for economic, environmental, and social benefits, these solutions come up against the difficulties of actual implementation of the designed systems [9]. One of the most critical issues concerns the environment in which agents (robots, drones, or microcontrollers) operate [10], as it often spans vast geographical areas that are unstructured and poorly supported by communication infrastructure [7]; this poses significant challenges to the implementation of distributed algorithms such as consensus, which requires continuous and synchronous exchanges of information between agents.

In a scenario of an unstructured environment with low communication capacity, energy constraints, and scarce computational resources, one of the key aspects concerns the preservation of bandwidth and the reduction of the length of messages exchanged. To address this issues, some methodologies have been proposed, such as the distributed consensus with quantized communications assuming fixed [11], [12], or even time variant topologies [13]. In [14], an impulsive consensus problem of nonlinear multiagent systems is addressed, but for this and the previous approaches, the selection of the number of quantization levels is required. A further strategy includes speeding up the convergence of the consensus algorithm, as in [15], where each agent can expand its knowledge by employing multi-hop paths. However, by allowing multi-hop communications, this methodology is not applicable in an environment with low communication capability and the need to safeguard bandwidth utilization. Another approach is based on the alternating direction multipliers method (ADMM) [16]; requiring the solving of convex optimization problems, this approach is difficult to apply in contexts with limited computational capabilities.

A recent and effective model for weak communication is the one proposed in [17], [18], which exploits beeps as communication exchanges between agents. This model is interesting from a practical point of view since it is possible to implement it (or emulate it) even in extremely restricted radio network environments. Interestingly, such kind of approaches have been successfully adopted in the context of power systems [19] and mobile devices [20]. Based on the beep model, it has been shown that several tasks can still be carried out in spite of the reduced communication bandwidth, e.g., interval coloring [17], distributed voting [21] or broadcasting [18]. In [22], the binary consensus problem is addressed, in which each agent is initialized with a value in the set $\{0, 1\}$ and aims to reach a common agreement with the network on a value in the same set. In [23], this framework is extended to the case the set of initial values is not binary but contains some values assumed by the network as initial conditions. However, to the best of our knowledge, this type of communication has never been exploited to propose a new bandwidth-saving iterative structure for accomplishing consensus tasks. In this paper, we aim to fill this gap by developing a protocol suitable for all the applications characterized by the lack of robust communication infrastructure and energy constraints, such as in the case of precision agricultural settings.

A. Contribution

Despite the growing need to ensure that a common agreement is reached in some real-world application scenarios where a solid communication infrastructure is lacking and energy constraints are in place (such as precision agriculture), to date, this aspect is still poorly investigated.

In this paper, we propose a low-bandwidth usage discrete-time average consensus algorithm for undirected graphs that

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exploits the beep communication model. Specifically, our approach is based on the alternation of two dynamics: the first involves traditional average consensus steps, while in the second the agents update their state locally, based on whether the states of their neighbors increased or decreased during the last consensus step. All the information exchanged for the two dynamics is transmitted via beeps, including both the values of the state variables and the variation indicator signal, which is encoded in a message of shorter length, allowing the algorithm to progress based on information derived from a minimal communication effort.

We theoretically characterize the practical convergence property [24] of the proposed protocol, that is the error can be made arbitrarily small by acting on the parameters of the dynamics. Additionally, we numerically demonstrate that, for a proper choice of such parameters, the protocol exhibits an interesting trade-off between convergence rate and achievable accuracy. This makes the proposed protocol a valuable alternative compared to the standard consensus algorithm, especially for all those scenarios lacking a robust communication infrastructure while requiring an energy-preserving solution, such as in the case of precision agriculture settings.

The outline of the paper is as follows: Section II collects some preliminary concepts and definitions; Section III details the proposed approach for distributed consensus algorithm and Section IV analyzes the error dynamics and its convergence properties. Section V provides some simulations to numerically evaluate convergence scenarios as parameters change; finally, Section VI draws some conclusions and discusses possible future work directions.

II. Preliminaries

A. Notation

We denote vectors with boldface lowercase letters and matrices with uppercase letters. We refer to the (i, j)-th entry of a matrix A by A_{ij}. We use $\texttt{ker}(A)$ to refer to the kernel of A. We represent by $\mathbf{0}_n$ and $\mathbf{1}_n$ vectors with n entries, all equal to zero and to one, respectively. Given a vector $\boldsymbol{p} \in \mathbb{R}^n$, we use $\texttt{diag}(\boldsymbol{p})$ to denote the $n \times n$ diagonal matrix such that $\texttt{diag}_{ii}(\boldsymbol{p}) = p_i$. We also use $\texttt{span}(\boldsymbol{p})$ to denote the linear span of the vector $\boldsymbol{p}$, i.e., the set of all vectors $\alpha\boldsymbol{p}$ with $\alpha \in \mathbb{R}$. We represent the elementwise signum function of a vector $\boldsymbol{p}$ using the notation $\texttt{sign}(\boldsymbol{p})$. We use $\|\cdot\|$ to denote the Euclidean norm and $\|\cdot\|_1$ to denote the Manhattan norm.

B. Graph Theory

Let $G = \{V, E\}$ be a *graph* with n nodes $V = \{v_1, \ldots, v_n\}$ and e edges $E \subseteq V \times V$, where $(v_i, v_j) \in E$ captures the existence of a link from node v_i to node v_j. A graph is said to be *undirected* if the existence of an edge $(v_i, v_j) \in E$ implies the existence of $(v_j, v_i) \in E$, while it is said to be *directed* otherwise. In this paper, we consider undirected graphs. Let the neighborhood $\mathcal{N}_i$ of a node v_i in an undirected graph be the set of nodes v_j such that $(v_j, v_i) \in E$. The *degree* d_i of a node v_i in an undirected graph is the number of its edges, i.e., $d_i = |\mathcal{N}_i|$. Given an undirected graph $G = \{V, E\}$ with n nodes, we define the Laplacian matrix L as the $n \times n$ matrix such that $L_{i,j} = d_i$ if $i = j$, $L_{i,j} = -1$ if $(v_j, v_i) \in E$, and $L_{i,j} = 0$, otherwise. It is well known [25] that when G is connected L has a unique eigenvalue equal to zero and that the corresponding left and right eigenvectors coincide and are equal to $\mathbf{1}_n$.

C. Distributed Consensus

Let us consider a network of n agents, interconnected by an undirected graph $G = \{V, E\}$. Let us suppose each agent i holds an initial condition $w_i(0) = w_{i0}$, and can interact according to the protocol

$$w_i(k+1) = w_i(k) + \tau \sum_{j \in \mathcal{N}_i} \left(w_j(k) - w_i(k)\right), \qquad (1)$$

which, in a compact form, corresponds to $\boldsymbol{w}(k+1) = (I - \tau L)\boldsymbol{w}(k)$, where $\boldsymbol{w}(k) \in \mathbb{R}^n$ collect the entries $w_i(k)$.

It is well known (see [2], [3]) that, if G is connected and undirected and $\tau < 1/max_i\{d_i\}$, then the agents asymptotically reach an agreement such that

$$\lim_{k \to \infty} \boldsymbol{w}(k) = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top \boldsymbol{w}(0),$$

i.e., the agents asymptotically reach the average of their initial conditions.

D. Beep Communication Model

The beep model, introduced in [17], is a weak network communication model in which information can be passed only in the form of a beep or a silence; it can be used to transmit logical values or binary-encoded messages sent through bit strings of defined length [18]. The network is modeled as an undirected connected graph $G = \{V, E\}$, where vertices in the graph represent devices in the network, and edges represent direct reachability. According to the model, each node is able to send a beep and all neighboring nodes will receive it: each node can both beep and listen, but, in order to avoid collisions, it is not possible to listen while beeping. Notably, each node is assumed to have minimal knowledge about the environment and severely limited communication capabilities, therefore it cannot even distinguish the number of beeps it receives if they are sent to it simultaneously. Time is divided into discrete steps, and the basic unit of time is referred to as the time slot; we assume that global synchronization among the agents' clocks has been achieved, as in the other works exploiting the beep model [17], [21], and that the nodes send the beep signals at the specified time slots. It is assumed that, at the beginning of the message, the sender node identifier is provided suitably encoded.

III. Proposed Approach

Let us consider a network of n agents interconnected by a connected and undirected graph topology $G = \{V, E\}$, each holding a scalar initial condition[1] $x_i(0) \in \mathbb{R}$. Moreover, let

[1]For the sake of clarity, scalar initial conditions are considered even though the result can be easily generalized for the vectorial case.

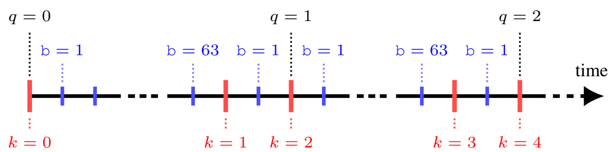


Fig. 1: Micro- and macro-iterations within the proposed approach. A consensus macro-iteration requires w beep steps $\mathtt{b}$ (e.g., $w = 64$) while a state variation macro-iteration requires two beep steps. We use k to index the k-th macro-iteration (either type), while we use q to index the overall-iterations (i.e., a consensus macro-iteration, followed by a state variation one).

us consider the average consensus problem, i.e., the agents aim to reach the average of their initial values. In doing so, we assume that agents are able to transmit information to others by exchanging only beep signals, according to the beep communication model.

Assumption 1: Each agent in the network is able to communicate with its neighbors by exchanging beep signals.

Within the proposed framework, we assume that the agents communicate with their neighbors by using sequences of beeps (referred to as *beep steps* and acting as micro-iterations) that are organized by alternating two conceptual phases (see Fig. 1), which we refer to as *macro-iterations*: a *consensus macro-iteration* and a *state variation macro-iteration*. In particular, the consensus macro-iteration amounts to the transmission of the agent's state (e.g., a double-precision[2] number) via a given number w of successive beep steps (i.e., $w = 64$ for double-precision numbers). Conversely, the state variation macro-iteration amounts to two beep steps, encoding the fact that the sender's state has increased or decreased, respectively. We use *overall-iteration* to denote a consensus macro-iteration followed by a state variation macro-iteration. In the following, for the sake of analysis only, we use the iterator k to model the k-th macro-iteration: at even-numbered macro-iterations k, each agent i transmits to its neighbors the current value of its state $x_i(k)$, while at odd-numbered macro-iterations each agent only transmits information $b_i(k) \in \{-1, 0, 1\}$, indicating whether that state has decreased or increased compared to the previous step, i.e., $b_i(k) = 1$ if $\boldsymbol{x}_i(k) > \boldsymbol{x}_i(k-1)$, $b_i(k) = 1$ if $\boldsymbol{x}_i(k) < \boldsymbol{x}_i(k-1)$, and $b_i(k) = 0$, otherwise. In order to better describe the dynamics, let us define the index function $\zeta(k)$ as $\zeta(k) = 1$ if k is even, while $\zeta(k) = 0$ if k is odd. In other words, $\zeta(k) = 1$ if k is a consensus macro-step and $\zeta(k) = 0$ if k is a state variation.

Overall, the dynamics of the i-th agent is given by

$$\begin{aligned} x_i(k+1) = x_i(k) + \zeta(k)\tau \sum_{j\in\mathcal{N}_i} (x_j(k) - x_i(k)) \\ + (1-\zeta(k))\beta\tau \sum_{j\in\mathcal{N}_i} (b_i(k) - b_j(k)), \end{aligned} \tag{2}$$

where β, τ are positive gains to be discussed next, while $b_j(k)$ is the variation signal transmitted by the j-th agent at time k. In a compact form, defining $\boldsymbol{x}(k), \boldsymbol{b}(k)$ as the stack of the terms $x_i(k)$ and $b_i(k)$, respectively, we get

$$\boldsymbol{x}(k+1) = \boldsymbol{x}(k) - \zeta(k)\tau L\boldsymbol{x}(k) + (1-\zeta(k))\beta\tau L\boldsymbol{b}(k), \tag{3}$$

where L is the Laplacian matrix associated to the graph G.

Remark 1: Assuming the *word length* to represent the agents' state is w bits, the actual beep counter associated with each iterator k (i.e., the number of beep steps at which the k-th macro-iteration terminates) is T_k, where $T_0 = 0$ and $T_{k+1} = T_k + w\zeta(k) + 2(1-\zeta(k))$. In fact, w beep steps are required during a consensus macro-iteration, while a state variation macro-step consists of two beep steps. Based on the above equation, it can be easily shown that $T_k = (w+2)\lfloor \frac{k}{2} \rfloor + 2(1-\zeta(k))$.

IV. Convergence Analysis

In this section, we aim to define an equivalent expression of the error dynamics, and then analyze its convergence properties. To this end, let us now express the dynamics in terms of an iterator $q = \lfloor k/2 \rfloor$ that represents the number of overall-iterations (see Fig. 1). Notice that the actual beep step associated with the end of each of such iterations q is $T_q = (w+2)q$.

Lemma 1: The overall dynamics of the error is equal to

$$\boldsymbol{e}(q+1) = (I - \tau L)\boldsymbol{e}(q) - \beta\tau L\mathtt{sign}(L\boldsymbol{e}(q)) \tag{4}$$

Proof: Firstly, we observe that, when k corresponds to a consensus macro-iteration (i.e., when k is even), Eq. (3) amounts to

$$\boldsymbol{x}(k+1) = (I - \tau L)\boldsymbol{x}(k) \tag{5}$$

while, during the subsequent state variation macro-iteration, it holds

$$\begin{aligned} \boldsymbol{x}(k+2) &= \boldsymbol{x}(k+1) + \beta\tau L\boldsymbol{b}(k+1) \\ &= \boldsymbol{x}(k+1) + \beta\tau L\mathtt{sign}(\boldsymbol{x}(k+1) - \boldsymbol{x}(k)) \\ &= \boldsymbol{x}(k+1) + \beta\tau L\mathtt{sign}(-\tau L\boldsymbol{x}(k)) \\ &= \boldsymbol{x}(k+1) - \beta\tau L\mathtt{sign}(L\boldsymbol{x}(k)), \end{aligned} \tag{6}$$

where we used the fact that, assuming $\alpha > 0$, it holds $\mathtt{sign}(-\alpha\boldsymbol{w}) = -\mathtt{sign}(\boldsymbol{w})$ by construction.

At this point, plugging Eq. (5) into Eq. (6), and using the iterator q to account only for the overall-iteration, we get the equivalent dynamics

$$\boldsymbol{x}(q+1) = (I - \tau L)\boldsymbol{x}(q) - \beta\tau L\mathtt{sign}(L\boldsymbol{x}(q)). \tag{7}$$

In other words, the above equation models the evolution of the overall dynamics after considering only the q steps.

Let us now consider the error $\boldsymbol{e}(q)$ between the state and the average of the initial conditions, i.e.,

$$\boldsymbol{e}(q) = \boldsymbol{x}(q) - \underbrace{\frac{1}{n}\sum_{i=1}^{n} x_i(0)\mathbf{1}_n}_{\boldsymbol{x}_{\text{ave}}},$$

and let us derive the equivalent error dynamics (i.e., the error dynamics when only overall-iterations are considered). Since

[2]For the sake of simplicity, in this paper we ignore quantization errors.

$L\mathbf{1}_n = \mathbf{0}_n$, we have that

$$\begin{aligned}
&\boldsymbol{e}(q+1) = \boldsymbol{x}(q+1) - \boldsymbol{x}_{\texttt{ave}} \\
&= (I-\tau L)\boldsymbol{x}(q) - \beta\tau L\texttt{sign}(L\boldsymbol{x}(q)) - \boldsymbol{x}_{\texttt{ave}} \\
&= (I-\tau L)(\boldsymbol{x}(q) - \boldsymbol{x}_{\texttt{ave}}) - \beta\tau L\texttt{sign}(L(\boldsymbol{x}(q) - \boldsymbol{x}_{\texttt{ave}})) \\
&= (I-\tau L)\boldsymbol{e}(q) - \beta\tau L\texttt{sign}(L\boldsymbol{e}(q)).
\end{aligned}$$

This completes our proof. ■

Let us now analyze the equivalent error dynamics in Eq. (4) to establish the convergence properties of the error. As a first step, let us show that the fixed points of Eq. (4) correspond to $\texttt{span}(\mathbf{1}_n)$, as in regular consensus.

Proposition 1: The fixed points of Eq. (4) all belong to $\texttt{span}(\mathbf{1}_n)$.

Proof: In order to prove the statement, let us consider the fixed points of Eq. (4), i.e., those vectors $\boldsymbol{z} \in \mathbb{R}^n$ that satisfy $\boldsymbol{z} = (I-\tau L)\boldsymbol{z} - \beta\tau L\texttt{sign}(L\boldsymbol{z})$, which is equivalent to

$$L(\boldsymbol{z} + \beta\texttt{sign}(L\boldsymbol{z})) = \mathbf{0}_n. \tag{8}$$

By construction, for $\boldsymbol{z}$ to be a fixed point, it must hold

$$\boldsymbol{z} + \beta\texttt{sign}(L\boldsymbol{z}) \in \texttt{ker}(L) = \texttt{span}(\mathbf{1}_n), \tag{9}$$

i.e., $\boldsymbol{z} = \alpha\mathbf{1}_n - \beta\texttt{sign}(L\boldsymbol{z})$, for some $\alpha \in \mathbb{R}$. Therefore, since $\texttt{sign}(L\boldsymbol{z}) \in \{-1,0,1\}^n$, we have that $z_i \in \{\alpha-\beta, \alpha, \alpha+\beta\}$, for all $i \in \{1,\dots,n\}$. We claim that the only possibility is $\boldsymbol{z}_i = \alpha$ for all i and we prove our claim by contradiction. Suppose that for some i it holds $z_i = \alpha+\beta$. By Eq. (9), we have that

$$\alpha + \beta + \texttt{sign}\left(\sum_{j\in\mathcal{N}_i}(\alpha+\beta-z_j)\right) = \alpha$$

i.e.,

$$\texttt{sign}\left(\sum_{j\in\mathcal{N}_i}(\alpha+\beta-z_j),\right) = -\beta$$

which holds true if and only if $\sum_{j\in\mathcal{N}_i}(\alpha+\beta-z_j) < 0$. However, since $z_j \in \{\alpha-\beta, \alpha, \alpha+\beta\}$, we have that $\alpha+\beta-z_j \geq 0$, which is a contradiction. Similarly, mutatis mutandis, we get the same kind of contradiction when $z_i = \alpha-\beta$. Hence it must hold $z_i = \alpha$ for all i, i.e., $\boldsymbol{z} = \alpha\mathbf{1}_n$, which is our claim. The proof is complete. ■

Let us now show that the error vector is orthogonal to $\mathbf{1}_n$.

Lemma 2: For all q it holds $\boldsymbol{e}(q) \perp \mathbf{1}_n$.

Proof: Let us prove the statement by induction. Specifically, we observe that, by construction, it holds $\mathbf{1}_n^\top\boldsymbol{e}(0) = \mathbf{1}_n^\top\boldsymbol{x}(0) - n\boldsymbol{x}_{\texttt{ave}} = 0$. At this point, let us assume that $\mathbf{1}_n^\top\boldsymbol{e}(q) = 0$ and let us show that, then, $\mathbf{1}_n^\top\boldsymbol{e}(q+1) = 0$. In particular, we have that, since $\mathbf{1}_n^\top L = \mathbf{0}_n^\top$, by construction it holds

$$\begin{aligned}
\mathbf{1}_n^\top\boldsymbol{e}(q+1) &= \mathbf{1}_n^\top\boldsymbol{e}(q) - \tau\mathbf{1}_n^\top L\boldsymbol{e}(q) - \tau\beta\mathbf{1}_n^\top L\texttt{sign}(L\boldsymbol{e}(q)) \\
&= \mathbf{1}_n^\top\boldsymbol{e}(q) = 0.
\end{aligned}$$

The proof is complete. ■

Let us now show that, within the proposed discrete-time scheme, the norm of the error decreases whenever it is above a given bound that depends on τ and β. Such bound can be made arbitrarily small for suitably small values of τ and β.

Theorem 1: Let $\Delta(q) = \|\boldsymbol{e}(q+1)\|^2 - \|\boldsymbol{e}(q)\|^2$ and assume that $\tau < 2/\lambda_2(L)$. Then, $\Delta(q) < 0$ whenever $\|\boldsymbol{e}(q)\| > \phi(\beta,\tau) > 0$, where

$$\phi(\tau,\beta) = \beta\frac{4\tau d_{\max}^2\sqrt{n} - \lambda_2(L) + \sqrt{\texttt{d}}}{\lambda_2(L)(2-\tau\lambda_2(L))},$$

and

$$\texttt{d} = (4\tau d_{\max}^2\sqrt{n} - \lambda_2(L))^2 + \tau\lambda_2(L)\lambda_n^2(L)n(2-\tau\lambda_2(L)).$$

Proof: In the following, where understood, we use $\texttt{e}$ to denote $\boldsymbol{e}(q)$ and $\texttt{s}$ to denote $\texttt{sign}(L\boldsymbol{e}(q))$. Moreover, we abbreviate $\lambda_2(L)$ and $\lambda_n(L)$ with λ_2 and λ_n, respectively. In order to prove the statement, we observe that, by Lemma 1, it holds

$$\begin{aligned}
\Delta(q) = \texttt{e}^\top(I-\tau L)^2\texttt{e} + \tau^2\beta^2\texttt{s}^\top L^2\texttt{s} - 2\tau\beta\texttt{s}^\top L(I-\tau L)\texttt{e} \\
- \texttt{e}^\top\texttt{e}.
\end{aligned} \tag{10}$$

At this point we remark that, by Lemma 2, it holds $\boldsymbol{e}(q) \perp \mathbf{1}_n$ for all q. Based on this fact, and on the fact that $\mathbf{1}_n$ is an eigenvector of L associated with the eigenvalue $\lambda_1(L) = 0$ with the smallest magnitude, we have that the following inequalities hold true:

$$\texttt{e}^\top(I-\tau L)^2\texttt{e} \leq (1-\tau\lambda_2)^2\|\texttt{e}\|^2 = (1+\tau^2\lambda_2^2 - 2\tau\lambda_2)\|\texttt{e}\|^2,$$

$$\texttt{s}^\top L^2\texttt{s} = \|L\texttt{s}\|^2 \leq \lambda_n^2\|\texttt{s}\|^2 \leq \lambda_n^2\|\mathbf{1}_n\|^2 = \lambda_n^2 n,$$

$$\texttt{s}L\texttt{e} = \|L\texttt{e}\|_1 \geq \|L\texttt{e}\| \geq \lambda_2\|\texttt{e}\|,$$

and

$$\texttt{s}L^2\texttt{e} \leq \|L^2\texttt{e}\|_1 \leq \|L\|_1^2\sqrt{n}\|\texttt{e}\| = 4d_{\max}^2\sqrt{n}\|\texttt{e}\|,$$

where we used the fact that by construction $\|L\|_1 = 2d_{\max}$, $d_{\max}$ being the maximum degree of a node in the graph underlying L. As a consequence, we have that

$$\Delta(q) \leq -\texttt{a}\|\texttt{e}\|^2 - \texttt{b}\|\texttt{e}\| - \texttt{c}, \tag{11}$$

where

$$\texttt{a} = \tau\lambda_2(2-\tau\lambda_2), \quad \texttt{b} = 2\tau\beta(\lambda_2 - 4\tau d_{\max}^2\sqrt{n}), \tag{12}$$

$$\texttt{c} = -\tau^2\beta^2\lambda_n^2 n. \tag{13}$$

Since we assumed $\tau < 2/\lambda_2$, we have that $\texttt{a} > 0$, and thus, by some algebra, $\Delta(q) < 0$ for $\|\texttt{e}\| > \phi(\tau,\beta)$. Notably, since by construction $\texttt{d} > \lambda_2^2$, we have that $\phi(\tau,\beta) > 0$. This completes our proof. ■

As a consequence of Theorem 1, the error norm is decreasing as long as it is above $\phi(\tau,\beta)$. We now show that, when the error falls below such a threshold for the first time, it becomes bounded by a quantity that can be made arbitrarily small based on the choice of τ or β.

Corollary 1: Let the assumptions of Theorem 1 hold true. Then, if at some q^* it holds $\|\boldsymbol{e}(q^*)\| \leq \phi(\tau,\beta)$, then for all $q \geq q^*$ it holds $\|\boldsymbol{e}(q)\| \leq \omega(\tau,\beta)$, where

$$\omega(\tau,\beta) = \sqrt{\phi^2(\tau,\beta) + |\texttt{b}|\phi(\tau,\beta) + |\texttt{c}|},$$

while b and c are defined in Eqs. (12) and (13), respectively.

Proof: In order to prove the statement, let us assume that at some q^* it holds $\|\boldsymbol{e}(q^*)\| \leq \phi(\tau, \beta)$. According to Theorem 1, we can not exclude that $\|\boldsymbol{e}(q^*+1)\| > \|\boldsymbol{e}(q^*)\|$. However, since $\tau < 2/\lambda_2(L)$, we have that $\mathrm{a} > 0$. Therefore, using Eq. (11), it holds

$$\begin{aligned}\|\boldsymbol{e}(q^*+1)\|^2 &= \|\boldsymbol{e}(q^*)\|^2 + \Delta(q^*)\\ &\leq \|\boldsymbol{e}(q^*)\|^2 + |\mathrm{b}|\|\boldsymbol{e}(q^*)\| + |\mathrm{c}|\\ &\leq \phi^2(\tau,\beta) + |\mathrm{b}|\phi(\tau,\beta) + |\mathrm{c}|.\end{aligned}$$

At this point we observe that, if $\|\boldsymbol{e}(q^*+1)\| > \phi(\tau, \beta)$, then by Theorem 1 it holds $\|\boldsymbol{e}(q^*+2)\| < \|\boldsymbol{e}(q^*+1)\|$, while if $\|\boldsymbol{e}(q^*+1)\| \leq \phi(\tau, \beta)$, using the same argument as above, we conclude that $\|\boldsymbol{e}(q^*+2)\| < \omega(\tau, \beta)$. This reasoning can be iterated for all $q > q^*$. The proof is complete. ■

A few remarks are now in order.

Remark 2: The bounds $\phi(\beta, \tau)$ and $\omega(\beta, \tau)$ can be made arbitrarily small for sufficiently small τ.

Remark 3: Let us assume that β is replaced by a vanishing $\beta(q)$. In this case, for any $\tau > 0$, we have that

$$\lim_{q\to\infty} \phi(\tau, \beta(q)) = 0, \quad \lim_{q\to\infty} \omega(\tau, \beta(q)) = 0,$$

and thus the error converges to zero asymptotically.

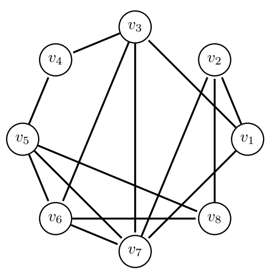


Fig. 2: Graph topology considered for the simulations.

V. Simulations

In this section, we illustrate the effectiveness of the proposed approach through simulations. In particular, we consider the undirected network topology reported in Figure 2.

We assume the agents have initial conditions equal to

$$\boldsymbol{x}_0 = \left[0.500, 5.359, 6.638, 5.149, 9.446, 5.866, 9.034, 1.375\right]^\top,$$

and we implement the proposed algorithm considering the overall-iteration provided in the equivalent dynamics in Eq. (7) with $\tau = 0.0005$ and $\beta = 1$. In Figure 3 we compare our protocol (Fig. 3a) with a standard consensus (Fig. 3b) in terms of executed beep steps (we assume $w = 64$ bits), and we observe that the latter is remarkably slower than the proposed approach. Figs. 4a and 4b compare the two approaches in terms of macro-iterations and beep steps, respectively, considering different choices for β. Interestingly, in Figure 4a, we identify an initial zone where the proposed approach is slightly slower than regular consensus, an intermediate zone where our approach is remarkably faster, and a zone where our algorithm remains bounded (while regular consensus keeps reducing its associated error). Moreover, within the proposed algorithm, there is an evident trade-off between the initial speed of convergence before becoming bounded (i.e., faster for higher values of β) and the final error (i.e., smaller for lower values of β). Notably, if we consider the actual beep steps (Fig. 4b), since the two algorithms require different amounts of beep steps to implement the macro-iterations, the proposed algorithm exhibits even better performance. We recall that our approach requires $w+2$ beep steps to implement one of such iterations, while standard consensus implements it in $2w$ beep steps (with $w = 64$, to model double-precision numbers).

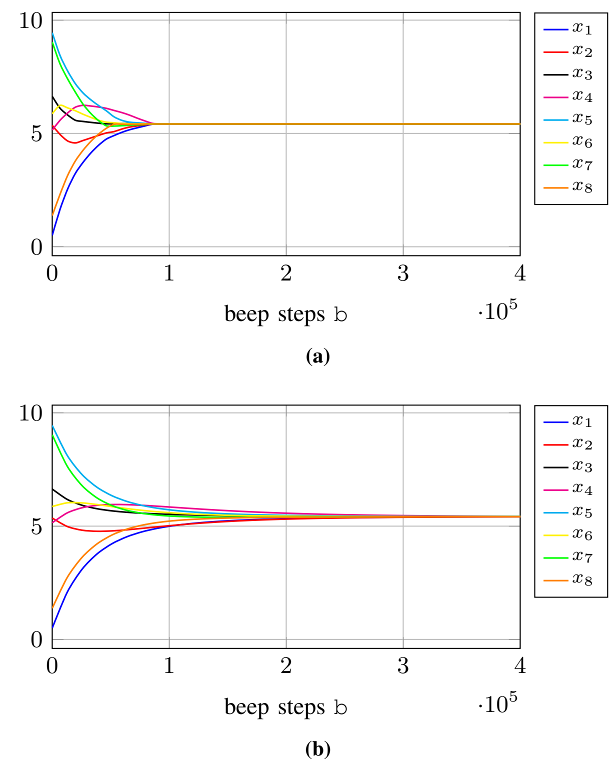


Fig. 3: State evolution over the topology in Figure 2: (a) of the proposed algorithm with $\beta = 1$, and (b) of the standard consensus.

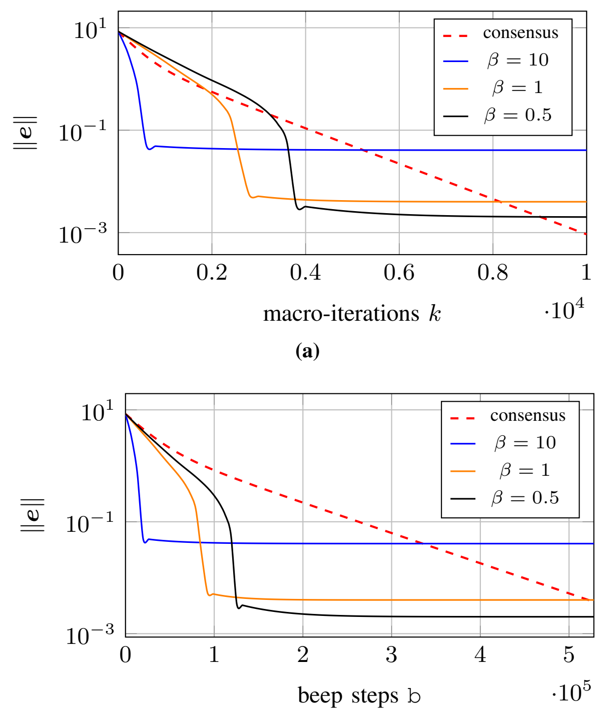
To conclude, Fig. 5 depicts the case where $\beta(k) = 0.999^{k/2}$ is vanishing. According to the figure, the proposed algorithm exhibits an error that is initially similar to standard consensus, but during an intermediate phase we observe a relevant error reduction. Finally, as $\beta(k)$ becomes negligible, the error converges with a rate that is comparable to consensus. Notably, in this case, the bound $\omega(\tau, \beta(k))$ becomes monotonically convergent to zero.

VI. Conclusions

In this paper, we present a novel consensus algorithm designed for undirected graphs, which prioritizes low-bandwidth usage by harnessing the capabilities of the beep communication model. Specifically, the algorithm amounts to the repetition of two conceptual phases: a traditional average consensus and a phase where the agents only inform their neighbors about the increase or decrease of their states.

Through theoretical analysis, we describe the practical convergence behavior of the proposed protocol, wherein adjustments to the dynamics parameters can significantly reduce errors. Furthermore, the numerical simulations illustrate that, by appropriately selecting these parameters, the protocol shows a trade-off between convergence rate and accuracy.

Future research directions include: (i) further characterizing the convergence speed and its trade-off against accuracy, also accounting for the duration of the macro-iterations in terms of beep steps; (ii) optimizing the gains and adapting them during the algorithm execution to improve accuracy; (iii) extending the approach to directed graphs and to distributed optimization problems.



(a)

(b)

Fig. 4: Norm of the equivalent error dynamics of the proposed algorithm as β varies, compared to the norm of the error dynamics of the standard consensus algorithm (dashed red line), with respect to macro-iterations (a) or beep steps (b).

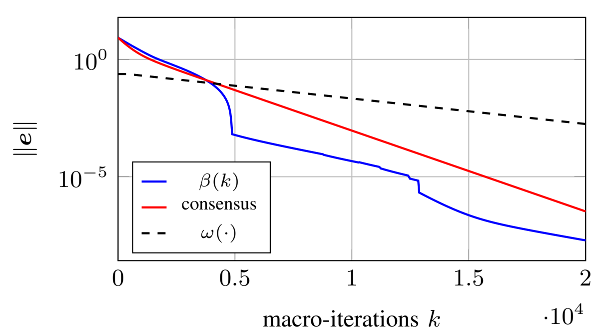


Fig. 5: Norm of the equivalent error dynamics of the proposed algorithm when β is vanishing, compared to the norm of the error dynamics of the standard consensus and to the bound $\omega(\tau, \beta(k))$.

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