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Graph-Theoretic Robustness Analysis of Log Learning Learning Dynamics

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***Abstract*— We investigate the propagation of stubborn behavior in a network coordination game where players update their strategies using log-linear learning dynamics. A network is considered robust if the stubborn players cannot impact the stable behavior of the other players. We present a graph-theoretic framework for analyzing the robustness of various networks, establishing conditions wherein all network nodes switch to a stubborn behavior. Our framework leverages the notion of graph closed-knittedness, which measures the strength of external influence on a set of nodes. Using closed-knittedness and a closely related notion of graph plumpness, we derive necessary and sufficient conditions for network robustness to stubborn behavior. We validate our analytical results and bound through extensive Monte-Carlo simulations.**

I. Introduction

As the use of social networks continues to grow, the decisions made by individuals within these networks are increasingly influenced by the choices of their peers. Peer influence plays a substantial role in shaping the behavior and decisions of individuals within the network. The process through which information permeates a social network has become a pivotal focus in research and has given rise to important inquiries regarding how behaviors and innovations diffuse within these interconnected communities.

Innovation diffusion is a theoretical framework and concept that explains the mechanisms through which new ideas, products, services, or practices are spread and adopted within various social contexts, including social systems, communities, organizations, and societies. In the existing literature of innovation diffusion, this problem has been formalized as a network coordination game, where players decide between maintaining the status quo and adopting innovation [1], [2]. Players update their strategies using log-linear learning dynamics ([3], [4], and [5]). The authors provide conditions under which the speed of convergence to innovative behavior is fast. The speed of convergence of behavior in binary-choice coordination games [6], [7], [8], [9], [10], [11], [12] varies under different interaction structures.

The framework used in the previous works assumes a homogeneous population where all players update their strategies using log-linear learning dynamics. In this homogeneous context, the long-term behavior remains unaffected by the interaction structure, as noted in [13]. Consequently, all players ultimately converge to adopt the innovation. The interaction structure's role is primarily in determining the speed at which players embrace innovative behavior. On the contrary, if we allow certain players to be heterogeneous, meaning they follow rules different from log-linear learning, the eventual outcome is affected by both the interaction structure and the quantity and placement of these heterogeneous players within the network.

This paper primarily addresses the propagation of stubborn behavior as opposed to the diffusion of innovative behavior. Specifically, it investigates whether, in a network coordination game, if an initial group of stubborn players adheres to the status quo, can their stubborn behavior hinder the remaining players, who update their actions using Log Linear Learning, from adopting innovation? Additionally, we explore the graph-theoretic properties that define and influence the spread of stubborn behavior within the network. We have previously initiated research on this problem and presented initial findings in [14] and [15]. In those works, we established conditions based on game parameters and network structures to ascertain the network's robustness or non-robustness, utilizing the Radius Coradius result of [7] and [5] for the characterization of stochastically stable states. In [14], we introduced a novel notion of robustness to quantify the impact of heterogeneous players. Three types of heterogeneous players were considered, namely stubborn, confused, and strategic adversaries. Later in [15], we derived qualitative conditions to determine the robustness of stochastically stable action profiles under various games and network settings. Building upon our previous work, this research aims to enhance the efficiency of robustness analysis, which can be computationally expensive for larger networks. We seek to identify graph theoretic properties that influence a network's robustness, making it either robust or non-robust. This capability empowers us to address complex inquiries regarding the robustness of diverse networks. Our primary contribution lies in introducing a graph theoretic method for quantifying network robustness.

The structure of the paper is as follows. Section II outlines the setup, while Section III introduces a graph-theoretic framework for assessing network robustness. In Section IV, we present results for diverse network classes, such as d-regular, path, tree, grid, Cartesian product, and clique chain networks. Finally, Section V concludes the paper.

II. Setup

We consider a symmetric coordination game played across a network, where each player interacts with a specific subset of others. This game offers two action choices for each player: A, typically denoting innovation, and B, symbolizing the status quo. The network is represented by a graph

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	A	B
A	$1+\alpha, 1+\alpha$	$0, 0$
B	$0, 0$	$1, 1$

Fig. 1. Payoff matrix of a 2×2 coordination game.

$G(\mathcal{N}, \mathcal{E})$, where the vertex set $\mathcal{N}$ represents the set of players and the edge set $\mathcal{E}$ represents the interaction connections among the players. N denotes the number of nodes in the set $\mathcal{N}$. Let $\mathcal{N}_i$ be the neighborhood set of player i and the number of neighbors of node i is its degree $d(i)$. Given a set $\mathcal{S} \subset \mathcal{N}$, $d(\mathcal{S})$ is the sum of the degrees of all the vertices in $\mathcal{S}$, i.e., $d(\mathcal{S}) = \sum_{i=1}^{\mathcal{N}_i} d(i)$. Given a pair of sets T and T', $d(T, T')$ denote the number of edges $(i, j) \in \mathcal{E}$ such that $i \in T$ and $j \in T'$. The action set of each player is the set of available actions and is denoted by A_i. The set of joint action profiles is $\mathcal{A} = A_1 \times A_2 \times \cdots \times A_N$. Let $\sigma \in A$ be a joint action profile. We will represent it as $\sigma = (\sigma_i, \sigma_{-i})$, where $\sigma_i \in A_i$ is the action of i and σ_{-i} is the join action of all the players other than i in the joint action profile σ. We also use the notation $\sigma(i) = \sigma_i$ to represent the action of player i in the profile σ. Let σ^A and σ^B represent the action profiles in which all the players play action A and B, respectively, i.e., $\sigma^A(i) = A$ and $\sigma^B(i) = B$ for all $i \in N$. We define $\mathcal{N}_A(\sigma) \subseteq \mathcal{N}$ and $\mathcal{N}_B(\sigma) \subseteq \mathcal{N}$ be the set of players with actions A and B in profile σ, respectively.

We model players' decisions through Log-Linear Learning dynamics (referred to as LLL), which is a noisy best response dynamics. Let $\mathcal{N}_i$ be the neighborhood set of player i, the utilities of player i for selecting actions A and B

$$\begin{aligned} U_i(A, \sigma_{-i}) &= \sum_{j \in \mathcal{N}_i} u(i,j) = (1+\alpha)|\mathcal{N}_A(\sigma_{-i}) \cap \mathcal{N}_i|. \\ U_i(B, \sigma_{-i}) &= \sum_{j \in \mathcal{N}_i} u(i,j) = |\mathcal{N}_B(\sigma_{-i}) \cap \mathcal{N}_i|. \end{aligned} \tag{1}$$

A salient feature of the above payoff structure is that the resulting game is a potential game that admits an exact potential function, which is given by

$$\phi(\sigma) = (1+\alpha) d(\mathcal{N}_A(\sigma), \mathcal{N}_A(\sigma)) + d(\mathcal{N}_B(\sigma), \mathcal{N}_B(\sigma)).$$

Note that the potential function for this setup is equal to half of the sum of the payoffs received by all the players. In general, any function that satisfies the following property can be used as a potential function.

$$\phi(\sigma_i, \sigma_{-i}) - \phi(\sigma_i', \sigma_{-i}) = U_i(\sigma_i, \sigma_{-i}) - U_i(\sigma_i', \sigma_{-i}).$$

A. Graph theoretic Notions

To analyze the robustness properties of the stochastically stable (denoted as SS) profiles in the network coordination game under LLL, we will build our framework on the graph-theoretic notions of closed knittedness and plumpness, which were presented in [1] and [16].

Definition 2.1: A set $\mathcal{S} \subseteq \mathcal{N}$ is r-closed-knit if it satisfies the following condition:

$$\forall\ \mathcal{S}' \subseteq \mathcal{S},\ \mathcal{S}' \neq \{\},\ d(\mathcal{S}', \mathcal{S})/d(\mathcal{S}') \geq r.$$

Closed-knittedness (denoted as CK) is a measure of how well integrated each member of the set $\mathcal{S}$ is with the other members of $\mathcal{S}$. Thus $CK(\mathcal{S}) = r$ implies that no subset $\mathcal{S}'$ of $\mathcal{S}$ has more than r fraction of its interactions outside of $\mathcal{S}$. The closed knittedness of a set is bounded between 0 and $1/2$, where zero corresponds to the situation in which a subset of $\mathcal{S}$ has no interactions with elements of $\mathcal{S}$ and $1/2$ means that no subset of $\mathcal{S}$ has any interactions outside of $\mathcal{S}$.

For defining set autonomy, consider a scenario in which all the members of $\mathcal{S}$ follow LLL, and all the members of the complement set $\mathcal{N} \setminus \mathcal{S}$ are allowed to select any decision strategy. We refer to this scenario as restricted LLL.

Definition 2.2: [1] A set $\mathcal{S}$ is $\sigma^*_{\mathcal{S}}$ autonomous if and only if $(\sigma^*_{\mathcal{S}}, \sigma_{\mathcal{N} \setminus \mathcal{S}})$ is stochastically stable under restricted LLL. For potential games, stochastically stable profiles are those that maximize a potential function. Therefore, autonomous sets can also be defined in terms of potential function.

Definition 2.3: [16] " A set $\mathcal{S} \subseteq \mathcal{N}$ is $\sigma^*_{\mathcal{S}}$-autonomous if, for all σ such that $\sigma_{\mathcal{S}} \neq \sigma^*_{\mathcal{S}}$

$$\phi(\sigma^*_{\mathcal{S}}, \sigma_{-\mathcal{S}}) > \phi(\sigma).$$

Finally, [1] connected the notion of set autonomy with closed knittedness with the following results.

Proposition 1: [1] A set $\mathcal{S} \subseteq \mathcal{N}$ is $\sigma^A_{\mathcal{S}}$ autonomous if and only if

$$CK(\mathcal{S}) > 1/(\alpha + 2).$$

III. Relationship between Robustness, Autonomy, and close-knittedness

We present our graph-theoretic framework for the robustness of stochastically stable profiles in graphical coordination games in log linear learning with a focus on the scenario in which a small set of stubborn players are present in the setup. In [14], we primarily considered the situation in which a single stubborn/heterogeneous player was introduced in the setup, and we proposed a Radius-Coradius based analysis framework for characterizing the long-term impact of the heterogeneous players on the entire population. The Radius and Coradius are primarily the properties of the learning dynamics and in this work, we will present graph-theoretic conditions for the robustness of stochastically stable profiles in LLL dynamics in graphical coordination games. For graphical coordination games, profile σ^A, in which all players play action A, is the unique stochastically stable profile [13]. This setup is said to be robust if all the players continue playing action A regardless of the behavior of the stubborn players. We start by connecting the notion of robustness with close-knittedness that we will analyze in this work.

Proposition 2: Consider a network coordination game with parameter α, which represents the payoff advance of an innovative practice. Let the interaction network topology be represented by a graph $G(\mathcal{N}, \mathcal{E})$. Let $H \subset \mathcal{N}$ be the set of stubborn players. Then, σ^A is robust to the stubborn players if and only if

$$\alpha > \frac{1}{CK(\mathcal{N} \setminus H)} - 2. \tag{2}$$

Proof: Consider the restricted LLL dynamics, in which all the players other than the stubborn players follow LLL dynamics. Moreover, we say that σ^A is robust to stubborn behavior if $\sigma^A_{\mathcal{N}\setminus H}$ is stochastically stable under the restricted LLL dynamics. From Def.2.2, the above condition implies that σ^A is robust if $\mathcal{N}\setminus H$ is an autonomous set. From Prop. 1, $\mathcal{N}\setminus H$ is autonomous if and only if $CK(\mathcal{N}\setminus H) > 1/(\alpha+2)$. Rearranging this expression yields the desired robustness condition on α. ■

An important factor in the practicality of the above result is the complexity of computing the closed knittedness of a set. Computing the closed knittedness of a given set $\mathcal{S}$ requires evaluating the ratio $d(\mathcal{S}',\mathcal{S})/d(\mathcal{S}')$ for all the possible $2^{|\mathcal{S}|}$ subsets of the $\mathcal{S}$, which can be computationally intractable for large networks. To address the computational issues, we refer to the notions of homogeneous sets and plumpness.

Definition 3.1: [17] A set $\mathcal{S}\subset\mathcal{N}$ is homogeneous if

$$CK(\mathcal{S}) = d(\mathcal{S},\mathcal{S})/d(\mathcal{S}).$$

Thus, a set is homogeneous if no subset of $\mathcal{S}$ is less integrated within the set than the set $\mathcal{S}$ is with itself.

Definition 3.2: [16] The plumpness of a set $\mathcal{S}\subset\mathcal{N}$ is defined as follows:

$$Pl(\mathcal{S}) = d(\mathcal{S},\mathcal{S})/d(\mathcal{S}).$$

In general, $CK(\mathcal{S}) \le Pl(\mathcal{S}) \le 1/2$. However, if $\mathcal{S}$ is homogeneous, then $Pl(\mathcal{S}) = CK(\mathcal{S})$. This equality is important because computing the closed knittedness of a set requires evaluating the ratio $d(\mathcal{S}',\mathcal{S})/d(\mathcal{S}')$ for all the possible $2^{|\mathcal{S}|}$ subsets of $\mathcal{S}$. Whereas, computing the plumpness of a set is a one-step operation.

Proposition 3: Consider the constrained LLL such that all the players in a set $\mathcal{S}\subset\mathcal{N}$ follow LLL dynamics and the players in $\mathcal{N}\setminus\mathcal{S}$ are fixed on action B. Then, the profile $\sigma = (\sigma^A_{\mathcal{S}}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})$ is not stochastically stable if

$$Pl(\mathcal{S}) < 1/(\alpha+2)$$

Proof: Suppose that $Pl(\mathcal{S}) < 1/(\alpha+2)$. Then, $d(\mathcal{S},\mathcal{S})/d(\mathcal{S}) \le 1/(\alpha+2)$. Substituting $d(\mathcal{S}) = 2d(\mathcal{S},\mathcal{S}) + d(\mathcal{S},\mathcal{N}\setminus\mathcal{S})$,we get

$$\alpha\ d(\mathcal{S},\mathcal{S}) \le d(\mathcal{S},\mathcal{N}\setminus\mathcal{S}). \tag{3}$$

In order to establish that $(\sigma^A_{\mathcal{S}}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})$ is not stochastically stable, it is essential to provide evidence of an alternative profile, such as $(\sigma^B_{\mathcal{S}}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})$, which acts as the potential maximizer according to the specified condition.

$$\begin{aligned}
&\phi(\sigma^B_{\mathcal{S}}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}) - \phi(\sigma^A_{\mathcal{S}}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}),\\
&= [d(\mathcal{S},\mathcal{S}) + d(\mathcal{S},\mathcal{N}\setminus\mathcal{S})] - d(\mathcal{S},\mathcal{S})(1+\alpha),\\
&= d(\mathcal{S},\mathcal{N}\setminus\mathcal{S}) - \alpha\ d(\mathcal{S},\mathcal{S}),\\
&\ge 0
\end{aligned}$$

The last step is based on the condition in (3). ■

A direct consequence of Prop. 2 and 3 is that a network is robust to placement of $|H|$ stubborn players if and only if the set $\mathcal{S} = \mathcal{N}\setminus H$ constitute an autonomous set for a given α. According to Prop. 1, an autonomous set $\mathcal{S}$ is r-close-knit, network is robust for $\alpha > 1/r - 2$. On the other hand, if the network is not homogeneous, the network is not robust if $\alpha < 1/Pl(\mathcal{S}) - 2$.

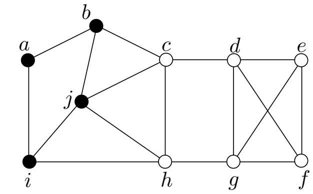


Fig. 2. A Graph Illustrating Prop. 2 and 3. Here $\mathcal{S} = \{c,d,e,f,g,h\}$ and $\mathcal{S}' = \{c,h\}$. The black nodes represent the stubborn nodes, while the rest of the nodes update their actions using LLL

To highlight the distinction between plumpness and close-knittedness, let's examine the network depicted in Fig.2, which was originally presented in [16]. For this network, if we choose $\mathcal{S} = \{c,d,e,f,g,h\}$. Let $H = \{a,b,j,i\}$ is the set of stubborn players. Close-knittedness of set $\mathcal{S}$ is $CK(\mathcal{S}) = \min_{\mathcal{S}'\subset\mathcal{S}} d(\mathcal{S}',\mathcal{S})/d(\mathcal{S}') = 3/8$. This minimum is achieved for $\mathcal{S}' = \{c,h\}$. Plumpness of set $\mathcal{S}$ is $Pl(\mathcal{S}) = d(\mathcal{S},\mathcal{S})/d(\mathcal{S}) = 9/22$. Based on Prop. 2, $\sigma = (\sigma^A_{\mathcal{S}}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})$ is robust to stubborn players if and only if $\alpha > (1/CK(\mathcal{S})) - 2 = 0.67$. From Prop. 3, $\sigma = (\sigma^A_{\mathcal{S}}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})$ is not stochastically stable if $\alpha < (1/Pl(\mathcal{S})) - 2 = 0.44$. Lets calculate the potential for three different action profiles: $\phi(\sigma^A_{\mathcal{S}'}, \sigma^A_{\mathcal{S}\setminus\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})$, $\phi(\sigma^B_{\mathcal{S}'}, \sigma^B_{\mathcal{S}\setminus\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})$, and $\phi(\sigma^B_{\mathcal{S}'}, \sigma^A_{\mathcal{S}\setminus\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})$.

$$\begin{aligned}
\phi(\sigma^A_{\mathcal{S}'}, \sigma^A_{\mathcal{S}\setminus\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}) = d(\mathcal{S}',\mathcal{S}')U_i(A,A) + d(\mathcal{S}',\mathcal{S}\setminus\mathcal{S}')\\
U_i(A,A) + d(\mathcal{S}-\mathcal{S}',\mathcal{S}\setminus\mathcal{S}')U_i(A,A)
\end{aligned}$$

We get $\phi(\sigma^A_{\mathcal{S}'}, \sigma^A_{\mathcal{S}-\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}) = 9 + 9\alpha$. Similarly, $\phi(\sigma^B_{\mathcal{S}'}, \sigma^B_{\mathcal{S}-\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}) = 13$, and $\phi(\sigma^B_{\mathcal{S}'}, \sigma^A_{\mathcal{S}-\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}) = 11 + 6\alpha$. Comparing the potential of all three states, we get the following conditions on α.

- $\phi(\sigma^B_{\mathcal{S}'}, \sigma^B_{\mathcal{S}-\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})$ is SS if $\alpha < 0.33$.
- $\phi(\sigma^B_{\mathcal{S}'}, \sigma^A_{\mathcal{S}-\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})$ is SS if $0.33 < \alpha < 0.67$.
- $\phi(\sigma^A_{\mathcal{S}'}, \sigma^A_{\mathcal{S}-\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})$ is SS if $\alpha > 0.67$.

Relating these to the bounds on α obtained from CK($\mathcal{S}$) and Pl($\mathcal{S}$), we confirm that Prop. 3 offers a sufficient condition to establish the non-robustness of a given network. This guarantees that σ_A can not become a stable action profile when α remains below a certain threshold. Additionally, Prop. 2 presents both a sufficient and necessary condition to establish the robustness of a given network, ensuring that σ_A remains a stable action profile when α exceeds a certain limit. In the intermediate range of α values between these thresholds, we observe an equilibrium action profile where some players choose action A while others select action B (referred to as co-existent equilibria in [18]).

As a starting point, we will utilize Proposition 3 in the subsequent sections of the paper to confirm the non-robustness of various networks, thereby simplifying the computational complexity of our analysis.

IV. Analytical Results

A. Results for single stubborn player

In this section, we show that in regular networks, robustness is independent of the number of edges in the network and only depends on the number of nodes present in the network.

1) k-regular network: A k-regular network is defined such that each node is connected to exactly k neighbors. These networks hold significance due to their inherent structural regularity, simplifying the analysis process.

Proposition 4: Any connected $k-$regular network with N nodes is not robust to the presence of a single stubborn player in the system if $\alpha < 2/(N-2)$.

Proof: In a $k-$regular network where $\mathcal{S} = \mathcal{N} \setminus H$ players update their action using LLL and one node is made stubborn. Pl($\mathcal{S}$) is given by

$$Pl(\mathcal{S}) = \frac{d(\mathcal{S},\mathcal{S})}{d(\mathcal{S})} = \frac{\frac{Nk}{2}-k}{Nk-k} = \frac{N-2}{2(N-1)}$$

Using Prop. 3, we obtain the desired bound on α. ■

2) Multi-regular Network: A multi-regular network consists of variable degree clusters, where the degrees range from 1 to p in ascending order as shown in Fig. 3(a). The number of nodes in each cluster is $\{N_1, N_2, ...N_p\}$, $N = \sum_{i=1}^{p} N_i$ and degrees of clusters are $\{k_1, k_2, ..., k_p\}$ and $\mathcal{S} = \mathcal{N} \setminus H$ is the set of nodes, which update their actions using LLL.

Proposition 5: A multi-regular network is not robust to the placement of a single stubborn player in the highest degree cluster if $\alpha < 2k_p/(N_1k_1+N_2k_2+....+N_pk_p-2k_p)$.

Proof: CK of a multi-regular network is given by

$$Pl(\mathcal{S}) = \frac{d(\mathcal{S},\mathcal{S})}{d(\mathcal{S})} = \frac{\frac{N_1k_1}{2} + \frac{N_2k_2}{2} + \ldots + \frac{N_pk_p}{2} - k}{N_1k_1 + N_2k_2 + \ldots + N_pk_p - k}$$

To expand the upper limit of α, CK must be minimized, given that α is less than $1/r-2$.
By applying the mediant inequality $\frac{a-x}{b-x} \leq \frac{a}{b}$, we can conclude that the smallest Pl($\mathcal{S}$) is achieved when k equals k_p i.e the stubborn node is placed in highest degree cluster. Thus, the Pl($\mathcal{S}$) after placing the stubborn node in the highest degree cluster is

$$\frac{d(\mathcal{S},\mathcal{S})}{d(\mathcal{S})} = \frac{N_1k_1 + N_2k_2 + \ldots + N_pk_p - 2k_p}{2(N_1k_1 + N_2k_2 + \ldots + N_pk_p - k_p)}$$

Using Prop. 3, we attain the desired bound on α. ■

In regular networks, we have observed that the addition of edges doesn't affect network robustness since α is exclusively dependent on the number of nodes. However, this is not the case for non-regular networks. Next, we will discuss the analysis of robustness in non-regular networks.

3) Path network: A path graph is a graph that can be visually represented in such a way that all its vertices and edges are positioned along a single straight line.

Proposition 6: A path network with N nodes is not robust to the presence of a single stubborn player placed at the corner of the network if $\alpha < 1/(N-2)$.

Proof: If we place a stubborn node at the corner of a path network with N total nodes, the resulting Pl($\mathcal{S}$)forS = N $\setminus H$ is

$$\frac{d(\mathcal{S},\mathcal{S})}{d(\mathcal{S})} = \frac{N-2}{2N-3}$$

Using Prop. 3, We get the desired condition on α. ■

4) Grid Network: A grid graph is characterized by a layout resembling a grid, where nodes are positioned at the intersections of rows and columns. When two nodes are adjacent, they are connected by an edge.

Proposition 7: An $m \times n$ grid network is not robust to the addition of a single stubborn player in the network if $\alpha < 4/(2mn-m-n-4)$.

Proof: An $m \times n$ grid network consists of $(m-1)n + (n-1)m$ edges. It has four corner nodes with degree 2, $2(m-2) + 2(n-2)$ boundary nodes with degree 3 and $(m-2)(n-2)$ internal nodes with degree 4. Pl($\mathcal{S}$) after adding a single stubborn node is

$$\frac{d(\mathcal{S},\mathcal{S})}{d(\mathcal{S})} = \frac{2mn-m-n-4}{4mn-2m-2n-4}$$

Applying Prop. 3, resuls in the aforementioned bound. ■

5) Cartesian Product of complete and path graph ($K_n \times P_m$): The Cartesian product of two graphs K_n and P_m is a graph that has a vertex set equal to the Cartesian product of the vertex sets of K_n and P_m, and an edge between two vertices (k_1, p_1) and (k_2, p_2) if and only if either $k_1 = k_2$ and there is an edge between p_1 and p_2 in P_m, or $p_1 = p_2$ and there is an edge between k_1 and k_2 in K_n.

Proposition 8: The Cartesian product of two graphs $(K_n \times P_m)$ is not robust to the addition of a single stubborn node if $\alpha \leq 2(n+1)/(mn^2+mn-4n-2)$.

Proof: The Cartesian product of two graphs $(K_n \times P_m)$ consists of $m[\frac{n(n-1)}{2}] + (m-1)n$ total edges and degree of this network is $2n(n) + n(m-2)(n+1)$. Pl($\mathcal{S}$) for $\mathcal{S} = \mathcal{N} \setminus H$ is

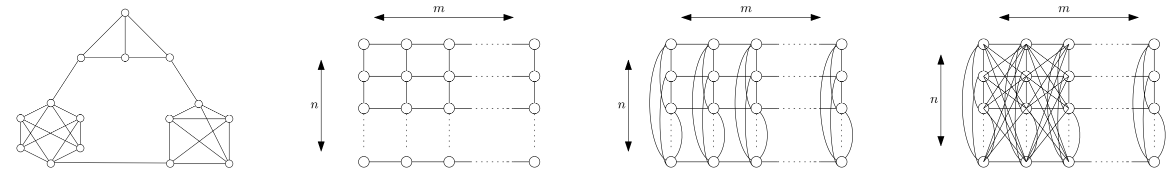
$$\frac{d(\mathcal{S},\mathcal{S})}{d(\mathcal{S})} = \frac{mn^2+mn-4n-2}{2(mn^2+mn-3n-1)}$$

Applying Prop. 3, results in the desired condition. ■

6) Clique Chains: A clique chain $G(K_n, m)$ of N nodes and diameter D is a graph obtained from a path graph of diameter $m-1$, by replacing each node with a clique of size n such that the vertices in distinct cliques are adjacent if and only if the corresponding original vertices in the path graph are adjacent.

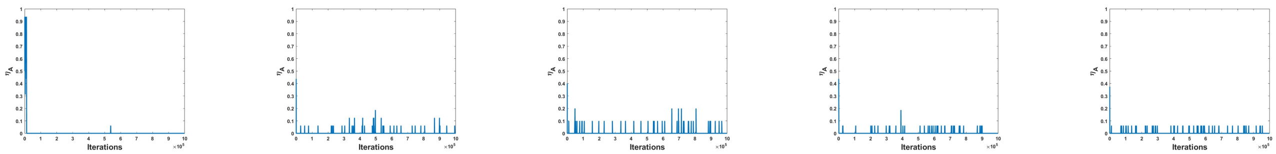
Proposition 9: A clique chain network$G(K_n, m)$ is not robust to the addition of a single stubborn node in the network, if $\alpha \leq (6n-2)/(3mn^2-mn-2n^2-6n+2)$.

Proof: Each clique in the clique chain network $G(K_n, m)$ has $n(n-1)/2$ edges and the edges within any two cliques are $n^2(m-1)$. Set $\mathcal{S} = \mathcal{N} \setminus H$, so $d(\mathcal{S},\mathcal{S})$ in this case becomes $m(n(n-1)/2) + mn^2(m-1) - (3n-1)$, where $3n-1$ are the edges associated with stubborn node. The degree of all the nodes other than boundary nodes is

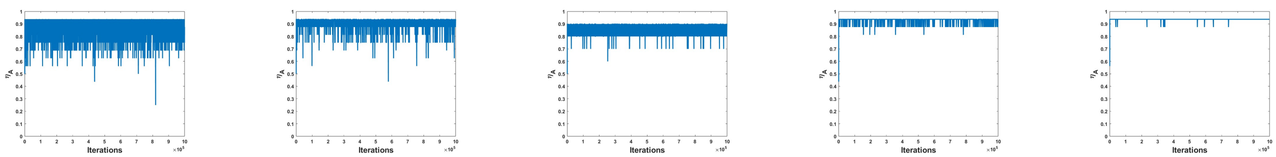


(a) Multi-regular example network (b) $m \times n$ Grid network (c) Cartesian Product network $K_n \times P_m$ (d) Clique chain network $K_{m,n}$

Fig. 3. Example networks



(a) Path graph with 12 nodes for $\alpha - \epsilon$ (b) 4×4 grid network for $\alpha - \epsilon$ (c) Peterson graph(k-regular) for $\alpha - \epsilon$ (d) Cartesian product $K_4 \times P_4$ for $\alpha - \epsilon$ (e) Clique Chain $G(K_4, 4)$ for $\alpha - \epsilon$



(f) Path graph with 12 nodes for $\alpha + \epsilon$ (g) 4×4 grid network for $\alpha + \epsilon$ (h) Peterson graph(k-regular) for $\alpha + \epsilon$ (i) Cartesian product $K_4 \times P_4$ for $\alpha + \epsilon$ (j) Clique Chain $G(K_4, 4)$ for $\alpha + \epsilon$

Fig. 4. Long run population behavior in graphical coordination game across various networks, considering parameters: noise $\tau = 0.1$, iterations $= 1 \times 10^6$, and following the different bounds on α specified in Prop. 4, 6, 7, 8, and 9. Vertical axes represent the fraction of players playing A (η_A)

$n(m-2)(3n-1)$ and the degree of boundary nodes is $2n(2n-1)$. Pl($\mathcal{S}$) for this graph is

$$\frac{d(\mathcal{S},\mathcal{S})}{d(\mathcal{S})} = \frac{3mn^2 - mn - 2n^2 - 6n + 2}{2(3mn^2 - mn - 2n^2 - 3n + 1)}$$

Using Prop. 3, we obtain the desired condition. ■

By examining the limits of α in grid networks, Cartesian product networks, and clique chains, we can see that the bound on α becomes smaller and smaller with the addition of further edges in the network.

To validate the findings outlined in Propositions. 4, 6, 7, 8, and 9, a coordination game was simulated for 1×10^6 iterations with noise parameter $\tau = 0.1$ and the outcomes are depicted in Fig. 4. The vertical axes represent the fractions of players opting for action A. Figs. 4(a) through 4(e) demonstrate that the various networks do not exhibit robustness in the presence of stubborn players when the payoff advantage is $\alpha - \epsilon$. A noteworthy finding when the payoff advantage is $\alpha + \epsilon$ is that, as depicted in Figs. 4(f) through 4(j), the network exhibits robustness to the presence of stubborn players. In this simulation, ϵ is set to 0.05 for all cases, and α is calculated using the threshold specified in Propositions [4-10].

B. Multiple stubborn players

In this section, our objective is to determine the relationship between the minimum number of disjoint stubborn players required to render the network non-robust for all values of $\alpha < 1$. Let $|H|$ be the number of disjoint stubborn agents required to make the network non-robust and let $\mathcal{S} = \mathcal{N} \setminus H$.

1) k-regular network: For a k-regular network with N nodes, the plumpness of this network after the addition of $|H|$ stubborn nodes is

$$\frac{d(\mathcal{S},\mathcal{S})}{d(\mathcal{S})} = \frac{N - 2|H|}{2(N - |H|)},$$

Then, $\alpha < 2|H|/(N - 2|H|)$.

2) Grid network: For an $m \times n$ grid network, Pl($\mathcal{S}$) after addition of $|H|$ disjoint stubborn nodes is

$$\frac{d(\mathcal{S},\mathcal{S})}{d(\mathcal{S})} = \frac{2mn - m - n - 4|H|}{4mn - 2m - 2n - 4|H|},$$

Bound on α is, $\alpha < 4|H|/(2mn - m - n - 4|H|)$.

3) Cartesian Product network: In a Cartesian product network ($K_n \times P_m$), Pl($\mathcal{S}$) after addition of $|H|$ disjoint stubborn nodes is

$$\frac{d(\mathcal{S},\mathcal{S})}{d(\mathcal{S})} = \frac{mn^2 + mn - 2n - 2|H|(n+1)}{2(mn^2 + mn - 2n) - 2|H|(n+1))},$$

Bound on α is $\alpha < (2|H|(n+1))/(mn^2 + mn - 2n - 2|H|(n+1))$.

4) Clique chain: In a clique chain network ($K_{n,m}$), Pl($\mathcal{S}$) after addition of $|H|$ disjoint stubborn nodes is

$$\frac{d(\mathcal{S},\mathcal{S})}{d(\mathcal{S})} = \frac{3mn^2 - mn - 2n^2 - 6n|H| + 2|H|}{2(3mn^2 - 2n^2 - mn - 3n|H| + |H|)},$$

Bound on α is $\alpha < (2|H|(3n-1))/(3mn^2 - mn - 2n^2 - 2|H|(3n-1))$.

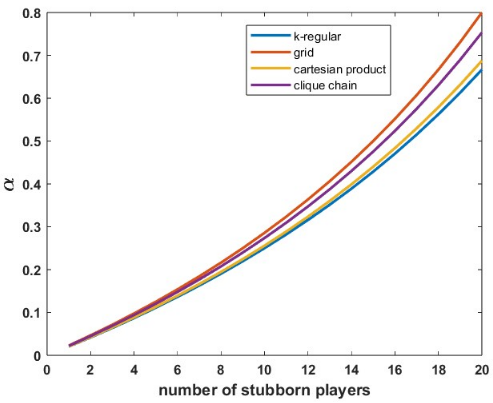


Fig. 5. Comparison of bound on α as a function of the minimum number of disjoint stubborn agents for k-regular, grid, cartesian product, and clique chain network. $m = n = 10$ and $N = 100$

When comparing the bounds on α with respect to the minimum number of disjoint stubborn agents needed to render the network non-robust as shown in Fig. 5, it is evident that the inclusion of edges may or may not influence the allowable α value for maintaining network non-robustness.

V. Conclusions

In this paper, we have presented a graph-theoretic framework to evaluate the robustness of various networks. Existing algorithms for addressing this issue are computationally intractable. Our analysis spans a diverse set of network structures, including sparse and dense networks, such as k-regular networks, multi-regular networks, path networks, grid networks, cartesian product networks, and clique chain networks. For our future research, we aim to bridge the gap between Radius-coradius analysis, which relies on the explicit computation of resistances, and the concept of close-knittedness, which is a purely network-related property.

VI. Appendix

A. Proof of Prop.1

If set $\mathcal{S}$ is $\sigma^*_{\mathcal{S}}$-autonomous, it achieves the maximum potential and $\sigma^*_{\mathcal{S}}$ is stochastically stable. Thus,

$$\begin{aligned}\phi(\sigma^A_{\mathcal{S}}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}) &> \phi(\sigma^B_{\mathcal{S}'}, \sigma^A_{\mathcal{S}\setminus\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}) \\ \phi(\sigma^A_{\mathcal{S}'}, \sigma^A_{\mathcal{S}\setminus\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}) &> \phi(\sigma^A_{\mathcal{S}'}, \sigma^B_{\mathcal{S}\setminus\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}})\end{aligned} \quad (4)$$

By definition of the potential function, $\phi(\sigma^A_{\mathcal{S}'}, \sigma^A_{\mathcal{S}\setminus\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}) = d(\mathcal{S}', \mathcal{S}')(1+\alpha) + d(\mathcal{N}\setminus\mathcal{S}, \mathcal{N}\setminus\mathcal{S}) + d(\mathcal{S}\setminus\mathcal{S}', \mathcal{S}\setminus\mathcal{S}')(1+\alpha) + d(\mathcal{S}', \mathcal{S}\setminus\mathcal{S}')(1+\alpha)$
$\phi(\sigma^A_{\mathcal{S}'}, \sigma^B_{\mathcal{S}\setminus\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}) = d(\mathcal{S}', \mathcal{S}')(1+\alpha) + d(\mathcal{S}\setminus\mathcal{S}', \mathcal{S}\setminus\mathcal{S}') + d(\mathcal{S}\setminus\mathcal{S}', \mathcal{N}\setminus\mathcal{S}) + d(\mathcal{N}\setminus\mathcal{S}, \mathcal{N}\setminus\mathcal{S})$
After substituting into Eq. 4 and simplifying,

$$\begin{aligned}d(\mathcal{S}', \mathcal{S}')(\alpha) + d(\mathcal{S}', \mathcal{S}\setminus\mathcal{S}')(1+\alpha) - d(\mathcal{S}', \mathcal{N}\setminus\mathcal{S}) > 0 \\ \implies d(\mathcal{S}', \mathcal{S})(2+\alpha) - d(\mathcal{S}') > 0\end{aligned}$$

which results in the desired condition.
To prove the other side, we need to show that If $d(\mathcal{S}', \mathcal{S})/d(\mathcal{S}') > 1/(2+\alpha)$, $\mathcal{S}$ is $\sigma^*_{\mathcal{S}}$-autonomous.

$$\begin{aligned}d(\mathcal{S}', \mathcal{S})(2+\alpha) - d(\mathcal{S}') > 0 \\ d(\mathcal{S}', \mathcal{S})(1+\alpha) - [d(\mathcal{S}') - d(\mathcal{S}', \mathcal{S})] > 0 \\ d(\mathcal{S}', \mathcal{S})(1+\alpha) - [d(\mathcal{S}') - d(\mathcal{S}', \mathcal{S}\setminus\mathcal{S}') - d(\mathcal{S}', \mathcal{S}') > 0 \\ d(\mathcal{S}', \mathcal{S}')(+\alpha) + d(\mathcal{S}', \mathcal{S}\setminus\mathcal{S}')(1+\alpha) - d(\mathcal{S}', \mathcal{N}\setminus\mathcal{S}) > 0\end{aligned}$$

After adding some redundant terms,

$$\begin{aligned}&d(\mathcal{S}', \mathcal{S}')(1+\alpha) + d(\mathcal{S}', \mathcal{S}\setminus\mathcal{S}')(1+\alpha) + d(\mathcal{N}\setminus\mathcal{S}, \mathcal{N}\setminus\mathcal{S}) \\ &+ d(\mathcal{S}\setminus\mathcal{S}', \mathcal{S}\setminus\mathcal{S}')(1+\alpha) > d(\mathcal{S}', \mathcal{S}')(1+\alpha) + \\ &d(\mathcal{S}\setminus\mathcal{S}', \mathcal{S}\setminus\mathcal{S}') + d(\mathcal{S}\setminus\mathcal{S}', \mathcal{N}\setminus\mathcal{S}) + d(\mathcal{N}\setminus\mathcal{S}, \mathcal{N}\setminus\mathcal{S}).\end{aligned}$$

which gives the condition,

$$(\sigma^A_{\mathcal{S}'}, \sigma^A_{\mathcal{S}\setminus\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}) > \phi(\sigma^A_{\mathcal{S}'}, \sigma^B_{\mathcal{S}\setminus\mathcal{S}'}, \sigma^B_{\mathcal{N}\setminus\mathcal{S}}).$$

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