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Numerical Discretization Methods for the Extended Linear Quadratic Control Problem

Zhanhao Zhang, Jan Lorenz Svensen, Morten Wahlgreen Kaysfeld,
Anders Hilmar Damm Christensen, Steen Hørsholt, John Bagterp Jørgensen

***Abstract*— In this study, we introduce numerical methods for discretizing continuous-time linear-quadratic optimal control problems (LQ-OCPs). The discretization of continuous-time LQ-OCPs is formulated into differential equation systems, and we can obtain the discrete equivalent by solving these systems. We present the ordinary differential equation (ODE), matrix exponential, and a novel step-doubling method for the discretization of LQ-OCPs. Utilizing Euler-Maruyama discretization with a fine step, we reformulate the costs of continuous-time stochastic LQ-OCPs into a quadratic form, and show that the stochastic cost follows the χ^2 distribution. In the numerical experiment, we test and compare the proposed numerical methods. The results ensure that the discrete-time LQ-OCP derived using the proposed numerical methods is equivalent to the original problem.**

I. Introduction

In optimal control theory, linear-quadratic (LQ) optimal control problems are considered the fundamental but core problem. They involve a quadratic cost function that needs to be optimized, and the controlled system is linear. Due to the simplicity and analytical solvability of LQ-OCPs, it has widespread practical applications in numerous fields, including engineering, aerospace, biology and economics [5], [7], [24], [25]. Implementing continuous-time LQ-OCPs in real-world scenarios might be infeasible, primarily because many practical applications operate on digital platforms in digital form. Consequently, discretization techniques become imperative in practice.

There is rich research on discretization and numerical solution methods for optimal control problems [8], [10], [13], [15], [20], [26], [35]. However, this is not the case for how to obtain the discrete-time equivalent from a continuous-time LQ-OCP. Based on existing literature, there are two popular ways for LQ discretization: 1) design a continuous-time LQ-OCP and subsequently derive its discrete-time equivalent, 2) initially discretizing the continuous-time system, followed by designing a discrete-time LQ-OCP with the discrete-time system. Compared with the latter, the first method provides a better approximation, which eventually leads to the solution of the original problem.

The continuous-time LQ-OCP can be converted into discrete-time by assuming zero-order hold (ZOH) on the state vector. However, this approximation is a crude approximation that is often inaccurate for large sample times [16]. In [3], [4], the cost equivalent is obtained by extending the continuous-time LQ-OCP's cost function. It leads to the analytic expressions of the desired equivalent, discrete weighting matrices. Further, a matrix exponential method is introduced for calculating the cost equivalent [1], [2], [11], [29], [34]. Modelling sampled-data systems with traditional approaches has fundamental difficulties, which can be resolved using incremental models. Incremental models provide a seamless connection between continuous- and discrete-time systems, and they can be implemented for optimal filtering and control [14], [27], [31], [36]. In addition, LQ-OCPs are associated with advanced control algorithms, such as model predictive control (MPC). Prediction-Error-Methods for identification of these models exist [19], [21]–[23]. Also efficient numerical methods for the solution of extended LQ-OCPs exists, e.g. structure exploiting algorithms that use a Riccati recursion [12], [19], [20]. LQ-OCPs become hard to solve and analyze when the controlled systems are stochastic. The analytic expression of the stochastic cost and its expectation can be calculated using It$\hat{o}$ calculus [3], [4]. Many stochastic LQ-OCPs aim to optimize the mean value of their costs, which may not be true for some scenarios, e.g., conditional Value-at-Risk optimization problems (CVaR optimization) [6], [18], [30], [32], [33]. Therefore, it is critical to investigate the cost function distribution of stochastic LQ-OCPs.

The key problem that we address in this paper:

1. Formulation of differential equation systems for LQ discretization
2. Numerical methods for solving the resulting systems of differential equations
3. Distribution of stochastic cost functions

In Section II, we introduce deterministic and stochastic LQ-OCPs and propose differential equation systems for LQ discretization. For stochastic LQ-OCPs, we reformulate their cost function and describe the distribution of the stochastic cost. Section III introduces three numerical methods for solving proposed differential equation systems. Section IV presents a numerical experiment comparing the proposed numerical methods, and conclusions are given in Section V.

II. Linear-Quadratic Optimal Control Problems

In this section, we introduce deterministic and stochastic LQ-OCPs and describe the differential equation systems for LQ discretization.

A. Deterministic linear-quadratic optimal control problem

Consider the deterministic LQ-OCP

$$\min_{x,u,z,\tilde{z}} \quad \phi = \int_{t_0}^{t_0+T} l_c(\tilde{z}(t))dt \tag{1a}$$

$$\begin{aligned}
s.t. \quad & x(t_0) = \hat{x}_0, && && (1b)\\
& u(t) = u_k, && t_k \le t < t_{k+1},\ k \in \mathcal{N}, && (1c)\\
& \dot{x}(t) = A_c x(t) + B_c u(t), && t_0 \le t < t_0 + T, && (1d)\\
& z(t) = C_c x(t) + D_c u(t), && t_0 \le t < t_0 + T, && (1e)\\
& \bar{z}(t) = \bar{z}_k, && t_k \le t < t_{k+1},\ k \in \mathcal{N}, && (1f)\\
& \tilde{z}(t) = z(t) - \bar{z}(t), && t_0 \le t < t_0 + T, && (1g)
\end{aligned}$$

with the stage cost function

$$l_c(\tilde{z}(t)) = \frac{1}{2}\|W_z\tilde{z}(t)\|_2^2 = \frac{1}{2}\tilde{z}(t)'Q_c\tilde{z}(t), \tag{2}$$

where $Q_c = W_z'W_z$ is a semi-positive definite matrix. This problem is in continuous-time with decision variables $x(t), u(t), z(t)$, and $\tilde{z}(t)$. The control horizon $T = NT_s$ with sampling time T_s and $N \in \mathbb{Z}^+$, and $\mathcal{N} = 0, 1, \ldots, N-1$. We assume piecewise constant inputs, $u(t) = u_k$ and target variables $\bar{z}(t) = \bar{z}_k$ for $t_k \le t < t_{k+1}$.

Remark 1: Note that the case $\bar{z}(t) = [\bar{x}(t); \bar{u}(t)]$, $C_c = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $D_c = \begin{bmatrix} 0 \\ I \end{bmatrix}$, and $Q_c = \begin{bmatrix} Q_{c,xx} & 0 \\ 0 & Q_{c,uu} \end{bmatrix}$ corresponds to

$$\begin{aligned} l_c(\tilde{z}(t)) = &\frac{1}{2}[x(t) - \bar{x}(t)]' Q_{c,xx} [x(t) - \bar{x}(t)] \\ &+ \frac{1}{2}[u(t) - \bar{u}(t)]' Q_{c,uu} [u(t) - \bar{u}(t)]. \end{aligned} \tag{3}$$

■

The corresponding discrete-time LQ-OCP is

$$\begin{aligned}
\min_{x,u} \quad & \phi = \sum_{k\in\mathcal{N}} l_k(x_k, u_k) && (4a)\\
s.t. \quad & x_0 = \hat{x}_0, && (4b)\\
& x_{k+1} = Ax_k + Bu_k, \quad k \in \mathcal{N}, && (4c)
\end{aligned}$$

with the stage costs

$$l_k(x_k, u_k) = \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}' Q \begin{bmatrix} x_k \\ u_k \end{bmatrix} + q_k' \begin{bmatrix} x_k \\ u_k \end{bmatrix} + \rho_k, \quad k \in \mathcal{N}, \tag{5}$$

where the coefficient in the affine term and the constant term are

$$q_k = M\bar{z}_k, \quad \rho_k = \int_{t_k}^{t_{k+1}} l_c(\bar{z}_k)dt = l_c(\bar{z}_k)T_s, \quad k \in \mathcal{N}. \tag{6}$$

Proposition 1 (Discretization of the deterministic LQ-OCP): The system of differential equations

$$\begin{aligned}
\dot{A}(t) &= A_c A(t), & A(0) &= I, && (7a)\\
\dot{B}(t) &= A(t)B_c, & B(0) &= 0, && (7b)\\
\dot{Q}(t) &= \Gamma(t)'Q_c\Gamma(t), & Q(0) &= 0, && (7c)\\
\dot{M}(t) &= -\Gamma(t)'Q_c, & M(0) &= 0, && (7d)
\end{aligned}$$

where

$$\Gamma(t) = \begin{bmatrix} C_c & D_c \end{bmatrix} \begin{bmatrix} A(t) & B(t) \\ 0 & I \end{bmatrix}, \tag{7e}$$

may be used to compute ($A = A(T_s)$, $B = B(T_s)$, $Q = Q(T_s)$, $M = M(T_s)$).

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B. Certainty equivalent LQ control for a stochastic system

Consider an initial state and an input noise modelled by the following random variables,

$$\boldsymbol{x}(t_0) \sim N(\hat{x}_0, P_0), \qquad d\boldsymbol{\omega}(t) \sim N_{iid}(0, Idt). \tag{8}$$

The stochastic system can be described as continuous-time linear stochastic differential equations (SDEs) in the form

$$\begin{aligned}
d\boldsymbol{x}(t) &= (A_c\boldsymbol{x}(t) + B_c u(t))\,dt + G_c d\boldsymbol{\omega}(t), && (9a)\\
\boldsymbol{z}(t) &= C_c\boldsymbol{x}(t) + D_c u(t). && (9b)
\end{aligned}$$

The corresponding discrete-time stochastic system is

$$\begin{aligned}
\boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + Bu_k + \boldsymbol{w}_k, && (10a)\\
\boldsymbol{z}_k &= C\boldsymbol{x}_k + Du_k, && (10b)
\end{aligned}$$

where

$$\boldsymbol{x}_0 \sim N(\hat{x}_0, P_0), \qquad \boldsymbol{w}_k \sim N_{iid}(0, R_{ww}). \tag{10c}$$

Proposition 2 (Discretization of the linear SDE): The system of differential equations

$$\begin{aligned}
\dot{A}(t) &= A_c A(t), & A(0) &= I, && (11a)\\
\dot{B}(t) &= A(t)B_c, & B(0) &= 0, && (11b)\\
\dot{R}_{ww} &= \Phi(t)\Phi(t)', & R_{ww}(0) &= 0, && (11c)
\end{aligned}$$

where

$$\Phi(t) = A(t)G_c, \tag{11d}$$

can be used to compute ($A = A(T_s)$, $B = B(T_s)$, $R_{ww} = R_{ww}(T_s)$).

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C. Stochastic linear-quadratic optimal control problem

Consider the stochastic LQ-OCP

$$\min_{\boldsymbol{x},u,\boldsymbol{z},\tilde{\boldsymbol{z}}} \quad \psi = E\left\{\phi = \int_{t_0}^{t_0+T} l_c(\tilde{\boldsymbol{z}}(t))dt\right\} \tag{12a}$$

$$\begin{aligned}
s.t. \quad & \boldsymbol{x}(t_0) \sim N(\hat{x}_0, P_0), && (12b)\\
& d\boldsymbol{\omega}(t) \sim N_{iid}(0, Idt), && (12c)\\
& u(t) = u_k, \quad t_k \le t < t_{k+1},\ k \in \mathcal{N}, && (12d)\\
& d\boldsymbol{x}(t) = (A_c\boldsymbol{x}(t) + B_c u(t))dt + G_c d\boldsymbol{\omega}(t), && (12e)\\
& \boldsymbol{z}(t) = C_c\boldsymbol{x}(t) + D_c u(t), && (12f)\\
& \bar{z}(t) = \bar{z}_k, \quad t_k \le t < t_{k+1},\ k \in \mathcal{N}, && (12g)\\
& \tilde{\boldsymbol{z}}(t) = \boldsymbol{z}(t) - \bar{z}(t). && (12h)
\end{aligned}$$

The corresponding discrete-time stochastic LQ-OCP is

$$\min_{x,u} \quad \psi = E\left\{\phi = \sum_{k\in\mathcal{N}} l_k(\boldsymbol{x}_k, u_k) + l_{s,k}(\boldsymbol{x}_k, u_k)\right\} \tag{13a}$$

$$\begin{aligned}
s.t. \quad & \boldsymbol{x}_0 \sim N(\hat{x}_0, P_0), && && (13b)\\
& \boldsymbol{w}_k \sim N_{iid}(0, R_{ww}), && k \in \mathcal{N}, && (13c)\\
& \boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k + \boldsymbol{w}_k, && k \in \mathcal{N}, && (13d)
\end{aligned}$$

where the stage cost function $l_k(\boldsymbol{x}_k, u_k)$ is

$$l_k(\boldsymbol{x}_k, u_k) = \frac{1}{2}\begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix}' Q \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix} + q_k' \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix} + \rho_k, \tag{14a}$$

and the stochastic stage cost function $l_{s,k}(\boldsymbol{x}_k, u_k)$ is

$$l_{s,k}(\boldsymbol{x}_k, u_k) = \int_{t_k}^{t_{k+1}} \frac{1}{2}\boldsymbol{w}(t)' Q_{c,ww} \boldsymbol{w}(t) + \boldsymbol{q}_{s,k}' \boldsymbol{w}(t) dt. \quad (14b)$$

Q, q_k, and ρ_k in (14a) are identical to the deterministic case. The state variables and system matrices of $l_{s,k}(\boldsymbol{x}_k, u_k)$ are

$$\boldsymbol{w}(t) = \int_0^t A(s) G_c d\boldsymbol{\omega}(s), \qquad Q_{c,ww} = C_c' Q_{c,xx} C_c, \quad (15a)$$

$$\tilde{\boldsymbol{z}}_k = \Gamma(t) \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix} - \bar{z}_k, \qquad \boldsymbol{q}_{s,k} = \begin{bmatrix} C_c' & 0 \end{bmatrix} Q_c \tilde{\boldsymbol{z}}_k. \quad (15b)$$

The expectation of the stochastic LQ-OCP is [3]

$$\min_{x,u} \psi = \sum_{k \in \mathcal{N}} l_k(x_k, u_k) + \frac{1}{2}\left[\mathrm{tr}\left(Q \bar{P}_k\right) + \int_{t_k}^{t_{k+1}} \mathrm{tr}\left(Q_{c,ww} P_w\right) dt \right] \quad (16a)$$

$$s.t. \quad x_0 = \hat{x}_0, \quad (16b)$$

$$x_{k+1} = A x_k + B u_k, \quad k \in \mathcal{N}, \quad (16c)$$

where

$$\begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix} \sim N(m_k, \bar{P}_k), \quad m_k = \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \quad \bar{P}_k = \begin{bmatrix} P_k & 0 \\ 0 & 0 \end{bmatrix}, \quad (17a)$$

$$P_{k+1} = A P_k A' + R_{ww}, \qquad P_w = \mathrm{Cov}(\boldsymbol{w}(t)). \quad (17b)$$

Proposition 3 (Discretization of the stochastic LQ-OCP): The system of differential equations

$$\dot{A}(t) = A_c A(t), \qquad A(0) = I, \quad (18a)$$

$$\dot{B}(t) = A(t) B_c, \qquad B(0) = 0, \quad (18b)$$

$$\dot{R}_{ww}(t) = \Phi(t)\Phi(t)', \qquad R_{ww}(0) = 0, \quad (18c)$$

$$\dot{Q}(t) = \Gamma(t)' Q_c \Gamma(t), \qquad Q(0) = 0, \quad (18d)$$

$$\dot{M}(t) = -\Gamma(t)' Q_c, \qquad M(0) = 0, \quad (18e)$$

where

$$\Phi(t) = A(t) G_c, \qquad \Gamma(t) = \begin{bmatrix} C_c & D_c \end{bmatrix} \begin{bmatrix} A(t) & B(t) \\ 0 & I \end{bmatrix}, \quad (18f)$$

can be used to compute $(A = A(T_s),\ B = B(T_s),\ R_{ww} = R_{ww}(T_s),\ Q = Q(T_s),\ M = M(T_s))$.

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Proposition 4 (Distribution of the stochastic costs): Using Euler-Maruyama (EM) discretization, we can reformulate (13a) into isolated stochastic form as

$$\begin{aligned} \phi &= \sum_{k \in \mathcal{N}} l_k(\boldsymbol{x}_k, u_k) + l_{s,k}(\boldsymbol{x}_k, u_k) \\ &= \frac{1}{2} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{W}_{\mathcal{N}} \end{bmatrix}' Q_{\mathcal{N}} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{W}_{\mathcal{N}} \end{bmatrix} + q_{\mathcal{N}}' \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{W}_{\mathcal{N}} \end{bmatrix} + \rho_{\mathcal{N}}, \end{aligned} \quad (19a)$$

where

$$\begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{W}_{\mathcal{N}} \end{bmatrix} \sim N(\bar{m}, \bar{P}), \quad \bar{m} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \quad \bar{P} = \begin{bmatrix} P_0 & 0 \\ 0 & P_w \end{bmatrix}. \quad (19b)$$

$\boldsymbol{W}_{\mathcal{N}}$ is a vector of sub-sampling random variables over the horizon, and its covariance is $P_w = \mathrm{diag}([I\delta t, I\delta t, \ldots, I\delta t])$.

Based on [9], the reformulated stochastic cost follows a generalized χ^2 distribution, and its expectation and variance can be expressed as

$$E\{\phi\} = \frac{1}{2}\bar{m}' Q_{\mathcal{N}} \bar{m} + q_{\mathcal{N}}' \bar{m} + \rho_{\mathcal{N}} + \frac{1}{2}\mathrm{tr}\left(Q_{\mathcal{N}} \bar{P}\right), \quad (20a)$$

$$\begin{aligned} V\{\phi\} &= q_{\mathcal{N}}' \bar{P} q_{\mathcal{N}} + 2\bar{m}' Q_{\mathcal{N}} \bar{P} q_{\mathcal{N}} + \bar{m}' Q_{\mathcal{N}} \bar{P} Q_{\mathcal{N}} \bar{m} \\ &\quad + \frac{1}{2}\mathrm{tr}(Q_{\mathcal{N}} \bar{P} Q_{\mathcal{N}} \bar{P}). \end{aligned} \quad (20b)$$

The stochastic cost ϕ in (12a) follows a generalized χ^2 distribution with mean and variance given by (20) when taking the limit of integration steps $n \to \infty$. See Appendix VI for more details.

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III. Numerical Methods of LQ Discretization

This section introduces numerical methods for the discretization of continuous-time LQ-OCPs.

A. Ordinary differential equation methods

Consider an s-stage ODE method with $N \in \mathbb{Z}^+$ integration steps and the time step $h = \frac{T_s}{N}$. We compute $(A,\ B,\ R_{ww},\ Q,\ M)$ as

$$A_{k+1} = \Lambda A_k, \qquad k \in \mathcal{N}, \quad (21a)$$

$$B_{k+1} = B_k + \Theta A_k \bar{B}_c, \qquad k \in \mathcal{N}, \quad (21b)$$

$$\Gamma_{k+1} = \Omega \Gamma_k, \qquad k \in \mathcal{N}, \quad (21c)$$

$$M_{k+1} = M_k + \Gamma_k' \bar{M}_c, \qquad k \in \mathcal{N}, \quad (21d)$$

$$Q_{k+1} = Q_k + \Gamma_k' \bar{Q}_c \Gamma_k, \qquad k \in \mathcal{N}, \quad (21e)$$

$$R_{ww,k+1} = R_{ww,k} + \sum_{i=1}^{s} b_i \Lambda_i A_k \bar{R}_{ww,c} A_k' \Lambda_i', \qquad k \in \mathcal{N}, \quad (21f)$$

where $\bar{B}_c = h B_c$, $H = \begin{bmatrix} C_c & D_c \end{bmatrix}$, $\bar{M}_c = -h\sum_{i=1}^{s} b_i \Omega_i' H' Q_c$, $\bar{Q}_c = h\sum_{i=1}^{s} b_i \Omega_i' H' Q_c H \Omega_i$, and $\bar{R}_{ww,c} = h G_c G_c'$ are constant matrices. Λ, Θ, Ω are functions of the coefficients, $a_{i,j}$ and b_i for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, s$ in the Butcher

TABLE I

Λ and Θ of the ODE method with discretization methods

Methods	Λ	Θ
Expl. Euler	$I + hA_c$	I
Impl. Euler	$(I - hA_c)^{-1}$	$(I - hA_c)^{-1}$
Expl. Trape.	$I + hA_c + 0.5h^2 A_c^2$	$I + 0.5hA_c$
Impl. Trape.	$(I + 0.5hA_c)^{-1}$ $(I - 0.5hA_c)$	$(I - 0.5hA_c)^{-1}$
ESDIRK34	$(I - 0.44hA_c)^{-3}$ $(I - 0.31hA_c - 0.24h^2 A_c^2)$	$(I - 0.44hAc)^{-3}$ $(I - 0.81hAc + 0.08h^2 Ac^2)$
Classic RK4	$I + hA_c + 0.5h^2 A_c^2 +$ $0.17h^3 A_c^3 + 0.04h^4 A_c^4$	$I + 0.5hA_c + 0.17h^2 A_c^2 +$ $0.04h^3 A_c^3$

Algorithm 1 ODE method for LQ Discretization

Input: $(A_c, B_c, G_c, C_c, D_c, Q_c, T_s, h)$
Output: $(A(T_s), B(T_s), C, D, Q(T_s), M(T_s), R_{ww}(T_s))$

Set initial states
$(k=0,\ A_k = I,\ B_k = 0,\ Q_k = 0,\ M_k = 0,\ R_{ww,k} = 0)$
Use (22) to compute $(\Lambda_i,\ \Theta_i,\ \Omega_i,\ \Lambda,\ \Theta,\ \Omega)$
Compute integration steps $N = \frac{T_s}{h}$
while $k < N$ **do**
  Use (21) to update $(A_k, B_k, \Gamma_k, Q_k, M_k, R_{ww,k})$
  Set $k \leftarrow k+1$
end while
Get system matrices $(A(T_s) = A_k, B(T_s) = B_k, C = C_c, D = D_c, Q(T_s) = Q_k, M(T_s) = M_k, R_{ww}(T_s) = R_{ww,k})$

tableau of the ODE method. They are computed as

$$A_{k,i} = A_k + h\sum_{j=1}^{s} a_{i,j}\dot{A}_{k,j} = \Lambda_i A_k, \tag{22a}$$

$$B_{k,i} = B_k + h\sum_{j=1}^{s} a_{i,j}\dot{B}_{k,j} = B_k + \Theta_i A_k \bar{B}_c, \tag{22b}$$

$$\Gamma_{k,i} = \Omega_i \Gamma_k = \begin{bmatrix} \Lambda_i & \Theta_i \bar{B}_c \\ 0 & I \end{bmatrix} \Gamma_k, \tag{22c}$$

$$\Lambda = I + h\sum_{i=1}^{s} b_i A_c \Lambda_i, \tag{22d}$$

$$\Theta = \sum_{i=1}^{s} b_i \Lambda_i, \tag{22e}$$

$$\Omega = \begin{bmatrix} \Lambda & \Theta\bar{B}_c \\ 0 & I \end{bmatrix}, \tag{22f}$$

where Λ_i, Θ_i, Ω_i are coefficients of stage variables $A_{k,i}$, $B_{k,i}$, $\Gamma_{k,i}$.

Consequently, the differential equation systems $A(T_s) = A_N,\ B(T_s) = B_N,\ R_{ww}(T_s) = R_{ww,N},\ Q(T_s) = Q_N,\ M(T_s) = M_N$, has constant coefficients Λ_i, Θ_i, Ω_i, Λ, Θ and Ω when using fixed-time-step ODE methods. The constant coefficients can be computed offline. Table I describes Λ and Θ for different discretization methods. Algorithm 1 presents the ODE methods for LQ discretization.

B. Matrix exponential method

The matrix exponential method describes the LQ discretization by three matrix exponential problems

$$\begin{bmatrix} \Phi_{1,11} & \Phi_{1,12} \\ 0 & \Phi_{1,22} \end{bmatrix} = \exp\left(\begin{bmatrix} -H' & \bar{Q}_c \\ 0 & H \end{bmatrix} t \right), \tag{23a}$$

$$\begin{bmatrix} I & \Phi_{2,12} \\ 0 & \Phi_{2,22} \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & I \\ 0 & H' \end{bmatrix} t \right), \tag{23b}$$

$$\begin{bmatrix} \Phi_{3,11} & \Phi_{3,12} \\ 0 & \Phi_{3,22} \end{bmatrix} = \exp\left(\begin{bmatrix} -A_c & \bar{G}_c \\ 0 & A_c' \end{bmatrix} t \right), \tag{23c}$$

where

$$H = \begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix}, \qquad \bar{M}_c = -\begin{bmatrix} C_c & D_c \end{bmatrix}' Q_c, \tag{24a}$$

$$\bar{Q}_c = -\bar{M}_c \begin{bmatrix} C_c & D_c \end{bmatrix}, \qquad \bar{G}_c = G_c G_c'. \tag{24b}$$

The elements of matrix exponential problems are

$$\Phi_{1,22} = \Gamma(t) = \begin{bmatrix} A(t) & B(t) \\ 0 & I \end{bmatrix}, \tag{25a}$$

$$\Phi_{1,12} = \Gamma(-t)' \int_0^t \Gamma(\tau)' \bar{Q}_c \Gamma(\tau) d\tau, \tag{25b}$$

$$\Phi_{2,12} = \int_0^t \Gamma(\tau)' d\tau, \tag{25c}$$

$$\Phi_{3,22} = A(t)', \tag{25d}$$

$$\Phi_{3,12} = A(-t)' \int_0^t A(\tau) \bar{G}_c A(\tau)' d\tau. \tag{25e}$$

Set $t = T_s$, we can compute differential equations A, B, R_{ww}, Q and M as

$$A(T_s) = \Phi_{1,22}(1:n_x, 1:n_x), \tag{26a}$$

$$B(T_s) = \Phi_{1,22}(1:n_x, n_x+1:\text{end}), \tag{26b}$$

$$Q(T_s) = \Phi_{1,22}' \Phi_{1,12}, \tag{26c}$$

$$M(T_s) = \Phi_{2,12} \bar{M}_c, \tag{26d}$$

$$R_{ww}(T_s) = \Phi_{3,22}' \Phi_{3,12}. \tag{26e}$$

The matrix exponential method is inspired by formulas from [28], [29], [34].

C. Step-doubling method

Consider an s-stage ODE method with $N \in \mathbb{Z}^+$ integration steps and the time step $h = \frac{T_s}{N}$. The matrices

$$\tilde{A}(N) = \Lambda^N, \qquad \tilde{A}(1) = \Lambda, \tag{27a}$$

$$\tilde{B}(N) = \sum_{i=0}^{N-1} \Lambda^i, \qquad \tilde{B}(1) = I, \tag{27b}$$

$$\tilde{\Gamma}(N) = \Omega^N, \qquad \tilde{\Gamma}(1) = I_{xu}, \tag{27c}$$

$$\tilde{M}(N) = \sum_{i=0}^{N-1} \left(\Gamma^i\right)', \qquad \tilde{M}(1) = I_{xu}, \tag{27d}$$

$$\tilde{Q}(N) = \sum_{i=0}^{N-1} \left(\Gamma^i\right)' \bar{Q}_c \left(\Gamma^i\right), \qquad \tilde{Q}(1) = \bar{Q}_c, \tag{27e}$$

$$\tilde{R}(N) = \sum_{i=0}^{N-1} \left(A^i\right) \bar{R}_{ww,c} \left(A^i\right)', \qquad \tilde{R}(1) = \bar{R}_{ww,c}, \tag{27f}$$

can be used to solve (A, B, M, Q, R_{ww})

$$A(T_s) = \tilde{A}(N), \tag{28a}$$

$$B(T_s) = \Theta \tilde{B}(N) \bar{B}_c, \tag{28b}$$

$$M(T_s) = \tilde{M}(N) \bar{M}_c, \tag{28c}$$

$$Q(T_s) = \tilde{Q}(N), \tag{28d}$$

$$R_{ww}(T_s) = \sum_{i=1}^{s} b_i \Lambda_i \tilde{R}(N) \Lambda_i', \tag{28e}$$

where Λ, Θ, Ω, $\bar{B}_c$, $\bar{M}_c$ and $\bar{Q}_c$ are the same as in the ODE method case. $I_{xu} \in \mathbb{R}^{n_{xu} \times n_{xu}}$ is an identity matrix with the size $n_{xu} = n_x + n_u$.

Fig. 1 shows an example of $\tilde{A}(t) = e^{tA_c}$, where the step-doubling method (red dots) uses the $\frac{n}{2}^{th}$ step result as the initial state to compute the double step's result $\tilde{A}(n)$. We then get the step-doubling expression as

$$\tilde{A}(1) \rightarrow \tilde{A}(2) \rightarrow \tilde{A}(4) \rightarrow \ldots \rightarrow \tilde{A}(\frac{N}{4}) \rightarrow \tilde{A}(\frac{N}{2}) \rightarrow \tilde{A}(N), \tag{29a}$$

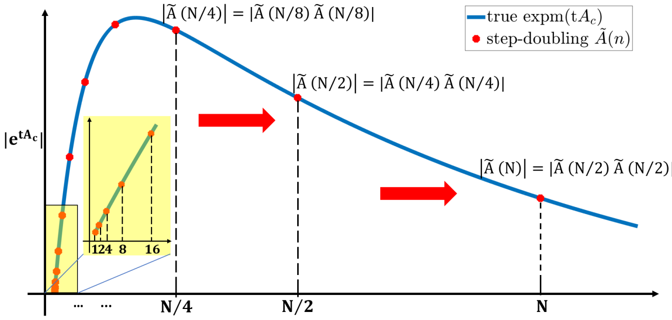

Fig. 1. The exponential of a matrix A_c. The blue line is the true result computed with expm() in MATLAB. The red dots are the results of the step-doubling method.

TABLE II

NUMERICAL EXPRESSIONS OF THE STEP-DOUBLING METHOD

ODEs	ODE expression	Step-doubling expression
$\tilde{A}(N)$	Λ^N	$\tilde{A}(\frac{N}{2})\tilde{A}(\frac{N}{2})$
$\tilde{B}(N)$	$\sum_{i=0}^{N-1}\Lambda^i$	$\tilde{B}(\frac{N}{2})\left(I+\tilde{A}(\frac{N}{2})\right)$
$\tilde{\Gamma}(N)$	Ω^N	$\tilde{\Gamma}(\frac{N}{2})\tilde{\Gamma}(\frac{N}{2})$
$\tilde{M}(N)$	$\sum_{i=0}^{N-1}\Omega^i\tilde{M}_c$	$\tilde{M}(\frac{N}{2})\left(I+\tilde{\Gamma}(\frac{N}{2})'\right)$
$\tilde{Q}(N)$	$\sum_{i=0}^{N-1}\left(\Gamma^i\right)'\tilde{Q}_c\left(\Gamma^i\right)$	$\tilde{Q}(\frac{N}{2})+\tilde{\Gamma}(\frac{N}{2})'\tilde{Q}(\frac{N}{2})\tilde{\Gamma}(\frac{N}{2})$
$\tilde{R}(N)$	$\sum_{i=0}^{N-1}\left(A^i\right)\tilde{R}_{ww,c}\left(A^i\right)'$	$\tilde{R}(\frac{N}{2})+\tilde{A}(\frac{N}{2})\tilde{R}(\frac{N}{2})\tilde{A}(\frac{N}{2})'$

where

$$\tilde{A}(n)=\tilde{A}(\frac{n}{2})\tilde{A}(\frac{n}{2}), \qquad n\in[2,4,\ldots,\frac{N}{2},N]. \tag{29b}$$

Eq. (29) is inspired by the scaling and squaring algorithm for solving matrix exponential problem presented in [1], [2], [17]. We apply the same idea for other differential equations. Table II describes the step-doubling expressions for ($\tilde{A}$, $\tilde{B}$, $\tilde{\Gamma}$, $\tilde{Q}$, $\tilde{M}$, $\tilde{R}_{ww}$). The step-doubling method takes only j steps to get the same result as the ODE method with $N=2^j$ integration steps. Algorithm 2 describes the step-doubling method for LQ discretization.

IV. Numerical Experiments

Consider a continuous-time stochastic LQ-OCP with system matrices

$$A_c=\begin{bmatrix}-49 & 24\\ -64 & 31\end{bmatrix}, \quad B_c=\begin{bmatrix}2 & 0.5\\ 1 & 3\end{bmatrix}, \quad G_c=\begin{bmatrix}0.1 & 0.0\\ 0.0 & 0.1\end{bmatrix}. \tag{30}$$

The system output matrices are $C_c=[1.0,1.0]$, $D_c=[0.0,0.0]$, and the system references and system inputs are

$$\bar{z}(t)=\bar{z}_k=3.0, \qquad t_k\le t\le t_{k+1}, \quad k\in\mathcal{N}, \tag{31a}$$
$$\bar{u}(t)=\bar{u}_k=0.0, \qquad t_k\le t\le t_{k+1}, \quad k\in\mathcal{N}, \tag{31b}$$
$$u(t)=u_k=\begin{bmatrix}1.0 & 1.0\end{bmatrix}^T, \quad t_k\le t\le t_{k+1}, \quad k\in\mathcal{N}. \tag{31c}$$

The weights are $Q_{c,xx}=1.0$ and $Q_{c,uu}=\text{diag}([1.0,1.0])$, the sampling time $T_s=1.0$, and initial state vector is $\boldsymbol{x}_0=\begin{bmatrix}0.0 & 1.0\end{bmatrix}^T$ with the covariance $P_0=\text{diag}([0.1,0.1])$.

Algorithm 2 Step-doubling method for LQ Discretization

Input: $(A_c,B_c,G_c,C_c,D_c,Q_c,T_s,j)$
Output: $(A(T_s),B(T_s),C,D,Q(T_s),M(T_s),R_{ww}(T_s))$

Compute the number of step $N=2^j$
Compute the step size $h=\frac{T_s}{N}$
Use (22) to compute $(\Lambda_i,\ \Theta_i,\ \Omega_i,\ \Lambda,\ \Theta,\ \Omega)$
Set initial states of step doubling matrices ($i=1,\tilde{A}(i)=\Lambda,\ \tilde{B}(i)=I,\ \tilde{Q}(i)=\tilde{Q}_c,\ \tilde{M}(i)=I_{xu},\ \tilde{R}(i)=\tilde{R}_{ww,c}$)
while $i\le j$ **do**
  Use equations from Table II to update ($\tilde{\Gamma}(i)$,$\tilde{M}(i)$, $\tilde{Q}(i)$,$\tilde{R}(i)$)
  Use equations from Table II to update ($\tilde{A}(i)$, $\tilde{B}(i)$)
  Set $i=i+1$
end while
Use (28) to compute $(A(T_s),B(T_s),Q(T_s),M(T_s),R_{ww}(T_s))$

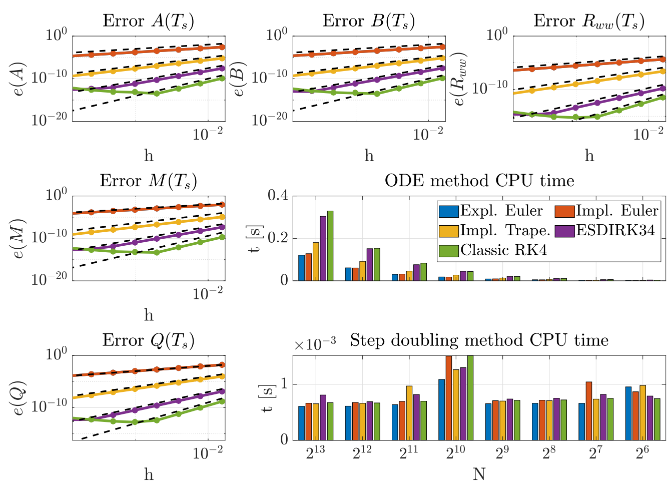

Fig. 2. The error and CPU time of the ODE methods and the step-doubling methods with different discretization methods. The error is $e(i)=|i(T_s)-i(N)|$ for $i\in[A,B,R_{ww},M,Q]$, where $i(T_s)$ is the true result from the matrix exponential method.

In this section, we test and compare the proposed numerical methods and investigate the distribution of the stochastic costs via Monte Carlo simulations.

A. Discretization of LQ-OCP

Fig. 2 describes the error and CPU time of ODE and step-doubling methods. The true solution of (A,B,R_{ww},Q,M) is calculated using the matrix exponential method. The results of the step-doubling method (dot plots) have the same error as the results of the ODE method (line plots). All methods have the correct convergence order (indicated by dashed lines). In bar plots, the CPU time of the ODE method increases as the integration steps and the stages of the discretization method increase. However, the CPU time of the step-doubling method is stable at around 0.6 ms.

Table III describes the error and CPU time of ODE and step-doubling methods with the classic RK4 method applied with the integration step $N=2^8$. The step-doubling method has the same error as the ODE method. The

TABLE III

CPU TIME AND ERROR OF THE SCENARIO USING CLASSIC RK4 WITH $N = 2^8$

	Unit	Matrix Exp.	ODE Method	Step-doubling
$e(A)$	[-]	-	$7.49 \cdot 10^{-12}$	$7.49 \cdot 10^{-12}$
$e(B)$	[-]	-	$8.33 \cdot 10^{-12}$	$8.33 \cdot 10^{-12}$
$e(R_{ww})$	[-]	-	$9.73 \cdot 10^{-11}$	$9.73 \cdot 10^{-11}$
$e(M)$	[-]	-	$1.25 \cdot 10^{-11}$	$1.25 \cdot 10^{-11}$
$e(Q)$	[-]	-	$2.03 \cdot 10^{-13}$	$2.03 \cdot 10^{-13}$
CPU Time	[ms]	0.74	9.5	0.68

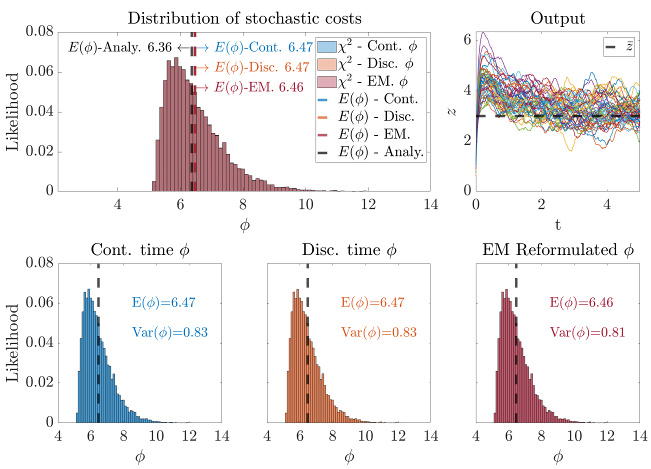

Fig. 3. The likelihood of the cost functions of continuous-time and discrete-time stochastic LQ-OCPs with 30000 Monte Carlo simulations. The Cont., Disc. EM. indicate the continuous-time, discrete-time, and EM reformulated stochastic costs, respectively. $E\{\phi\}$-Analy. is the analytic expectation described by (16), where the continuous-time element $\text{tr}(Q_{c,ww}P_w)$ is solved using the EM method with $N = 2^8$.

ODE method is the slowest among the three methods and takes 9.5 ms, while the matrix exponential and the step-doubling methods spend 0.74 ms and 0.68 ms, respectively.

B. Distribution of stochastic LQ-OCP

Fig. 3 shows the likelihood of the continuous-time stochastic costs (12a), discrete stochastic costs (13a), and EM reformulated stochastic costs (19a) via 30,000 Monte Carlo simulations. We apply the EM method to solve these stochastic cost functions with $N = 2^8$ integration steps.

The simulation results show that the continuous-time cost has the same distribution as the discrete and EM reformulated stochastic costs. There is an offset between the analytic expectation ($E(\phi) = 6.36$, obtained by (16)) and the other numerical expectations (6.47 for continuous- and discrete-time cases and 6.46 for the EM case). We consider it reasonable to have numerical errors since we cannot take the limit of $N \to \infty$ in experiments.

V. CONCLUSIONS

In this paper, we have discussed the discretization of both deterministic and stochastic LQ-OCPs and proposed three numerical methods. In our propositions, LQ discretization is converted into explicit, neat differential equation systems. We further extend the problem from the deterministic to the stochastic case and illustrate the stochastic cost adheres to a generalized χ^2 distribution. The proposed numerical methods are tested and compared in the numerical experiment, and its results indicate: 1) the step-doubling method is the fastest among the three methods while retaining the same accuracy and convergence order as the ODE method, 2) the discrete-time LQ-OCP derived by the proposed numerical methods is equivalent to the original problem in both the deterministic and stochastic cases.

VI. APPENDIX

DISTRIBUTION OF STOCHASTIC COSTS

To evaluate the distribution of the costs, consider the EM discretization of the stochastic system with a fine time step $\delta t = \frac{T_s}{n}$, $t_i = i\delta t$ for $i = 1, \dots, n$ is

$$x_{k,i} = \overbrace{(I + \delta t A_c)^i}^{A_i} x_k + \overbrace{\left(\sum_{j=0}^{i-1} A(j)\delta t B_c\right)}^{B_i} u_k + G_i \boldsymbol{w}_k, \quad (32a)$$

where

$$G_i = \begin{bmatrix} A_{i-1}G_c & A_{i-2}G_c & \dots & 0_{n-i} \end{bmatrix}, \quad (32b)$$

$$\boldsymbol{w}_k = \begin{bmatrix} \Delta \boldsymbol{w}'_{k,1} & \Delta \boldsymbol{w}'_{k,2} & \dots & \Delta \boldsymbol{w}'_{k,n} \end{bmatrix}'. \quad (32c)$$

The EM expression for the extended state vector $[x_k; u_k]$ is

$$\begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix} = \begin{bmatrix} A_n^k \\ 0 \end{bmatrix} \boldsymbol{x}_0 + \begin{bmatrix} \Theta_{u,k} \\ I_{u,k} \end{bmatrix} U_{\mathcal{N}} + \begin{bmatrix} \Theta_{w,k} \\ 0 \end{bmatrix} \boldsymbol{W}_{\mathcal{N}}, \quad (33a)$$

where $\boldsymbol{W}_{\mathcal{N}}$ and $U_{\mathcal{N}}$ are vectors of the random $\boldsymbol{w}_k = I_{w,k}\boldsymbol{W}_{\mathcal{N}}$ and the input $u_k = I_{u,k}U_{\mathcal{N}}$ over the horizon $\mathcal{N}$, and

$$\Theta_{u,k} = \begin{bmatrix} A_n^{k-1}B_n & A_n^{k-2}B_n & \dots & 0_{\mathcal{N}-k} \end{bmatrix}, \quad (33b)$$

$$\Theta_{w,k} = \begin{bmatrix} A_n^{k-1}G_n & A_n^{k-2}G_n & \dots & 0_{\mathcal{N}-k} \end{bmatrix}. \quad (33c)$$

The corresponding EM expression of the discrete stochastic cost function ϕ is

$$\begin{aligned} \phi &= \sum_{k \in \mathcal{N}} l_k(\boldsymbol{x}_k, u_k) + l_{s,k}(\boldsymbol{x}_k, u_k) \\ &= \sum_{k \in \mathcal{N}} \frac{1}{2} \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix}' Q \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix} + q'_k \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix} + \rho_k, \end{aligned} \quad (34)$$

where

$$q_k = M\bar{z}_k + \Omega \boldsymbol{w}_k, \quad (35a)$$

$$\begin{aligned} \rho_k = \sum_{i=1}^{n} \frac{1}{2} \boldsymbol{w}'_k (G'_i \bar{Q}_{c,ww} G_i) \boldsymbol{w}_k + (-\delta t G'_i \bar{Q}_c \bar{z}_k)' \boldsymbol{w}_k \\ + \frac{1}{2} \bar{z}'_k \bar{Q}_c \bar{z}_k. \end{aligned} \quad (35b)$$

$$\Gamma_i = \begin{bmatrix} C_c & D_c \end{bmatrix} \begin{bmatrix} A_i & B_i \\ 0 & I \end{bmatrix}, \quad (35c)$$

$$\Omega = \sum_{i=1}^{n} \delta t \Gamma'_i Q_c \begin{bmatrix} C'_c & 0 \end{bmatrix}' G_i, \quad (35d)$$

Using (33a) to substitute $[x_k; u_k]$, we reformulate the quadratic problem into isolated stochastic form (36a) with deterministic weights $Q_{\mathcal{N}}$, $q_{\mathcal{N}}$ and $\rho_{\mathcal{N}}$.

$$\phi = \frac{1}{2}\begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{W}_{\mathcal{N}} \end{bmatrix}' Q_{\mathcal{N}} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{W}_{\mathcal{N}} \end{bmatrix} + q'_{\mathcal{N}} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{W}_{\mathcal{N}} \end{bmatrix} + \rho_{\mathcal{N}}, \tag{36a}$$

$$\begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{W}_{\mathcal{N}} \end{bmatrix} \sim N(\bar{m}, \bar{P}), \quad \bar{m} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \quad \bar{P} = \begin{bmatrix} P_0 & 0 \\ 0 & P_w \end{bmatrix}, \tag{36b}$$

where the state vector of the reformulated cost function is normally distributed, and $\boldsymbol{W}_{\mathcal{N}}$ has the covariance $P_w = \mathrm{diag}(I\delta t, I\delta t, \ldots, I\delta t)$.

Based on the theory of integrating the normal in the quadratic domain [9], the stochastic cost ϕ follows a generalized χ^2 distribution. The quadratic-form of its expectation and variance can be computed as (20a) and (20b).

The original cost (12a) is equivalent to the discrete cost when taking the limit

$$\int_{t_0}^{t_0+T} l_c(\tilde{z}(t))dt = \lim_{n\to\infty} \sum_{j=1}^{n} l_c(\tilde{z}(t_0 + j\delta t))\delta t, \quad \delta t = \frac{T}{n}. \tag{37}$$

Thus, the stochastic cost ϕ of the continuous-time LQ-OCP is a generalized χ^2-distribution variable, with mean and variance taken in the limit of (20).

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