2024 European Control Conference (ECC)
June 25-28, 2024. Stockholm, Sweden

Quantized Periodic Event-triggered Control for LTI Systems with 1-bit Data Transmission

Mahmoud Abdelrahim, Dhafer Almakhles and Abdalla Swikir

Abstract— **This paper presents a novel framework for periodic event-triggered control (PETC) coupled with dynamic quantization for linear systems. Unlike traditional time-driven control methods, our approach leverages event-based mechanisms to judiciously update control actions, thus minimizing computational load and network traffic. We introduce a two-level dynamic quantizer for encoding feedback information with a single bit, thereby enhancing resource efficiency. The proposed PETC mechanism decides the transmission instants based on the quantized output samples. The resulting system is modeled as a hybrid dynamical system to capture both continuous and discrete dynamics. Sufficient conditions for ensuring the stability of the closed-loop system are presented in the form of a linear matrix inequality. Through numerical simulations, we demonstrate that our approach captures the initial output within a finite time and significantly reduces data transmissions compared to traditional methods. This paper makes key contributions in the integration of dynamic quantization with PETC, leading to resource-efficient and stable control systems.**

I. INTRODUCTION

Event-triggered control (ETC) is an adaptive control approach aimed at optimizing resource utilization while mitigating the computational and communication load, see [1]–[3] and the references therein. ETC diverges from conventional time-driven control by not executing control actions at fixed time intervals; instead, it triggers actions based on specific events or system conditions. This approach ensures that control updates only occur when necessary, thereby enhancing resource efficiency and potentially reducing network traffic. ETC finds particular relevance in networked control systems (NCS), primarily due to limited bandwidth constraints. A major challenge in ETC is the prevention of Zeno events (infinite transmissions within finite time) [4]–[6]. A viable solution to this issue involves verifying the triggering rule at periodic time points, leading to what is known as periodic event-triggered control (PETC), as discussed in [7]–[9].

M. Abdelrahim and Dhafer Almakhles are with the Renewable Energy Laboratory, College of Engineering, Prince Sultan University, 11586 Riyadh, Saudi Arabia {mabdelrahim, dalmakhles}@psu.edu.sa. M. Abdelrahim is also with the Department of Mechatronics Engineering, Assiut University, 71515 Assiut, Egypt m.abdelrahim@aun.edu.eg. A. Swikir is with the Munich Institute of Robotics and Machine Intelligence, 80992 München, Germany abdalla.swikir@tum.de.

* This work was supported by the Prince Sultan Defense Studies and Research Center (PSDSARC), Project No: PID-000085-01-04. The authors were also supported by the Prince Sultan University, Riyadh, Saudi Arabia.

Additionally, in digital implementations, the design of quantizers becomes crucial because feedback information is encoded using a finite number of bits. When the initial signal magnitude to be quantized is unknown, preventing quantizer saturation poses a technical challenge. An appealing approach in such scenarios is the dynamic quantizer introduced by [10]–[12], where the quantizer range and levels dynamically adjust based on state measurements. Notably, the simultaneous consideration of dynamic quantization and event-triggered implementation, though practical, has been addressed in only a limited number of works within the literature, such as [13]–[19]. All of these methods are based on the dynamic quantizer introduced of [10]–[12] and the information transmission is organized in a packet-based fashion

Motivated by the above results and their limitations, this work proposes a novel periodic event-triggering mechanism based on quantized output information for linear systems. In particular, we assume that the output measurement is first quantized and sampled periodically while the PETC decides which samples of the quantized output need to be submitted to the controller. The proposed quantizer has been established in our recent work [20] and is exploited here to handle both the effect of sampling and quantization rather as opposed to [20] where only the quantization aspect has been investigated. The developed quantization approach is dynamic and capable of capturing the unknown initial output within a finite time. Furthermore, the quantizer comprises solely two levels, making it possible to transmit feedback information using only a single bit, which is an attractive feature for practical implementation. Then, a novel PETC is synthesized to produce the sequence of transmission instants. The proposed PETC is dynamic in the sense that the triggering threshold involves an internal dynamic variable, which has been proven as efficient solution to further reduce the amount of transmissions. In addition, as the triggering condition is evaluated exclusively at periodic intervals, the Zeno phenomenon is inherently averted because all inter-transmission times are constrained to be greater than or equal to the constant sampling period. Furthermore, the sampling period is designed as the maximal allowable transmission interval (MATI) of time-triggered controllers based on the approach of [21].

The entire system is depicted as a hybrid dynamical system, accommodating both continuous and discrete dynamics, as commonly encountered in NCS. Sufficient conditions are provided to ensure the stability of the closed-loop system in the form of a linear matrix inequality (LMI). Numerical simulations have been conducted to verify the implementation and advantages of this approach. The simulation outcomes validate the quantizer potential to effectively capture the initial output within a finite time and while the developed PETC significantly reduces data transmissions compared to periodic sampling.

The main contributions of this paper include:

- A novel periodic event-triggering mechanism is proposed to generate the sequence of transmission instants based only on quantized output measurements.
- A two-level dynamic quantizer is developed to encode the feedback information using a single bit.
- The combined quantized periodic event-triggered control ensures an asymptotic stability of the closed-loop system.
- The overall problem is described as a hybrid dynamical system to account for the existing continuous-time and discrete-time dynamics.

II. Preliminaries and Notations

We employ the symbols $\mathbb{R}, \mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{N}$, and $\mathbb{N}_{>0}$, to represent the sets of real numbers, positive real numbers, non-negative real numbers, non-negative integers and positive integers, respectively. For any $c \in \mathbb{R}$, $|c|$ represents the absolute value of c, and for any $x \in \mathbb{R}^n$, the Euclidean norms of x is denoted by $|x|$.The identity matrix with dimension $n \in \mathbb{N}_{>0}$ is denoted by $\mathbb{I}_n$.

III. Problem formulation

Consider a linear plant with the following dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the state of the plant, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $y(t) \in \mathbb{R}^{n_y}$ is the available output for measurement, for some $n_x, n_u, n_y \in \mathbb{N}_{>0}$, and $t \in \mathbb{R}_{\geq 0}$. The plant matrices A, B, C are constant with appropriate dimensions. Assuming that the pair (A, B) is stabilizable and the pair (A, C) is detectable, the following dynamic output feedback controller is constructed to stabilize the plant in continuous-time

$$\begin{aligned} \dot{\zeta}(t) &= A_K\zeta(t) + B_K y(t) \\ u(t) &= C_K\zeta(t) + D_K y(t), \end{aligned} \tag{2}$$

where $\zeta(t) \in \mathbb{R}^{n_\zeta}, n_\zeta \in \mathbb{N}_{>0}$, denotes the state of the controller. The controller matrices A_K, B_K, C_K, D_K are constant with appropriate dimensions.

A. Implementation Scenario

The depicted deployment scenario is presented in Figure 1. In this context, it is assumed that the full state of the plant, denoted as x, cannot be directly measured. Instead, only an output $y(t)$ can be conveyed to the controller through a digital network. In particular, the output $y(t)$ is first quantized to $\bar{y}(t)$ and then sampled at periodic sampling times $t_k, k \in \mathbb{N}$. Subsequently, an event-triggering mechanism is utilized to determine the upcoming transmissions moment represented as $\hat{t}_k, k \in \mathbb{N}$, which is known as periodic event-triggered control (PETC), and we refer by $\bar{y}(\hat{t}_k)$ to value of $\bar{y}(t_k)$ available at the controller at time $\hat{t}_k, k \in \mathbb{N}$, see Figure 1. For later use, define $\mathcal{T} := \{t_k\}, \widehat{\mathcal{T}} := \{\hat{t}_k\}, k \in \mathbb{N}$, and note that $\widehat{\mathcal{T}} \subseteq \mathcal{T}$.

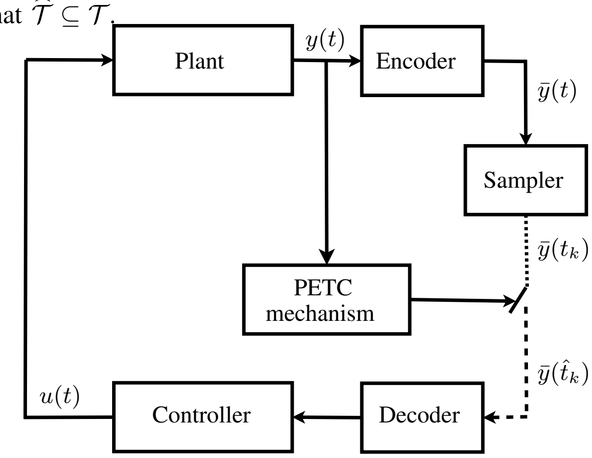

Fig. 1: Periodic event-triggered control with quantized output feedback. (solid line) continuous-time; (dotted line) periodic instants; (dash line) event-triggered instants.

Additionally, it is assumed that the initial value of the output measurement is not known, which poses challenges for designing a quantizer while avoiding saturation. While static quantizers evade this issue by assuming an unlimited range, this assumption is impractical. Instead, we adopt dynamic quantizers that adjust their range dynamically based on the magnitudes of the input signal to prevent saturation. We also assume that both $y(t)$ and $\bar{y}(\hat{t}_k)$ are available to the PETC mechanism.

B. Dynamic quantization

We assume that the output measurement is first subject to quantization before periodical sampling. Since the output measurement is assumed to be known, a dynamic quantizer is needed to capture the initial output using a finite quantization level. We adopt the novel dynamic quantizer developed in [20], which has the following dynamics

$$\begin{aligned} s(t) &= \bar{y}(t) - y(t) \\ \bar{y}(t) &= z\left(t\right) e^{\beta(t)} \\ \beta\left(t\right) &= \lambda_q \left|z(t)\right| \left|\int_{t_0}^{t} z(\tau)d\tau\right| \quad \lambda_q > 0 \end{aligned} \tag{3}$$

with

$$\ell\dot{z}(t) = z(t) - \operatorname{sgn}\left(s(t)\right), \tag{4}$$

$z(t)$ is the state of the low pass filter (LPF) as described in Equation (4). The parameter $\ell > 0$ denotes the time constant of the LPF.

The first equation in (3) describes the quantization error. The second and third equations in (3) illustrate the computation of the quantized value $\bar{y}(t)$ through an exponential function. The dynamics in (4) represent a linear first-order filter, which calculates the average of the switching signal $\operatorname{sgn}(s(t))$, see [22]. The use of the exponential function in (3) facilitates the rapid evolution of the quantized output $\bar{y}(t)$

to capture the output $y(t)$ within a finite time. It is important to note that the quantizer (3)-(4) is solely dependent on the sign of the quantization error $s(t)$, which enables the transmission of quantized information using only a single bit.

We employ a quantization approach similar to that in [12], where we initialize the control input $u = 0$ in (1) until the quantizer captures the output $y(t)$. Subsequently, we operate the system in closed-loop mode by setting $u = C_K\zeta(t) + D_K\bar{y}(t)$. Hence, we have

$$u(t) = \begin{cases} 0 & t \le t_c \\ C_K\zeta(t) + D_K\bar{y}(t) & t > t_c, \end{cases} \tag{5}$$

where $t_c \in \mathbb{R}_{\ge 0}$ is the capturing time of the state. Note that only the quantized value $\bar{y}(t)$ is available at the controller not the actual value $y(t)$ as we will explain further. To that end, similar to [12], we assume that plant dynamics (1) is forward complete [23], which is an essential requirement to ensure that the plant state does not explode in finite time during the transient stage.

The following Lemma establishes the main characteristics of the 2-level quantizer (3)-(4).

Lemma 1: Consider system (1) and the dynamic quantizer (3). Pick the constant α in (3) such that $\alpha > Re(\lambda_{\max}(A))$ and set $z(t_0) = \operatorname{sgn}(y_0)$. Then, there exists a finite capturing time $t_c \ge 0$ such that

1) $|z(t)| = 1 \quad \forall t \le t_c \quad$ and $\quad |z(t)| < 1 \quad \forall t > t_c$;
2) $|y(t) - \bar{y}(t)| \le \varepsilon \quad \forall t \ge t_c$ and for some $\varepsilon \ge 0$;
3) the capturing time t_c is upper bounded by $t_c \le \frac{\ln(|x_0|)}{|Re(\lambda_{\max}(A)) - \lambda_q|} + 5\ell$.

□

The proof of Lemma 1 follows from [24] and it is therefore omitted. Property (1) means that the magnitude of LPF state $z(t) = 1$ during the capturing stage and the magnitude of $z(t) < 1$ after the state is captured. Property (2) implies that the quantization error remains bounded after the state has been successfully captured by the quantizer, unlike [12] where zoom-in and zoom-out actions can be persistently activated. Property (3) is the derived upper bound on the estimated capturing time, which is finite. Note that the implementation of the dynamic quantizer (3) only requires the knowledge of the sign of the initial output measurement, i.e. $\operatorname{sgn}(y(t_0))$ rather than its magnitude, which is not restrictive. Moreover, this requirement can be relaxed without significant modification of the analysis. We chose to keep with this requirement for convenience, see [24] for more detail.

Problem statement. The aim of this paper is to design the periodic sampling interval, the periodic event-triggered controller, and the dynamic quantizer to ensure the closed-loop stability while minimizing the volume of transmissions.

IV. Hybrid model

In this section, we extract the dynamic characteristics of the closed-loop system and represent it as a hybrid dynamical system. To achieve this, it is important to emphasize that, because of the network, the controller can solely acquire the quantized output measurement at specific discrete-time instances. Accordingly, when considering the effect of sampling and quantization, the controller dynamics (2) becomes

$$\begin{aligned} \dot{\zeta}(t) &= A_K\zeta(t) + B_K\bar{y}(\hat{t}_k) \\ u(t) &= C_K\zeta(t) + D_K\bar{y}(\hat{t}_k), \end{aligned} \tag{6}$$

where $\bar{y}(\hat{t}_k)$ is the last submitted value of y to the controller. We define

$$e_y(t) := \bar{y}(\hat{t}_k) - \bar{y}(t) \qquad \forall t \in [\hat{t}_k, \hat{t}_{k+1}), k \in \mathbb{N}, \tag{7}$$

where $e_y(t)$ is the measurement error due to event-triggered implementation. During the time interval between two consecutive periodic sampling points $[t_k, t_{k+1}]$, the sampled output $\bar{y}(t_k)$ remains constant due to zero-order hold (ZOH) implementation. At each periodic sampling moment $t_{k+1}, k \in \mathbb{N}$, the value of $\bar{y}(t_k)$ is reset to $\bar{y}(t_{k+1})$. Additionally, between two consecutive transmission events $[\hat{t}_k, \hat{t}_{k+1}]$, the last transmitted value of the output $\bar{y}(\hat{t}_k)$, is held constant using ZOH.

Let the total error between the most recent transmitted output value $\bar{y}(\hat{t}_k)$ and the current output measurement $y(t)$ expressed as follows:

$$\begin{aligned} e(t) &:= \bar{y}(\hat{t}_k) - y(t) \qquad \forall t \in [\hat{t}_k, \hat{t}_{k+1}) \\ &= s(t) + e_y(t), \end{aligned} \tag{8}$$

where $s(t)$ is the quantization error as defined in (3). Then, it holds that

$$\begin{aligned} \dot{e}(t) &= -\dot{y}(t) = -C\dot{x}(t) \qquad t \in [\hat{t}_k, \hat{t}_{k+1}) \\ e(\hat{t}_k^+) &= s(\hat{t}_k^+) + e_y(\hat{t}_k^+) \\ &= s(\hat{t}_k). \end{aligned} \tag{9}$$

Since $e(t)$ is not reset to zero at every transmission instant $\hat{t}_k$, stability guarantees becomes more challenging and requires careful handling.

Let $\tilde{x} = (x, \zeta) \in \mathbb{R}^{n_{\tilde{x}}}$ with $n_{\tilde{x}} := n_x + n_\zeta$. Then, in view of (1), (6), (8), we obtain

$$\begin{aligned} \dot{\tilde{x}} &= \mathcal{A}_1\tilde{x} + \mathcal{B}_1 e \\ \dot{e} &= \mathcal{A}_2\tilde{x} + \mathcal{B}_2 e, \end{aligned} \tag{10}$$

where

$$\mathcal{A}_1 = \begin{bmatrix} A + BD_KC & BC_K \\ B_KC & A_K \end{bmatrix}, \qquad \mathcal{B}_1 = \begin{bmatrix} BD_K \\ B \end{bmatrix}$$

$$\mathcal{A}_2 = \begin{bmatrix} -C(A + BD_KC) & -CBC_K \end{bmatrix}, \ \mathcal{B}_2 = \begin{bmatrix} -CBD_K \end{bmatrix}. \tag{11}$$

We establish two supplementary time variables, denoted as $\tau, \hat{\tau} :: \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, which are defined as follows

$$\begin{aligned} \dot{\tau}(t) &= 1 \qquad t \in [t_k, t_{k+1}) \\ \tau(t_k^+) &= 0 \qquad t \in \{t_k\}_{k \in \mathbb{N}} \end{aligned} \tag{12}$$

and

$$\begin{aligned}\dot{\hat{\tau}}(t) &= 1 && t \in [\hat{t}_k, \hat{t}_{k+1}) \\ \hat{\tau}(\hat{t}_k^+) &= 0 && t \in \{\hat{t}_k\}_{k\in\mathbb{N}}.\end{aligned} \tag{13}$$

The time variable τ is employed to measure the duration between consecutive periodic sampling intervals $[t_k, t_{k+1}]$, and it undergoes a reset to zero at each periodic moment t_k. Similarly, $\hat{\tau}$ is utilized to monitor the time between successive transmission events $[\hat{t}_k, \hat{t}_{k+1}]$, and it is reset to zero at each transmission event $\hat{t}_k$.

To provide a comprehensive overview of the entire system, we present the fundamental structure of the proposed periodic event-triggering mechanism in the following. A detailed elaboration of this mechanism will be presented in the subsequent section. We formulate a Periodic Event-Triggered Control (PETC) system utilizing a dynamic variable η which follows the dynamics described below, see also [25]–[27],

$$\begin{aligned}\dot{\eta}(t) &= \Psi(y,\eta) && t \in [t_k, t_{k+1}) \\ \eta(t_k^+) &= g(\eta, e) && t_k \in \mathcal{T} \setminus \widehat{\mathcal{T}} \\ \eta(\hat{t}_k^+) &= \hat{g}(\eta, e) && \hat{t}_k \in \widehat{\mathcal{T}},\end{aligned} \tag{14}$$

where the details of functions $\Psi(y,\eta), g(\eta, e)$ and $\hat{g}(\eta, e)$ will be provided later. It is worth noting that these functions are determined solely based on locally accessible information (y, e, η) at the ETC mechanism. The triggering moments are determined by the following criterion:

$$\hat{t}_{k+1} = \min\{t > \hat{t}_k \mid t \in \mathcal{T} \wedge g(e,\eta) \leqslant 0\} \tag{15}$$

with $\hat{t}_0 = 0$. The ETC condition $t \in \mathcal{T} \wedge g(e,\eta) \leqslant 0$ indicates that we only verify the rule $g(e,\eta) \leqslant 0$ at periodic sampling times to implement the periodic ETC, which automatically ensures that Zeno sampling can never occur.

In view of (10)-(14), we derive the subsequent impulsive model

$$\left.\begin{aligned}\dot{\tilde{x}}(t) &= \mathcal{A}_1\tilde{x}(t) + \mathcal{B}_1 e(t) \\ \dot{e}(t) &= \mathcal{A}_2\tilde{x}(t) + \mathcal{B}_2 e(t) \\ \dot{\eta}(t) &= \Psi(y,\eta) \\ \dot{\tau}(t) &= 1 \\ \dot{\hat{\tau}}(t) &= 1\end{aligned}\right\} \quad t \notin \mathcal{T}$$

$$\left.\begin{aligned}\tau(t_k^+) &= 0 \\ \eta(t_k^+) &= g(e,\eta)\end{aligned}\right\} \quad t_k \in \mathcal{T} \setminus \widehat{\mathcal{T}} \tag{16}$$

$$\left.\begin{aligned}e(\hat{t}_k^+) &= s(t) \\ \eta(\hat{t}_k^+) &= \hat{g}(e,\eta) \\ \tau(\hat{t}_k^+) &= 0 \\ \hat{\tau}(\hat{t}_k^+) &= 0\end{aligned}\right\} \quad \hat{t}_k \in \widehat{\mathcal{T}}$$

Let $\xi := (\tilde{x}, e, \eta, \tau, \hat{\tau}) \in \mathbb{R}^{n_\xi}$. Next, the hybrid dynamical system is obtained

$$\begin{aligned}\dot{\xi} &= \mathcal{F}(\xi) && \xi \in \mathcal{C} \\ \xi^+ &\in \mathcal{G}(\xi) && \xi \in \mathcal{D},\end{aligned} \tag{17}$$

where $\mathcal{C} := \mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{D} := \mathcal{D}_1 \cup \mathcal{D}_2$ with

$$\begin{aligned}\mathcal{C}_1 &:= \{\xi \in \mathbb{X} : |z| = 1\} \\ \mathcal{C}_2 &:= \{\xi \in \mathbb{X} : |z| < 1 \quad \text{and} \quad \tau \leq T\}\end{aligned} \tag{18}$$

and

$$\begin{aligned}\mathcal{D}_1 &:= \Big\{\xi \in \mathbb{R}^{n_\xi} : \tau = T \text{ and } g(e,\eta) \geqslant 0\Big\} \\ \mathcal{D}_2 &:= \Big\{\xi \in \mathbb{R}^{n_\xi} : \tau = T \text{ and } g(e,\eta) \leqslant 0\Big\},\end{aligned} \tag{19}$$

where the periodic sampling interval $T > 0$ will be given later. The flow map $\mathcal{F}(\xi)$ is given by

$$\mathcal{F}(\xi) := \begin{cases}\Big\{F_1(\xi)\Big\} & \text{for } \xi \in \mathcal{C}_1 \\ \Big\{F_2(\xi)\Big\} & \text{for } \xi \in \mathcal{C}_2\end{cases} \tag{20}$$

with

$$F_1(\xi) := \begin{pmatrix}\mathcal{A}_1\tilde{x}(t) \\ \mathcal{A}_2\tilde{x}(t) \\ \Psi(y,\eta) \\ 1 \\ 1\end{pmatrix}, \quad F_2(\xi) := \begin{pmatrix}\mathcal{A}_1\tilde{x}(t) + \mathcal{B}_1 e(t) \\ \mathcal{A}_2\tilde{x}(t) + \mathcal{B}_2 e(t) \\ \Psi(y,\eta) \\ 1 \\ 1\end{pmatrix} \tag{21}$$

and the jump map $\mathcal{G}(\xi)$ in (17) is given by

$$\mathcal{G}(\xi) := \begin{cases}\Big\{G_1(\xi)\Big\} & \xi \in \mathcal{D}_1 \setminus \mathcal{D}_2 \\ \Big\{G_2(\xi)\Big\} & \xi \in \mathcal{D}_2 \\ \emptyset & \xi \notin \mathcal{D}_1\end{cases} \tag{22}$$

with

$$G_1(\xi) := \begin{pmatrix}\tilde{x}(t_k) \\ e(t_k) \\ g(e,\eta) \\ 0 \\ \hat{\tau}(t_k)\end{pmatrix}, \qquad G_2(\xi) := \begin{pmatrix}\tilde{x}(\hat{t}_k) \\ s(\hat{t}_k) \\ \hat{g}(e,\eta) \\ 0 \\ 0\end{pmatrix} \tag{23}$$

The system operates along the trajectory on $\mathcal{F}(\xi)$ as long as there is no zoom event, and the triggering condition remains intact; otherwise, the system undergoes a discontinuous transition. The jump map in (23) can be interpreted as follows. When $\xi \in \mathcal{D}_1 \setminus \mathcal{D}_2$, this implies that a periodic sampling occurs but not a transmission and hence only $\eta(t)$ and $\tau(t)$ are updated. When $\xi \in \mathcal{D}_2$, this implies that both a periodic sampling and a transmission occur.

V. Event-triggering Mechanism

The construction of the event-triggering mechanism is presented in this section. We first state the following result, which will be used later to prove our main theorem.

Lemma 2: Consider system (17)-(23) and define $\widetilde{C} := [C \quad 0]$. Suppose there exist positive scalars $\varepsilon_x, \varepsilon_y, \tilde{\sigma} > 0$

and a symmetric positive definit P such that

$$\begin{bmatrix} \mathcal{A}_1^T P + P\mathcal{A}_1 + \varepsilon_x \mathbb{I}_{n_{\tilde{x}}} + \mathcal{A}_2^T \mathcal{A}_2 + \varepsilon_y \widetilde{C}^T \widetilde{C} & P\mathcal{B}_1 \\ \mathcal{B}_1^T P & -\tilde{\sigma}\mathbb{I}_{n_e} \end{bmatrix} \leqslant 0, \tag{24}$$

then the storage function $V(\tilde{x}) = \tilde{x}^T P \tilde{x}$ satisfies

$$\langle \nabla V(\tilde{x}), \mathcal{A}_1\tilde{x} + \mathcal{B}_1 e\rangle \leq -\varepsilon_x|\tilde{x}|^2 - |\mathcal{A}_2\tilde{x}|^2 - \varepsilon_y|y|^2 + \tilde{\sigma}|e|^2 \tag{25}$$

for all $e \in \mathbb{R}^{n_e}$ and $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$. ■

The proof of Lemma 2 has been omitted due to space limit. Lemma 2 establishes a stability property based on the $\mathcal{L}_2$-gain for $\dot{\tilde{x}} = \mathcal{A}_1\tilde{x} + \mathcal{B}_1 e$ from $|e|$ to $(|\mathcal{A}_2\tilde{x}|, |y|)$, see also e.g. [28], [29].

Let $W(e) := |e|$. Consequently, in light of (17), we have that $\forall \tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$ and almost all $e \in \mathbb{R}^{n_e}$

$$\langle \nabla W(e), \mathcal{A}_2\tilde{x} + \mathcal{B}_2 e\rangle \leqslant |\mathcal{A}_2\tilde{x}| + L|e|, \tag{26}$$

where $L := |\mathcal{B}_2|$. The required functions for the dynamics of η in (14) are

$$\begin{aligned} \Psi(y,\eta) &:= \varepsilon_y|y|^2 - \vartheta\eta \\ g(e,\eta) &:= \sigma(\mu - \tfrac{1}{\mu})|e|^2 + \eta \\ \hat{g}(e,\eta) &:= \max(\epsilon, \sigma\mu|e|^2 + \eta) \end{aligned} \tag{27}$$

for some small $\vartheta, \epsilon > 0$ arbitrarily chosen, $\mu \in (0,1)$, $\sigma = \sqrt{\tilde{\sigma}}$ and $\tilde{\sigma}$ comes from Lemma 2.

The selection of the sampling period T is such that it falls within the range of $(0, \mathcal{T})$, where the upper limit $\mathcal{T}$ is defined as

$$\mathcal{T}(\sigma, L) := \begin{cases} \frac{1}{Lr}\arctan(r) & \sigma > L \\ \frac{1}{L} & \sigma = L \\ \frac{1}{Lr}\operatorname{arctanh}(r) & \sigma < L \end{cases} \tag{28}$$

with $r := \sqrt{\left|(\frac{\sigma}{L})^2 - 1\right|}$. To illustrate how the time T is derived, let ϕ be the solution to the differential system

$$\frac{d\phi}{d\tau} := \begin{cases} -2\tilde{L}\phi(\tau) - \sigma(\phi^2(\tau) + 1) & \tau \in [0, T] \\ 0 & \tau > T, \end{cases} \tag{29}$$

where $\tilde{L}$ is defines as $\tilde{L} := L + \nu$, where $L = |\mathcal{B}_2|$ for any $\nu > 0$, and the constant σ is determined based on the findings of Lemma 2. Subsequently, the time constant T represents the duration it takes for ϕ to transition from $\phi(0) = \frac{1}{\mu}$ to $\phi(T) = \mu$, with $\mu \in (0,1)$. The calculation of this time constant is presented in (28), which can be obtained by following a methodology similar to that employed in [28].

Now we have all the ingredients to formulate the main result of this paper. The proof has been omitted due to space constraints.

Theorem 1: Consider the hybrid system (17). Let the sampling period designed as in (28), the triggering rule constructed based on (27) and the dynamic quantizer is designed as in (3). Suppose that the required conditions in Lemma 2 are verified. Then it holds that the set $\mathcal{A} = \{\xi \in \mathbb{R}^{n_\xi} : \frac{\sigma}{\mu}|e_q|^2 < \epsilon\}$ is globally asymptotically stable for some $\epsilon > 0$.

VI. Illustrative Example

Consider the following LTI control system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1.3 & -0.2 & 6.7 & -5.6 \\ -0.6 & -4.3 & 0 & 0.7 \\ 1.1 & 4.3 & -6.7 & 5.9 \\ 0.1 & 4.3 & 1.3 & -2.1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 5.7 & 0 \\ 1.1 & -3.1 \\ 1.1 & 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 1 & -1 \end{bmatrix} x. \end{aligned} \tag{30}$$

We design the controller gain K to set the eigenvalues of $A - BK$ at $\{-1, -2, -3, -4\}$ and we design the observer gain L to set the eigenvalues of $A - LC$ at $\{-5, -8, -11, -20\}$.

We develop the hybrid model (17) as described in Section IV. We check the required conditions of Lemma 2 and found that (24) is feasible and leads to $T = 0.0206$. Finally, we pick $\vartheta = 0.01$ and thus all parameters of the ETM (27) are set. For the dynamic quantizer, we take $\alpha = 4$ and $\ell = 0.1$. We examine the approach on MATLAB simulation with $\xi(0,0) = (0.10, -0.1, -0.20, 0.20, 0, 0, 0.35, 0, 0, 0)$. When the simulation is executed for a duration of 5 seconds, the resulting minimum sampling interval τ_{min} and the average time between samples τ_{avg} were found to be $\tau_{\text{min}} = 0.0206$ and $\tau_{\text{avg}} = 0.1661$, respectively. We note that the minimum time between transmissions $\tau_{\text{min}} = T$ while the average time between transmissions τ_{avg} tends to be greater than T. This observation aligns with our analysis and serves to validate the advantages of this approach over traditional periodic sampling. The closed-loop response is depicted in Figures 2-3 shown below.

The plant output $y(t)$ and the quantized output $\bar{y}(t)$ are presented in Figure 2 where we see that the quantizer captures the output in a short time thanks to the zoom-out actions.

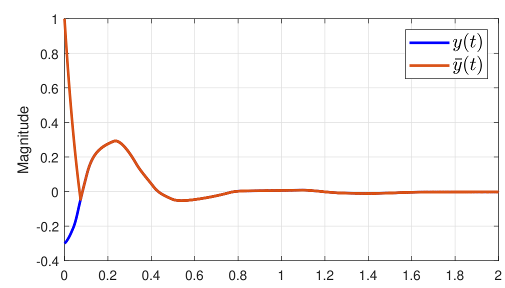

Fig. 2: Trajectories of the Plant output and its quantized value.

Figure 3 shows the periodic time instants and the transmission instants.

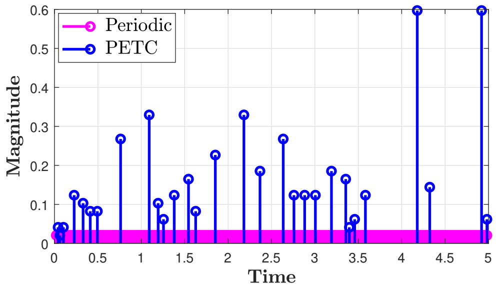

Fig. 3: Periodic and transmissions instants.

VII. Conclusion

We have tackled the challenge of stabilizing output feedback linear time-invariant systems with unknown initial conditions and output measurement subject to both sampling and quantization. To overcome these communication limitations, we have developed a novel quantized periodic event-triggering approach to ensure closed-loop system stability. The quantizer captures the initial output within a finite time using only two quantization levels, allowing for 1-bit data transmission. Additionally, the periodic event-triggering mechanism guarantees that time intervals between transmissions consistently exceed a predefined threshold, preventing Zeno phenomena. Numerical results confirm an over 80% reduction in transmissions, highlighting the benefit of the approach.

Future work includes extending this approach to nonlinear plant models and the investigation of different implementation scenarios such as distributed control architectures with asynchronous data transmissions.

References

[1] W.P.M.H. Heemels, K.H. Johansson, and P. Tabuada. An introduction to event-triggered and self-triggered control. *In Proceedings of the 51st IEEE Conference on Decision and Control, Maui, U.S.A.*, pages 3270–3285, 2012.

[2] Chen Peng and Fuqiang Li. A survey on recent advances in event-triggered communication and control. *Information Sciences*, 457-458:113–125, 2018.

[3] Jiancun Wu, Chen Peng, Hongchenyu Yang, and Yu-Long Wang. Recent advances in event-triggered security control of networked systems: a survey. *International Journal of Systems Science*, 53(12):2624–2643, 2022.

[4] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nešić. Event-triggered dynamic feedback controllers for nonlinear systems with asynchronous transmissions. *In Proceedings of the 54th IEEE Conference on Decision and Control, Osaka, Japan, 15–18 December*, pages 5494–5499, 2015.

[5] H. Yu and T. Chen. A new zeno-free event-triggered scheme for robust distributed optimal coordination. *Automatica*, 129:109639, 2021.

[6] D.P. Borgers and W.P.M.H. Heemels. Event-separation properties of event-triggered control systems. *IEEE Transactions on Automatic Control*, 59(10):2644–2656, 2014.

[7] W.P.M.H. Heemels, M.C.F. Donkers, and A.R. Teel. Periodic event-triggered control for linear systems. *IEEE Transactions on Automatic Control*, 58(4):847–861, 2013.

[8] E. Aranda-Escolástico, M. Abdelrahim, M. Guinaldo, S. Dormido, and W.P.M.H. Heemels. Design of periodic event-triggered control for polynomial systems: A delay system approach. *IFAC-PapersOnLine*, 50(1):7887–7892, 2017.

[9] M. H. Dhullipalla, H. Yu, and T. Chen. A framework for distributed control via dynamic periodic event-triggering mechanisms. *Automatica*, 146:110548, 2022.

[10] R.W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. 45(7):1279–1289, 2000.

[11] D. Liberzon. Hybrid feedback stabilization of systems with quantized signals. *Automatica*, 39(9):1543–1554, 2003.

[12] D. Liberzon and D. Nešić. Input-to-state stabilization of linear systems with quantized state measurements. *IEEE Transactions on Automatic Control*, 52(5):767–781, 2007.

[13] Na Lin and Qiang Ling. Bit-rate conditions for the consensus of quantized multiagent systems based on event triggering. *IEEE Transactions on Cybernetics*, 52(1):116–127, 2022.

[14] M. Abdelrahim, V.S. Dolk, and W.P.M.H. Heemels. Input-to-state stabilizing event-triggered control for linear systems with output quantization. *In Proceedings of the 55th IEEE Conference on Decision and Control, Las Vegas, NV, U.S.A., 12–14 December*, pages 483–488, 2016.

[15] M. Abdelrahim, V.S. Dolk, and W.P.M.H. Heemels. Event-triggered quantized control for input-to-state stabilization of linear systems with distributed output sensors. *IEEE Transactions on Automatic Control*, 61(9):2682–2687, 2019.

[16] L. Xing, C. Wen, Z. Liu, H. Su, and J. Cai. Event-triggered output feedback control for a class of uncertain nonlinear systems. *IEEE Transactions on Automatic Control*, 64(1):290–297, 2019.

[17] S. Liu, L. Xie, and D.E. Quevedo. Event-triggered quantized communication-based distributed convex optimization. *IEEE Transactions on Control of Network Systems*, 5(1):167–178, 2018.

[18] B. Zheng, X. Yu, and Y. Xue. Quantized feedback sliding-mode control: An event-triggered approach. *Automatica*, 91:126–135, 2018.

[19] T. Liu and Z. Jiang. Event-triggered control of nonlinear systems with state quantization. *IEEE Transactions on Automatic Control*, 64(2):797–803, 2019.

[20] D. Almakhles and M. Abdelrahim. Asynchronous dynamic quantization for nonlinear systems with one-bit data transmission. *In Proceedings of the 2023 American Control Conference (ACC 2023), San Diego, California, USA, 2023*, 2023.

[21] D. Nešić, A.R. Teel, and D. Carnevale. Explicit computation of the sampling period in emulation of controllers for nonlinear sampled-data systems. *IEEE Transactions on Automatic Control*, 54(3):619–624, 2009.

[22] Vadim I. Utkin. *Sliding Modes in Control and Optimization*. Springer Berlin Heildelberg, 1992.

[23] D. Angeli, E.D. Sontag, and Y. Wang. Forward completeness, unboundedness observability, and their lyapunov characterizations. *Systems & Control Letters*, 38(4):209–217, 1999.

[24] D. Almakhles and M. Abdelrahim. Asynchronous dynamic quantization for nonlinear systems with one-bit data transmission. *Systems & Control Letters*, 181:105630, 2023.

[25] A. Girard. Dynamic triggering mechanisms for event-triggered control. *IEEE Transactions on Automatic Control*, 60(7):1992–1997, 2015.

[26] V.S. Dolk, D.P. Borgers, and W.P.M.H. Heemels. Output-based and decentralized dynamic event-triggered control with guaranteed $\mathcal{L}_p$-gain performance and Zeno-freeness. *IEEE Transactions on Automatic Control*, 62(1):34–49, 2017.

[27] R. Postoyan, P. Tabuada, D. Nešić, and A. Anta. A framework for the event-triggered stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 60(4):982–996, 2015.

[28] D. Carnevale, A.R. Teel, and D. Nešić. A Lyapunov proof of an improved maximum allowable transfer interval for networked control systems. *IEEE Transactions on Automatic Control*, 52(5):892–897, 2007.

[29] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nešić. Input-to-state stabilization of nonlinear systems using event-triggered output feedback controllers. *In Proceedings of the 14th European Control Conference, Linz, Austria, 15–17 July*, pages 2185–2190, 2015.

□