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State observer design for bilinear Persidskii systems

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***Abstract*— This paper proposes a state estimator for a class of nonlinear systems that includes the Persidskii systems with bilinear cross-terms. The estimation error analysis is based on the input-to-output stability theory and formulated using linear matrix inequalities. Simulations are provided for a model of consumer-resource interaction.**

I. Introduction

The state estimation problem appears in many engineering applications, including complex physical and biological systems, where the models are characterized by significant nonlinearity and uncertainty [7], [10], [2]. This problem can be formulated in many ways, according to the application considered. In the case of systems containing an unknown input (exogenous perturbations, measurement noises, unmodeled dynamics), it is often helpful to formulate it in terms of input-to-output stability (IOS) [16], [4], where we regard the estimated error as the output (as a converging variable of interest). As with the regular stability notions, IOS analysis can be performed by designing special real-valued functions, called IOS-Lyapunov functions, following crucial results introduced in [15]. Although those results guarantee the existence of an IOS-Lyapunov function for a large class of IOS systems, no universal method for constructing such a function exists, which is why we have to rely on particular canonical forms of the system models.

The model we consider in this paper is a generalization of the important class of Persidskii systems, first introduced in [1], [13] for stability analysis, also related with Lur'e systems studied in the absolute stability theory [17], [9]. Thus, the advantage of Persidskii systems consists in the existence of a well-investigated form of candidate Lyapunov functions, which was used in many cases [6], [12], [11], often resulting in stability conditions that can be formulated in the form of linear matrix inequalities. Moreover, this class of models has been found valuable in representing many physical and biological phenomena, and therefore, it has been studied from many viewpoints, including that of IOS-stability [12]. In this paper, we consider a more significant generalization of this class of systems, motivated by biological and physical examples [8], [5], [3], where some new classes of nonlinearities are taken into account, in particular allowing for the addition of bilinear cross-products.

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The goal of this work is to consider a more complex Persidskii-like dynamics as in [12], which include bilinear terms (products of the state components and the nonlinearities or inputs), and design a state observer using the IOS theory. The existing results in the literature study only particular classes of bilinear systems, see [14] and references therein. We will assume that nonlinear state measurements are possible, and in this preliminary study, we will formulate local stability conditions (which are, however, not based on linearization).

Notation

- The set of reals is denoted by $\mathbb{R}$ and $\mathbb{R}_+ := \{s \in \mathbb{R} \mid s \geq 0\}$.
- The usual Euclidian norm in $\mathbb{R}^n$ is denoted by $\|\cdot\|$.
- For a Lebesgue measurable function $u : \mathbb{R}_+ \to \mathbb{R}^\ell$, define the norm $\|u\|_{[t_1,t_2)} = \text{ess sup}_{t\in[t_1,t_2)}\|u(t)\|$ for $[t_1, t_2) \subset \mathbb{R}_+$. We denote by $\mathcal{L}_\infty^\ell$ the set of functions u such that $\|u\|_\infty := \|u\|_{[0,+\infty)} < +\infty$.
- A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$. If in addition $\lim_{t\to+\infty} \sigma(t) = +\infty$, we say that σ belongs to the class $\mathcal{K}_\infty$. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r, \cdot)$ is a decreasing function going to zero for any fixed $r > 0$.

II. Observers and Input to Output Stability

In what follows, the dynamics considered is

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^\ell$ is the input, and $y(t) \in \mathbb{R}^p$ is the output. The function $f : \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^n$ is assumed to be locally Lipschitz with $f(0, 0) = 0$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is a continuously differentiable function. For any $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{L}_\infty^\ell$, we denote the respective solution of (1) by $x(t, x_0, u)$ that we assume to be defined for all $t \geq 0$. Then $y(t, x_0, u) = h(x(t, x_0, u))$.

Definition II.1. *We say that the system* (1) *is* Input-to-Output Stable (IOS) *if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that*

$$\|y(t, x_0, u)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_\infty)$$

for every $t \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$, and $u \in \mathcal{L}_\infty^\ell$.

To check this stability property, the Lyapunov function method can be used.

Definition II.2. *A continuously differentiable positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an* IOS-Lyapunov function *for (1) if there exist functions $\alpha_1, \alpha_2, \chi \in \mathcal{K}_\infty$, and $\alpha_3 \in \mathcal{KL}$ such that*

$$\alpha_1(\|h(x)\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{2}$$

and

$$V(x) \geq \chi(\|u\|) \implies \nabla V(x) f(x,u) \leq -\alpha_3\left(V(x), \|x\|\right) \tag{3}$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^\ell$.

Note that the right-hand side in (3) is not a definite function of $V(x)$ if the state norm is not bounded. To avoid this issue the following property will be used.

Definition II.3. *The system (1) is said to be* Uniformly Bounded-Input-Bounded-State Stable (UBIBS) *if there exists $\sigma \in \mathcal{K}$ such that*

$$\|x(t, x_0, d)\| \leq \max\left\{\sigma\left(\|x_0\|\right), \sigma\left(\|u\|_\infty\right)\right\},$$

for every $x_0 \in \mathbb{R}^n$ and every $u \in \mathcal{L}_\infty^\ell$.

The key result connecting the IOS property and the respective Lyapunov function is given below.

Theorem II.4. *[15] A UBIBS system (1) is IOS if and only if it admits an IOS-Lyapunov function.*

Both notions, the IOS property and the IOS-Lyapunov function, are defined globally above. The local counterparts can also be formulated by restricting the domain of validity.

Note that the BIBS property can be replaced with the forward completeness requirement (the existence of solutions for all $t \in \mathbb{R}_+$) while establishing IOS if (3) is strengthened as follows:

$$V(x) \geq \chi(\|u\|) \implies \nabla V(x) f(x,u) \leq -\alpha_4(V(x)) \tag{4}$$

for some $\alpha_4 \in \mathcal{K}$.

In this paper, we will use the IOS theory to define and design a state observer for (1) as it is formalized in the following definition:

Definition II.5. *We say that the system*

$$\dot{\hat{x}}(t) = \widetilde{f}(\hat{x}(t), u(t), y(t)) \tag{5}$$

is an observer *for the system (1) if the combined system $\begin{pmatrix}\dot{x} & \dot{\hat{x}}\end{pmatrix} = \begin{pmatrix} f(x,u) & \widetilde{f}(\hat{x},u,y)\end{pmatrix}$ is IOS with respect to the output $e(x,\hat{x}) = x - \hat{x}$ (the estimation error).*

III. Bilinear Persidskii Systems

Let us consider the following class of systems: for all $t \in \mathbb{R}_+$,

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + \sum_{j=1}^{M} A_j f^j\left(H_j x(t)\right) + G w(t) \\ &+ \sum_{i=1}^{n} x_i(t) \sum_{j=1}^{M} B_j^i f^j(H_j x(t)) + \sum_{i=1}^{n} x_i(t) D_i w(t),\end{aligned} \tag{6}$$

$$y(t) = \begin{pmatrix} C_0 x(t) \\ C_1 f^1(H_1 x(t)) \\ \vdots \\ C_M f^M\left(H_M x(t)\right)\end{pmatrix},$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^\ell$ is the unknown input, $w \in \mathcal{L}_\infty^\ell$, $y(t) \in \mathbb{R}^z$ is the measured output, $A_0 \in \mathbb{R}^{n\times n}$, $A_j \in \mathbb{R}^{n\times k_j}$, $H_j \in \mathbb{R}^{k_j \times n}$, $B_j^i \in \mathbb{R}^{n \times k_j}$, $C_j \in \mathbb{R}^{z_j \times k_j}$ and G, $D_i \in \mathbb{R}^{n\times\ell}$ for $i = 1, ..., n$ and $j = 1, ..., M$, $C_0 \in \mathbb{R}^{z_0 \times n}$, $z = \sum_{s=0}^{M} z_s$, $f^j : \mathbb{R}^{k_j} \to \mathbb{R}^{k_j}$ for $j = 1, ..., M$ with the hypothesis that $f^j(s) = \begin{pmatrix} f_1^j(s_1) & \dots & f_{k_j}^j(s_{k_j})\end{pmatrix}^\top$ and

$$\tau f_i^j(\tau) > 0, \text{ for every } \tau \neq 0. \tag{7}$$

For brevity of notation, we will use the convention $k_0 = n$, $H_0 = I_n$ and $f^0(x) = x$.

The objective of this paper is to design a robust observer for the system (6) in the sense of Definition II.5.

The system (6) includes the generalized Persidskii systems studied in [6], [12], [11] (the first line in (6)), but also additional bilinear terms multiplied by the state components (the second line). The motivation for considering this class is that it encapsulates some models of interest, which have been studied both in the context of observer design and of stabilization [5], [3].

Example III.1. *The following model describes the bacterial growth of two distinct species inside a chemostat with a single limiting substrate [5]:*

$$\begin{aligned}\dot{x}_i &= (\mu_i(S) - D)x_i, i = 1, 2, \\ \dot{S} &= (S_{in} - S)D - \sum_{i=1}^{2} \rho_i \mu_i(S) x_i,\end{aligned} \tag{8}$$

where $x_i \geq 0$ is the concentration of the ith species, $S \geq 0$ is the concentration of the substrate, $S_{in} \geq 0$ is the nutrient inflow concentration, ρ_i are positive constants and the functions μ_i are given by

$$\mu_i(S) = \frac{a_i S}{b_i + S},$$

where a_i, b_i are positive constants.

Renaming S as x_3, denoting the state vector $x = \begin{pmatrix} x_1 & x_2 & x_3\end{pmatrix}^\top \in \mathbb{R}_+^3$ and reparametrizing the input as $w = \begin{pmatrix} D & u\end{pmatrix}^\top$, where $u = S_{in} D$, we can see that it fits the model for $n = 3$ and $M = 2$. Indeed, following the notation of (6), this system is obtained by replacing

$A_0 = A_1 = A_2 = 0_{3\times3}$; $H_1 = H_2 = I_3$,

$$G = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, D_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{pmatrix},$$

$$B_1^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -\rho_1 \end{pmatrix}, B_2^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\rho_2 \end{pmatrix},$$

$$B_1^2 = B_2^1 = B_1^3 = B_2^3 = 0_{3\times3}$$

and $f^j(x) = \begin{pmatrix} \mu_j(x_1) & \mu_j(x_2) & \mu_j(x_3) \end{pmatrix}^\top$ *for* $j = 1, 2$.

Example III.2. *The following is a classic model of a chemical exothermic reactor [3]:*

$$\begin{aligned} \dot{X}_{in} &= 0 \\ \dot{X} &= D(X_{in} - X) - k\exp\left(\frac{-E}{RT}\right)X \\ \dot{T} &= D(T_{in} - T) + c\exp\left(\frac{-E}{RT}\right)X + v, \end{aligned} \tag{9}$$

where $\begin{pmatrix} X_{in} & X & T \end{pmatrix}^\top \in \mathbb{R}_+^3$ *is the state (representing, respectively, the inlet composition, the reactor composition and the temperature inside the reactor),* $\begin{pmatrix} E & R & k & c \end{pmatrix}^\top \in \mathbb{R}_+^4$ *is the vector of constant parameters, and* D, v *are nonnegative control inputs.*

Renaming $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^\top = \begin{pmatrix} X_{in} & X & T \end{pmatrix}^\top$ *and* $w = \begin{pmatrix} D & v & u \end{pmatrix}^\top$ *where* $u = T_{in}D$, *we have*

$$\begin{aligned} \dot{x} &= x_2 \begin{pmatrix} 0 & -k\exp\left(-\frac{E}{Rx_3}\right) & c\exp\left(-\frac{E}{Rx_3}\right) \end{pmatrix}^\top \\ &+ \begin{pmatrix} 0 & 0 & u+v \end{pmatrix}^\top + x_1 \begin{pmatrix} 0 & D & 0 \end{pmatrix}^\top \\ &+ x_2 \begin{pmatrix} 0 & -D & 0 \end{pmatrix}^\top + x_3 \begin{pmatrix} 0 & 0 & -D \end{pmatrix}^\top. \end{aligned}$$

As with the previous example, this system is obtained from (6) for $n = 3$ *and* $M = 1$, *by replacing* $A_0 = A_1 = 0_{3\times3}$ *and* $H_1 = I_3$,

$$B_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -k \\ 0 & 0 & c \end{pmatrix}, G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$B_1^1 = B_1^3 = 0_{3\times3}$$

and $f^1(x) = \begin{pmatrix} \varphi(x_1) & \varphi(x_2) & \varphi(x_3) \end{pmatrix}^\top$, *where* $\varphi(x_i) = \exp\left(-\frac{E}{Rx_i}\right)$.

Example III.3. *The following dynamical system models a particular kind of consumer-resource interaction [8]:*

$$\begin{aligned} \dot{M}_1 &= M_1\left(r_1 + c_1\frac{M_2}{h_1 + M_2} + c_2\frac{M_2}{h_2 + M_1} - c_3M_1\right) + w_1 \\ \dot{M}_2 &= M_2\left(r_2 + c_4\frac{M_1}{h_2 + M_1} + c_5\frac{M_1}{h_1 + M_2} - c_6M_2\right) + w_2, \end{aligned} \tag{10}$$

where M_1, $M_2 \in \mathbb{R}$ *are the population compartments,* r_1, r_2, h_1, h_2, $c_i \in \mathbb{R}$ $(i = 1, \ldots, 6)$ *and* w_1, w_2 *are bounded inputs.*

This system is again in the form (6) for $n = 2$ *and* $M = 2$, *with* $x = \begin{pmatrix} M_1 & M_2 \end{pmatrix}^\top$, *and*

$$A_0 = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix},$$

$$B_1^1 = \begin{pmatrix} -c_3 & 0 \\ 0 & 0 \end{pmatrix}, B_1^2 = \begin{pmatrix} 0 & 0 \\ 0 & -c_6 \end{pmatrix}$$

$$B_2^1 = \begin{pmatrix} 0 & c_1 \\ 0 & c_5 \end{pmatrix}, B_2^2 = \begin{pmatrix} c_2 & 0 \\ c_4 & 0 \end{pmatrix}$$

$$f^1(M) = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, f^2(M) = \begin{pmatrix} \frac{M_1}{h_2 + M_1} \\ \frac{M_2}{h_1 + M_2} \end{pmatrix},$$

and $H_1 = H_2 = G = I_2$, *other matrices are zero.*

Beyond their practical interest, those systems have some intrinsic symmetry, which could be further exploited as concrete examples of the theory of equivariant systems [3], [5]. By its analysis, we can understand the role that this kind of symmetry plays in the design of simpler and better-performing observers.

IV. Observer design

The proposed observer for the system (6) is

$$\begin{aligned} \dot{\hat{x}}(t) &= A_0\hat{x}(t) + \sum_{j=1}^{M} A_j f^j(H_j\hat{x}(t)) \\ &+ \sum_{i=1}^{n} \hat{x}_i(t) \sum_{j=1}^{M} B_j^i f^j(H_j\hat{x}(t)) + L_0(y_0(t) - C_0\hat{x}(t)) \\ &+ \sum_{j=1}^{M} L_j(y_j(t) - C_j f^j(H_j\hat{x}(t))) \\ &+ \sum_{i=1}^{n} \hat{x}_i \sum_{j=1}^{M} R_j^i(y_j(t) - C_j f^j(H_j\hat{x}(t))), \end{aligned} \tag{11}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state and the estimate of $x(t)$, and L_j, $R_j^i \in \mathbb{R}^{n\times z_j}$ are matrices to be designed.

Denoting $e = x - \hat{x}$ and $\delta f^j(x,\hat{x}) = f^j(H_j x) - f^j(H_j\hat{x})$, we have that the error dynamics is given by

$$\dot{e} = (A_0 - L_0C_0)e + \sum_{j=1}^{M}(A_j - L_jC_j)\delta f^j$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{M} B_j^i(x_i f^j(H_jx) - \hat{x}_i f^j(H_j\hat{x}))$$
$$+Gw + \sum_{i=1}^{n} D_i e_i w - \sum_{i=1}^{n}\hat{x}_i\sum_{j=1}^{M} R_jC_j\delta f^j.$$

The joint state variable $X = \begin{pmatrix} x & \hat{x}\end{pmatrix}^\top$ for the system (6) and (11) follows the dynamics equation

$$\dot{X} = \tilde{A}_0X + \sum_{j=1}^{M}\tilde{A}_jF^j(\tilde{H}_jX) + \sum_{i=1}^{2n}X_i\sum_{j=1}^{M}\tilde{B}_i^jF_j(\tilde{H}_jX)$$
$$+\tilde{G}w + \sum_{i=1}^{2n}X_i\tilde{D}_iw \tag{12}$$

where $X_j = x_j$ and $X_{n+j} = \hat{x}_j$, for $j = 1, ..., n$ and

$$\tilde{A}_j = \begin{pmatrix} A_j & 0_{n\times k_j} \\ L_jC_j & A_j - L_jC_j\end{pmatrix}, \tilde{H}_j = \begin{pmatrix} H_j & 0_{k_j\times n} \\ 0_{k_j\times n} & H_j\end{pmatrix},$$

$$\tilde{B}_j^i = \begin{pmatrix} B_j^i & 0_{n\times k_j} \\ 0_{n\times k_j} & 0_{n\times k_j}\end{pmatrix}, \tilde{B}_j^{n+i} = \begin{pmatrix} 0_{n\times k_j} & 0_{n\times k_j} \\ R_j^iC_j & B_j^i - R_j^iC_j\end{pmatrix},$$

$$F^j(\tilde{H}_jX) = \begin{pmatrix} f^j(H_jx) \\ f^j(H_j\hat{x})\end{pmatrix}, \tilde{G} = \begin{pmatrix} G \\ 0\end{pmatrix}$$

$$\tilde{D}_i = \begin{pmatrix} D_i \\ 0\end{pmatrix}, \tilde{D}_{n+i} = 0$$

for $j = 1, \ldots, M$ and $i = 1, \ldots, n$. We remark that the system (11) is still under the considered class of bilinear Persidskii systems and that the error can be written as $e(X) = \Gamma X$, where $\Gamma = \begin{pmatrix} I_n & -I_n\end{pmatrix}$.

Consider the following Lyapunov function candidate:

$$V(X) = X^\top P_\Gamma X + 2\sum_{j=1}^{M}\sum_{i=1}^{2k_j}\Lambda_i^j\int_0^{\tilde{H}_j^iX}F_i^j(\tau)\,d\tau, \tag{13}$$

where $\tilde{H}_j^i$ is the ith row of the matrix $\tilde{H}_j$, $P_\Gamma = \Gamma^\top P_1\Gamma + P_2$, for matrices $P_1 \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{2n\times 2n}$ and $\Lambda^j = \text{diag}(\Lambda_1^j, ..., \Lambda_{2k_j}^j)$ to be defined.

The time derivative of V along the trajectories of the system (12) can be computed as follows:

$$\dot{V}(X) = \dot{X}^TP_\Gamma X + X^TP_\Gamma\dot{X} + 2\sum_{j=1}^{M}\dot{X}^T\tilde{H}_j^T\Lambda^jF^j(\tilde{H}_jX)$$
$$= X^T(\tilde{A}_0^TP_\Gamma + P_\Gamma\tilde{A}_0)X + \left(\sum_{j=1}^{M}F^j(\tilde{H}_jX)^T\tilde{A}_j{}^T\right.$$
$$\left.+\sum_{i=1}^{2n}X_i\sum_{j=1}^{M}F^j(\tilde{H}_jX)^T(\tilde{B}_i^j)^T\right)P_\Gamma X$$
$$+X^TP_\Gamma\left(\sum_{j=1}^{M}\tilde{A}_jF^j(\tilde{H}_jX) + \sum_{i=1}^{2n}X_i\sum_{j=1}^{M}\tilde{B}_i^jF^j(\tilde{H}_jX)\right)$$
$$+2\sum_{j=1}^{M}[X^T\tilde{A}_0{}^T\tilde{H}_j^T\Lambda^jF^j(\tilde{H}_jX)$$
$$+\left(\sum_{s=1}^{M}F^s(\tilde{H}_sX)^T\tilde{A}_s{}^T\right)\tilde{H}_j^T\Lambda^jF^j(\tilde{H}_jX)$$
$$+\left(\sum_{i=1}^{2n}X_i\sum_{s=1}^{M}F^s(\tilde{H}_sX)^T(\tilde{B}_i^s)^T\right)\tilde{H}_j^T\Lambda^jF^j(\tilde{H}_jX)$$
$$+(\tilde{G}w + \sum_{i=1}^{2n}X_i\tilde{D}_iw)^T\tilde{H}_j^T\Lambda^jF^j(\tilde{H}_jX)]$$
$$+2X^TP_\Gamma(\tilde{G}w + \sum_{i=1}^{2n}X_i\tilde{D}_iw).$$

Note that regrouping the terms, we can write $\dot{V}$ as a quadratic form acting on the vector $Y = \begin{pmatrix} X^\top & F^1(\tilde{H}_jX)^\top & \ldots & F^M(\tilde{H}_jX)^\top & w^\top\end{pmatrix}^\top$:

$$\dot{V} = Y^TQY + \sum_{i=1}^{2n}X_iY^TS^{(i)}Y = Y^T\left(Q + \sum_{i=1}^{2n}X_iS^{(i)}\right)Y,$$

where the symmetric matrix $Q = Q^T \in \mathbb{R}^{2n\times 2n}$ is defined by the blocks

$$Q_{1,1} = \tilde{A}_0^TP_\Gamma + P_\Gamma\tilde{A}_0;$$
$$Q_{1,j+1} = P_\Gamma\tilde{A}_j + \tilde{A}_0^T\tilde{H}_j^T\Lambda^j;$$
$$Q_{1,M+2} = P_\Gamma G;$$
$$Q_{k+1,j+1} = \tilde{A}_k^T\tilde{H}_j^T\Lambda^j + \Lambda^k\tilde{H}_k\tilde{A}_j;$$
$$Q_{M+2,j+1} = G^T\tilde{H}_j^T\Lambda^j;$$
$$Q_{M+2,M+2} = 0;$$

for $j, k \in \{1, ..., M\}$ and the symmetric matrices $S^{(1)}, ..., S^{(2n)} \in \mathbb{R}^{(M+2)\times(M+2)}$ are defined by the blocks

$$S_{1,1}^{(i)} = 0;\ S_{1,j+1}^{(i)} = P_\Gamma\tilde{B}_i^j;$$
$$S_{k+1,j+1}^{(i)} = (\tilde{B}_i^k)^T\tilde{H}_j^T\Lambda^j;$$
$$S_{1,M+2}^{(i)} = P_\Gamma D_i;\ S_{M+2,j+1}^{(i)} = D_i^T\tilde{H}_j^T\Lambda^j;$$
$$S_{M+2,M+2}^{(i)} = 0,$$

for $i \in \{1, ..., 2n\}$ and $j, k \in \{1, ..., M\}$. This decomposition is crucial in determining the parameters of the candidate Lyapunov function, as shown in the following result.

Theorem IV.1. *Suppose that the system (6) is UBIBS. Suppose also that there exist matrices $\Xi^j \in \mathbb{D}_+^{2k_j}$ $(j = 0, ..., M)$, $\Psi \in \mathbb{D}_+^n$, $\Upsilon_{k,j} \in \mathbb{D}_+^{2n}$ $(k = 1, ..., M-1, j = k+1, ..., M)$,*

$0 < \Phi = \Phi^T \in \mathbb{R}^{2n\times 2n}$, $\Theta \in \mathbb{D}_+^{2n}$ *and* $\xi > 0$, $\rho > 0$ *satisfying the inequalities*

$$P_1 > 0 \text{ or } P_2^{11} - 2P_2^{12} + P_2^{22} - \rho P_1 > 0 \quad (14)$$
$$\text{or } \sum_{j=1}^{M} \tilde{\Lambda}^j + \rho P_1 > 0$$

where P_2^{11}, P_2^{12}, *and* $P_2^{22} \in \mathbb{R}^{n\times n}$ *are such that* $P_2 = \begin{pmatrix} P_2^{11} & P_2^{12} \\ P_2^{12} & P_2^{22} \end{pmatrix}$ *and* $\tilde{\Lambda}^j = \tilde{H}_j^T \operatorname{diag}(\tilde{\Lambda}_1^j, ..., \tilde{\Lambda}_{k_j}^j)\tilde{H}_j + \tilde{H}_j^T \operatorname{diag}(\tilde{\Lambda}_{k_j+1}^j, ..., \tilde{\Lambda}_{2k_j}^j)\tilde{H}_j$,

$$P_1 \le \xi\Psi, \quad (15)$$

$$P_2 + \sum_{j=1}^{M} \tilde{H}_j^T \Lambda^j \tilde{H}_j \le \xi(\sum_{k=0}^{M} \tilde{H}_k^T \Xi^k \tilde{H}_k \quad (16)$$
$$+2\sum_{r=1}^{M} \tilde{H}_r^T \Upsilon_{0,r} \tilde{H}_r + 2\sum_{s=1}^{M-1}\sum_{r=s+1}^{M} \tilde{H}_s^T \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^T \tilde{H}_r)$$

and such that

$$Q + \tilde{Q} < 0, \quad (17)$$

where $\tilde{Q}$ *is the symmetric matrix defined by the following blocks:*

$$\tilde{Q}_{1,1} = \Gamma^T \Psi \Gamma + \Xi^0;$$
$$\tilde{Q}_{j,j} = \Xi^j, j = 1, ..., M,$$
$$\tilde{Q}_{k+1,j+1} = \tilde{H}_k \Upsilon_{k,j} \tilde{H}_j^T, k = 1, ..., M-1, j = k+1, ..., M-1,$$
$$\tilde{Q}_{1,j+1} = \tilde{H}_j^T \Upsilon_{0,j}, j = 1, ..., M,$$
$$\tilde{Q}_{M+2,j} = 0, j = 1, ..., M+1,$$
$$\tilde{Q}_{M+2,M+2} = -\Phi.$$

Then there exists $\delta > 0$ *such that the system (12) is IOS with respect to the output* $e = x - \hat{x}$*, for* $\|X(0)\| \le \delta$ *and* $\|w\|_\infty < \delta$.

Proof: Consider the candidate Lyapunov function (13). The inequality (14) assures that if $V(X) = 0$ then $e(X) = \Gamma X = 0$ [12], which fulfills condition (2) with $h(X) = \Gamma X$. The upper bound α_2 in (2) exists by the continuity of V, the passivity of the nonlinearities and due to all matrices are nonnegative definite.

Once there exists $\tilde{Q} \in \mathbb{R}^{M+2\times M+2}$ verifying the inequality $Q + \tilde{Q} < 0$, we can find $\varepsilon > 0$ such that

$$Q + \tilde{Q} + \sum_{i=1}^{2n} X_i S^{(i)} \le 0 \quad (18)$$

provided that $|X_i| < \varepsilon$ for all $i = 1, \ldots, 2n$. Therefore, we can write

$$\dot{V} = Y^T \left(Q + \tilde{Q} + \sum_{i=1}^{2n} X_i S^{(i)} \right) Y - Y^T \tilde{Q} Y$$
$$\le -Y^T \tilde{Q} Y.$$

The inequalities (15),(16) make sure that

$$Y^T \tilde{Q} Y \ge \alpha(V(X)) - w^T \Phi w,$$

for some function $\alpha \in \mathcal{K}_\infty$. The computations to prove this fact are rather direct and are analogous to the ones performed in [12]. Hence, it follows that

$$\dot{V} \le -\alpha(V(X)) + w^T \Phi w,$$

which leads to

$$\dot{V} \le -\frac{1}{2}\alpha(V(X)), \text{ for } V(X) \ge 2\alpha^{-1}(w^T \Phi w),$$

showing that V is an IOS-Lyapunov function for the system. The computations to prove this fact are rather direct and are analogous to the ones performed in (12). Hence, by Theorem II.4 this system is IOS, and there exist related functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ given in Definition II.1.

Let us demonstrate the validity of (18), i.e., that the state X can be made arbitrarily bounded in the conditions of the theorem. Since (6) is UBIBS, there is $\sigma \in \mathcal{K}$ such that for any $\varepsilon' > 0$ we can chose $\delta > 0$ providing

$$\|x(t)\| \le \max\{\sigma(\delta), \sigma(\delta)\} \le \varepsilon', \ \forall t \ge 0$$

for any $\|x(0)\| \le \delta$ and $\|w\|_\infty \le \delta$. Further, due to the established IOS property of the system (12) with respect to $e(X) = x - \hat{x}$, this constant δ can be chosen sufficiently small guaranteeing

$$\|e(t)\| \le \beta(\alpha_2(2\delta), 0) + \gamma(\delta) < \varepsilon', \ \forall t \ge 0$$

for all $\|\hat{x}(0)\| \le \delta$, $\|x(0)\| \le \delta$ and $\|w\|_\infty \le \delta$. Combining these inequalities we obtain the boundedness of the observer state:

$$\|\hat{x}(t)\| \le \|x(t)\| + \|e(t)\| < 2\varepsilon',$$

and subsequently

$$\|X(t)\| < 3\varepsilon',$$

for all $t \in \mathbb{R}_+$.

Therefore if we chose $\varepsilon' > 0$ small enough, for the corresponding δ, we have $|X_i(t)| < \varepsilon$ for all $i = 1, ..., 2n$, and all $t \in \mathbb{R}_+$, provided that $\|X(0)\| < \delta$ and $\|w\|_\infty < \delta$, so the inequality (18) is satisfied.

The conditions given in the formulation of Theorem IV.1 are sufficient for existence of a Lyapunov function that establishes convergence of the estimation error to zero in the disturbance-free case.

Our result is local, and global estimation results were obtained in [12] without the bilinear cross-terms. More constructive and non-local conditions can be derived assuming that the state in the system (6) and the observer (11) is nonnegative (the latter property has to be ensured by the design of the observer), which is left for further investigations.

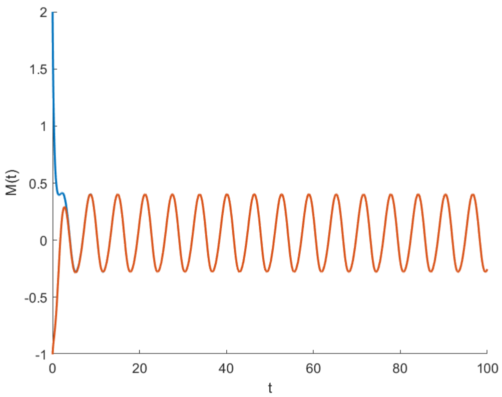

Fig. 1. State trajectory $M(t)$

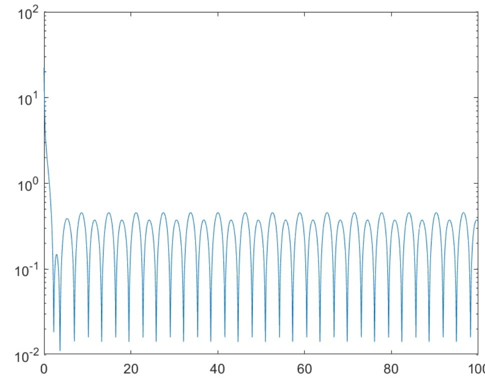

Fig. 2. Norm of the estimation error

V. Simulation Example

If we consider Example III.3 with parameters $r_1 = r_2 = -1$ and

$$C_0 = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

then the inequalities (16) and (17) become

$$P_2 + \Lambda^1 + \Lambda^2 \leq \xi(\Xi^0 + 2(\Upsilon_{0,1} + \Upsilon_{0,2}) + \Upsilon_{1,2}),$$

and

$$\begin{pmatrix} \tilde{A}_0^{\ T} P_\Gamma + P_\Gamma \tilde{A}_0 + \Xi^0 & \Upsilon_{0,1} \\ 2P_\Gamma^T + \Upsilon_{0,1} & -\Phi \end{pmatrix} < 0,$$

which, together with inequalities (14) and (15), constitute a feasible system of inequalities, if we chose the gain matrix as

$$L_0 = \begin{pmatrix} 20.43 & -5.01 \end{pmatrix}^T,$$

and $L_j = R_j^i = 0_{2\times 2}$, for $i = 1, 2$ and $j = 1, 2$. Hence Theorem IV.1 can be applied, and the observer can be chosen in the form (11).

For simulations, all system parameters are set to 1. Figures 1 and 2 show the behaviour of a trajectory of the system and the evolution of the error for $w_i(t) = 0.5\sin(t)$ for $i = 1, 2$ and the gain matrix L_0 chosen as above, illustrating that the error remains small for relatively small inputs and small initial conditions.

VI. Conclusion

In this paper, we have introduced a new class of systems that generalizes the Persidskii dynamics and allows the bilinear cross-product terms to be considered. Such an extension of the model is essential in many applications based on the mass-balance principle. For this class of systems, we have proposed a state estimator using the IOS notion, whose efficiency is illustrated through the example of a consumer-resource interaction model. The applicability conditions for the observer are formulated in terms of linear matrix inequalities, which is an advancement taking into account the complexity of the system.

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