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Approximation of Limit Cycles by Using Planar Switching Affine Systems with Guarantees for Uniqueness and Stability

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Abstract— **This paper proposes a novel method to approximate a stable and unique limit cycle, as obtained e.g. from simulation of a nonlinear model or from experimental data, by a planar switching affine system (PSAS) while preserving the properties of uniqueness and stability. In contrast to existing literature, which formulates elaborate procedures based on system transformation to conclude on stability and uniqueness, this paper provides easy to check sufficient conditions directly formulated for the PSAS. Those conditions then serve as equality and inequality constraints in an optimization problem to determine the model parameters of the PSAS which best approximate the data points measured for the given limit cycle. It is shown that the sufficient conditions lead to the same results as formulated in literature, while a system transformation is not necessary. Efficiency, flexibility, and performance of the proposed method are demonstrated for a numeric example.**

I. INTRODUCTION

Limit cycles have been a subject of research for a long period of time, since oscillations occur in a large variety of domains (technical, biological, physical, astronomical, etc.), see e.g. [1]–[3]. They have been analyzed in several papers devoted to different nonlinear oscillators [4], [5] and system theoretical investigations [6]. Many nonlinear oscillators generate stable and unique limit cycles [7], [8]. However, the analytical characterization as well as conditions ensuring uniqueness and stability are limited to very special cases. This circumstances motivated the use of PSAS to approximate, describe, and analyze limit cycles.

Techniques to approximate nonlinear oscillations while preserving its properties are barely considered in literature. Kai et al. provided stable limit cycles as polygonal curves, by connecting vertices of polygons through line segments determined by piecewise affine systems using state feedback [9], [10]. However, this approach quickly becomes costly in terms of the number of switching lines and subsystems, and is based on state feedback laws. Moreover, the connection of the vertices through line segments can limit the quality of approximation considerably. In [11], a method to synthesize the limit cycle of PSAS based just on four characteristic data points was proposed, but the objective of obtaining an optimized approximation was not considered.

In a different thread of investigations, literature has focused on studying the existence and number of limit cycles in the qualitative theory of planar differential equations, which are restricted to two planar piecewise linear systems separated by a straight line, leading to several challenges. A number of at most eight limit cycles has recently been proven by Carmona et al. [12] for a general setting, while specific variants may show at most two [13], three [14], or four [15] limit cycles. If, however, the existence of a single and unique limit cycle is the objective of the considerations, additional requirements need to be formulated: For two linear vector fields separated by a straight line and a continuity condition, the existence of one limit cycle was observed in [16], and first proven in [17]. For the case without continuity assumption for the gradients on the common boundary, a larger number of papers has applied bifurcation theory to limit cycles of discontinuous planar piecewise linear systems. These papers are devoted to different cases of the phase portraits (node, saddle, focus) and specific techniques to proof uniqueness [18], [19], [20]. Li and Llibre [21] first proved the uniqueness of limit cycles in the focus-saddle case, and recently Li et al. [22] proposed results on the uniqueness of limit cycles for the focus-node and focus-focus scenario. The saddle-saddle case [23] as well as the node-node case [24] have been studied by Huan and Yang. These investigations involve rather elaborate system transformations, and only refer a collection of analyzed specific scenarios examined with respect to their mathematical characteristics. None of these papers provides a system-theoretic interpretation, or addresses conditions which are suitable for approximating limit cycles observed from data or for nonlinear oscillator models without discontinuities of the state derivatives.

This paper, however, provides easy to evaluate sufficient conditions for PSAS in order to approximate given limit cycles, while ensuring the properties of uniqueness and stability for the constructed limit cycles. The proposed new conditions are set in context to those for the node-node case in [24], which can be related to the PSAS set-up considered here. Accordingly, Sec. II briefly recalls the methodology from [24] to conclude on unique and stable limit cycles, while also highlighting requirements, challenges and problems. In Sec. III, new sufficient conditions for the PSAS are introduced, and it is shown that these sufficient conditions allow the same conclusion on uniqueness and stability as the approach in [24]. Sec. III uses the proposed conditions to set up optimization problems for approximating unique and stable limit cycles from measured data, and extensions are reported to emphasize the versatility of the method. In Sec. IV, an illustrative numeric example is provided to illustrate the procedure, the work is concluded in Sec. V with an outlook on future work.

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II. Problem Description

The paper addresses periodic behavior represented by a limit cycle $\phi_c \subset \mathbb{R}^2$, as observed from experimental data or generated by a nonlinear dynamic system which is too complicated to allow for rigorous mathematical analysis. The standing assumption is that ϕ_c is unique and asymptotically stable on an appropriate region for the underlying oscillator, i.e., any trajectory initialized on this region converges to ϕ_c. The intuitive approach followed in this paper is that the periodic behavior represented by ϕ_c is approximated by a substitute system which shares an almost identical limit cycle (in form, period length, uniqueness and stability) while enabling mathematical analysis. For this purpose, the class of *planar switching affine system* (PSAS) is used [11], written here in the specific following form to allow later the transfer of results from [25] and [24]:

$$\dot{x} = \begin{cases} A^I x + B^I, \ x_1 < 0 \\ A^{II} x + B^{II}, \ x_1 > 0 \end{cases} \quad , \quad x = [x_1, x_2]^T \subset \mathbb{R}^2 \quad (1)$$

The coordinates x_1, x_2 are assumed to be chosen such that the given limit cycle ϕ_c oscillates around the origin. The matrices A^I and A^{II}, as well as the vectors B^I and B^{II} are parameters to be determined. It is pointed out, that, in general, a system of type (1) may establish more than one limit cycle (in fact, a number in between zero and eight, [12]), and that a limit cycle may also be unstable. Thus, even if (1) generates a limit cycle similar to ϕ_c, a lack of uniqueness and stability of the limit cycle may prevent it from being a suitable approximation of the original system.

A. Limit Cycles of PSAS

Given the system (1) with parameters:

$$A^I = \begin{bmatrix} a^I_{11} & a^I_{12} \\ a^I_{21} & a^I_{22} \end{bmatrix}, A^{II} = \begin{bmatrix} a^{II}_{11} & a^{II}_{12} \\ a^{II}_{21} & a^{II}_{22} \end{bmatrix}, B^I = \begin{bmatrix} b^I_1 \\ b^I_2 \end{bmatrix}, B^{II} = \begin{bmatrix} b^{II}_1 \\ b^{II}_2 \end{bmatrix}$$

(to be determined), a step often used in literature (see [25] for example) to prepare analysis is to transfer the system into Liénard-like canonical form (LCF):

$$\dot{x} = \begin{cases} \begin{bmatrix} T^I & -1 \\ D^I & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -m \end{bmatrix}, \ x_1 < 0 \\ \\ \begin{bmatrix} T^{II} & -1 \\ D^{II} & 0 \end{bmatrix} x + \begin{bmatrix} p \\ -n \end{bmatrix}, \ x_1 > 0 \end{cases} . \quad (2)$$

In here, T^I and D^I denote the trace and determinant of A^I (likewise T^{II} and D^{II}), while the other parameters are:

$$m := a^I_{12} b^I_2 - a^I_{22} b^I_1, \quad n := \frac{a^I_{12}}{a^{II}_{12}} (a^{II}_{12} b^{II}_2 - a^{II}_{22} b^{II}_1)$$

$$p := \frac{a^I_{12}}{a^{II}_{12}} b^{II}_1 - b^I_1.$$

The reason for getting to the LCF is that its limit cycle (if existing) shares the same properties regarding uniqueness and stability with the limit cycle of (1), while less parameters are involved. Note that the corresponding limit cycles of (1) and (2) are not the same (see [25] for more details), as is also shown in Fig. 1 for an example.

Next, the concept of a **sliding set** $\Sigma^s \subseteq \mathbb{R}^2$ of (1) is introduced, which is defined as:

$$\Sigma^s := \left\{ \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \ \middle| \ \left(\begin{bmatrix} a^I_{11} & a^I_{12} \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} + b^I_1 \right) \left(\begin{bmatrix} a^{II}_{11} & a^{II}_{12} \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} + b^{II}_1 \right) \leq 0 \right\}. \quad (3)$$

Note that the sliding set Σ^s denotes the part of the x_2-axis (i.e. the switching line of (1)) on which the x_1-components of the two affine dynamics of (1) point into opposite directions. If an LCF correspondence (2) is obtained for a system (1), the sliding set is transformed, too, but the following relation of uniqueness and stability of the limit cycles for the two representations was established in [24]:

Theorem 1: Let the system (1) with a sliding set Σ^s as well as an LCF according to (2) be given such that the following conditions are satisfied: $m > 0$, $n < 0$, $\min\{D^I, D^{II}\} > 0$, $(T^I)^2 > 4 \cdot D^I$, $(T^{II})^2 > 4 \cdot D^{II}$, $T^I \cdot T^{II} > 0$, and $p \neq 0$. Then the following statements hold:

- If $T^{II} \cdot p > 0$, (1) has no limit cycle without a point being also contained in Σ^s.
- If $T^{II} \cdot p < 0$, a limit cycle of (1) without a point being contained in Σ^s is unique. In addition, such a limit cycle is asymptotically stable if $p > 0$ and unstable if $p < 0$. □

This theorem can be used to examine based on the LCF whether a given system (1) has a unique and stable limit cycle without a point in Σ^s. However, in order to use these results to determine a system (1) such that the latter shows a limit cycle similar to ϕ_c, a number of problems need to be addressed first: The conditions above are tailored to the LCF (2), while the actual objective is to design a system of type (1) (containing a larger number of parameters). In addition, the existence, the uniqueness, and the stability properties in Theorem 1 are all referring to a limit cycle without points

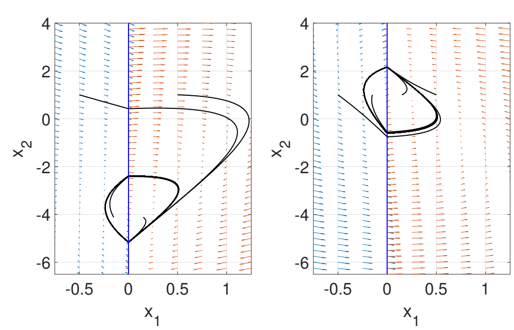

Fig. 1. For a system (1) with $A^I = \begin{bmatrix} -3 & 1 \\ 3 & -2 \end{bmatrix}$, $A^{II} = \begin{bmatrix} -4 & 1 \\ -3 & 0.25 \end{bmatrix}$, $B^I = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ and $B^{II} = \begin{bmatrix} 5 \\ 0.75 \end{bmatrix}$ on the left, and a corresponding LCF on the right, stabilizing (but different) limit cycles exist for both representations. The arrows in red and blue represent the phase portrait of the two systems.

in the sliding set Σ^s. As both, the limit cycle and the sliding set Σ^s of (1), are dependent upon the values of A^I, A^{II}, B^I and B^{II}, uniqueness and stability results for a limit cycle of (1) need to be related to these parameters, and the design of approximations of ϕ_c needs to consider that no intersection of the approximating cycle with the set Σ^s occurs.

With these aspects in mind, the objective is to formulate a set of new conditions tailored to the system (1), such that a unique and stable limit cycle is obtained. Furthermore, the additional degrees of freedom in (1) will be used to approximate ϕ_c in an optimal sense.

III. Determination of PSAS with Unique and Stable Limit Cycle

By temporarily neglecting the task of approximating the given limit cycle ϕ_c, a new theorem is formulated with the objective to ensure the existence, the uniqueness, and the stability of the limit cycle of (1):

Theorem 2: For a system of type (1), let the following conditions be satisfied:

1) The matrices A^I and A^{II} of (1) are Hurwitz and have distinct real-valued and negative eigenvalues.
2) Let $\bar{x}^I = [\bar{x}_1^I, \bar{x}_2^I]^T$ and $\bar{x}^{II} = [\bar{x}_1^{II}, \bar{x}_2^{II}]^T \in \mathbb{R}^2$ denote the equilibrium point of the dynamics $\dot{x} = A^I x + B^I$, and $\dot{x} = A^{II} x + B^{II}$ respectively, with:
$$\bar{x}_1^I > 0, \quad \bar{x}_1^{II} < 0; \tag{4}$$
3) Two points $\hat{x}_1 = [0, \hat{x}_{1,2}]^T$ and $\hat{x}_2 = [0, \hat{x}_{2,2}]^T$ on the x_2-axis exist with $\hat{x}_{1,2} < 0$ and $\hat{x}_{2,2} > 0$, as well as two time points $T_1, T_2 \in \mathbb{R}^{>0}$ satisfying:
$$\hat{x}_2 = e^{A^I T_1}\hat{x}_1 + \int_0^{T_1} e^{A^I (T_1-\tau)} B^I d\tau \tag{5}$$
$$\hat{x}_1 = e^{A^{II} T_2}\hat{x}_2 + \int_0^{T_2} e^{A^{II} (T_2-\tau)} B^{II} d\tau; \tag{6}$$
4) For the two points $\hat{x}_1$ and $\hat{x}_2$, the following inequalities hold in addition:
$$a_{12}^I \hat{x}_{1,2} + b_1^I < 0, \quad a_{12}^I \hat{x}_{2,2} + b_1^I > 0 \tag{7}$$
$$a_{12}^{II} \hat{x}_{1,2} + b_1^{II} < 0, \quad a_{12}^{II} \hat{x}_{2,2} + b_1^{II} > 0. \tag{8}$$

Then, the limit cycle of system (1) exists and contains no point in the sliding set Σ^s according to (3), and the limit cycle is unique and asymptotically stable. □

Proof. Since the conditions 1)-4) of Theorem 2 represent a special form of the existence conditions from Theorem 1 in [11], the existence of the limit cycle follows immediately.

As the eigenvalues of A^I and A^{II} are negative and real-valued, their determinants must satisfy:
$$D^I > 0, \quad D^{II} > 0, \tag{9}$$
and for the traces of A^I and A^{II} holds:
$$T^I < 0, \quad T^{II} < 0, \quad T^I \cdot T^{II} > 0. \tag{10}$$
As the eigenvalues of A^I and A^{II} are distinct, it applies that:
$$(T^I)^2 > 4 \cdot D^I, \quad (T^{II})^2 > 4 \cdot D^{II} \tag{11}$$
since $(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 > 0$ holds for any $\lambda_1 \neq \lambda_2$. Thus, the equalities in (5) and (6) determine a limit cycle of (1), which has a period of $T_1 + T_2$, and two switching points $\hat{x}_1$ and $\hat{x}_2$ on the x_2-axis exist.

The next part of the proof shows that the limit cycle through the points given by (5) and (6) contains no point in the sliding set Σ^s. Let the points $[0, \epsilon^I]^T$ and $[0, \epsilon^{II}]^T$ on the x_2-axis denote the system state satisfying:
$$a_{12}^I \epsilon^I + b_1^I = 0 \quad a_{12}^{II} \epsilon^{II} + b_1^{II} = 0. \tag{12}$$
Given the inequalities in (7), as well as the fact that the equilibrium point $\bar{x}^I$ of $\dot{x} = A^I x + B^I$ is located on the right side of the x_2-axis, the inequality:
$$\hat{x}_{1,2} < \epsilon^I < \hat{x}_{2,2} \tag{13}$$
hold and the relations:
$$a_{12}^I \epsilon + b_1^I < 0, \quad \forall\, \epsilon < \epsilon^I \tag{14}$$
$$a_{12}^I \epsilon + b_1^I > 0, \quad \forall\, \epsilon > \epsilon^I \tag{15}$$
hold for all points $[0, \epsilon]^T$ on the x_2-axis (See Fig. 2 for an illustration of these relations). Equivalently, the relations:
$$\hat{x}_{1,2} < \epsilon^{II} < \hat{x}_{2,2} \tag{16}$$
$$a_{12}^{II} \epsilon + b_1^{II} < 0, \quad \forall\, \epsilon < \epsilon^{II} \tag{17}$$
$$a_{12}^{II} \epsilon + b_1^{II} > 0, \quad \forall\, \epsilon > \epsilon^{II}. \tag{18}$$
hold for the dynamics $\dot{x} = A^{II} x + B^{II}$ and for all points $[0, \epsilon]^T$ on the x_2-axis.

Assume that $\epsilon^{II} \geq \epsilon^I$ applies (as in Fig. 2), then the sliding set Σ^s of (1) turns out to be:
$$\Sigma^s := \left\{ \begin{bmatrix} 0 & \epsilon \end{bmatrix}^T \in \mathbb{R}^2 \;\middle|\; \epsilon \in [\epsilon^I, \epsilon^{II}] \right\}. \tag{19}$$
As a result, both switching points $\hat{x}_1$ and $\hat{x}_2$ must be located outside of Σ^s, i.e., the limit cycle through (5) and (6) has no common point with the sliding set (the same implication also holds for $\epsilon^{II} \leq \epsilon^I$). Moreover, it is known from [25]

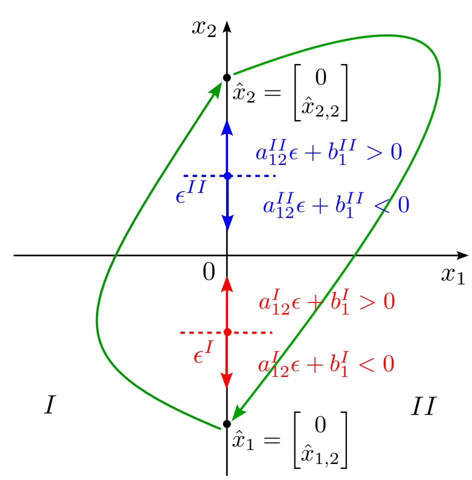

Fig. 2. The trajectory in green represents the limit cycle of (1) determined by (5) and (6), while the sliding set Σ^s is the part between ϵ^I and ϵ^{II} on the x_2-axis.

that the existence of such a limit cycle implies the following two facts:

1) The matrices A^I and A^{II}, satisfy the relation:
$$a_{12}^I a_{12}^{II} > 0. \quad (20)$$
2) As $T^I \cdot T^{II} > 0$, $T^I + T^{II} < 0$, a parameter:
$$p \neq 0 \quad (21)$$
must exist for the LCF (2) according to the Remark 3.8 of [25][1].

To show that $m > 0$ and $n < 0$ hold in (2), it is known that the equilibrium point $\bar{x}^I$ of $\dot{x} = A^I x + B^I$ satisfies the equation:
$$\bar{x}^I = \begin{bmatrix} \bar{x}_1^I \\ \bar{x}_2^I \end{bmatrix} = -(A^I)^{-1} B^I = \frac{1}{D^I} \cdot \begin{bmatrix} a_{12}^I b_2^I - a_{22}^I b_1^I \\ a_{21}^I b_1^I - a_{11}^I b_2^I \end{bmatrix}. \quad (22)$$
As $D^I > 0$ and $\bar{x}_1^I > 0$, the relation:
$$m = a_{12}^I b_2^I - a_{22}^I b_1^I > 0 \quad (23)$$
applies, and similarly, it can be shown that:
$$a_{12}^{II} b_2^{II} - a_{22}^{II} b_1^{II} < 0 \quad (24)$$
holds. This together with (20) implies that:
$$n = \frac{a_{12}^I}{a_{12}^{II}} \cdot (a_{12}^{II} b_2^{II} - a_{22}^{II} b_1^{II}) < 0. \quad (25)$$
Thus, all conditions in Theorem 1 are satisfied, as well as the fact that $T^{II} < 0$ and $p \neq 0$. Assuming $p < 0$, which implies $T^{II} \cdot p > 0$, there should exist no limit cycle that has no common point with Σ^s according to Theorem 1. This result, however, contradicts with the fact that a limit cycle from (5) and (6) indeed exists (premise of Theorem 2). Accordingly, the relation $p > 0$ must hold, which implies according to Theorem 1 that the limit cycle through (5) and (6) is unique and asymptotically stable. □

Note that by designing the system (1) such that it shows a limit cycle similar to ϕ_c, the values of $\hat{x}_{1,2}$, $\hat{x}_{2,2}$ as well as of T_1 and T_2 can be fixed in the first place, since they are determined by the points in which ϕ_c crosses the x_2-axis and by the period of ϕ_c. The conditions in Theorem 2, can be cast into a set of inequalities/equalities for A^I, A^{II}, B^I and B^{II} according to the following considerations:

- The requirement that A^I (like-wise for A^{II}) must have distinct and negative real-valued eigenvalues is ensured by the constraints:
$$a_{11}^I + a_{22}^I < 0, \quad (26)$$
$$a_{12}^I a_{21}^I - a_{11}^I a_{22}^I < 0, \quad (27)$$
$$\text{and } (a_{11}^I - a_{22}^I)^2 + 4 a_{12}^I a_{21}^I > 0. \quad (28)$$
- The requirement for the equilibrium points $\bar{x}^I$ and $\bar{x}^{II}$ in (4) can be ensured by enforcing:
$$a_{12}^I b_2^I - a_{22}^I b_1^I > 0 \text{ and } a_{12}^{II} b_2^{II} - a_{22}^{II} b_1^{II} < 0, \quad (29)$$
since A^I and A^{II} are Hurwitz.
- For given values of $\hat{x}_{1,2}$, $\hat{x}_{2,2}$, T_1, and T_2, the conditions (5), (6) as well as (7) and (8) represent constraints to be considered for the selection of the parameters in A^I, A^{II}, B^I and B^{II}.

[1]Note that $p \neq 0$ also implies that $\epsilon^{II} \neq \epsilon^I$, i.e., the sliding set Σ^s is not a singleton.

A. Approximation of a Given Limit Cycle ϕ_c

Assuming that a limit cycle ϕ_c was observed (e.g.) in an experiment, the objective addressed now is to determine a system of type (1) which constitutes a limit cycle that approximates ϕ_c with high accuracy, while ensuring the established properties of uniqueness and stability. First of all, based on the switching points $\hat{x}_1 = [0, \hat{x}_{1,2}]^T$ and $\hat{x}_2 = [0, \hat{x}_{2,2}]^T$ as well as the period length T_1 and T_2 of ϕ_c, a number of N different points $x_s^{[i]}(t_j^{[i]})$, $i \in \{1, \dots, N\}$, $j \in \{1, 2\}$ along ϕ_c are sampled on each side of the switching line with ordering:
$$0 < t_j^{[1]} < t_j^{[2]} < \dots t_j^{[N]} < T_j, \ j \in \{1, 2\}, \quad (30)$$
see Fig. 3. In order to measure the distance between the limit cycle of (1) and ϕ_c at each sampling time $t_j^{[i]}$, the following cost function is defined:
$$J := \sum_{i=1}^{N} (||e^{A^I t_1^{[i]}} \hat{x}_1 + \int_0^{t_1^{[i]}} e^{A^I (t_1^{[i]} - \tau)} B^I d\tau - x_s^{[i]}(t_1^{[i]})||_2$$
$$+ ||e^{A^{II} t_2^{[i]}} \hat{x}_2 + \int_0^{t_2^{[i]}} e^{A^{II} (t_2^{[i]} - \tau)} B^{II} d\tau - x_s^{[i]}(t_2^{[i]})||_2). \quad (31)$$
and used within the following optimization to determine the system (1):
$$\min_{A^I, B^I, A^{II}, B^{II}} J \quad (32)$$
s.t. constraints:
$$(26) - (28) \text{ for } A^I \text{ and } A^{II} \quad (33)$$
$$(5), (6), (7), (8), (29). \quad (34)$$
From the constraints included in this problem, the ones references as (5) and (6) are numerically difficult, since they contain the matrix exponential function and cannot

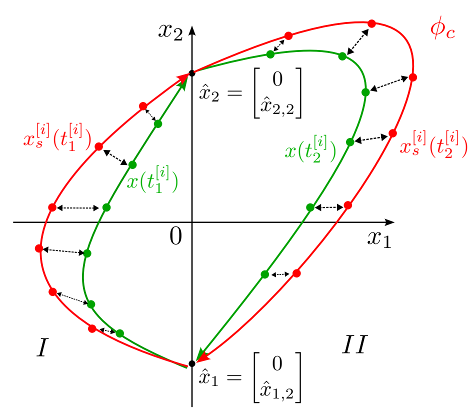

Fig. 3. The distance between ϕ_c (in red) and the limit cycle resulting from (1) (in green) at all sampled times $t_j^{[i]}$ is minimized by optimization.

be treated directly by solvers, such as YALMIP [26]. As countermeasures, one can either approximate the matrix exponential function by Taylor expansion with sufficiently large order, or use a diagonalized form of the matrices A^I and A^{II} (possible since the eigenvalues are distinct). For the latter option, the eigenvalues:

$$\lambda_1^I = \frac{1}{2}\left((a_{11}^I + a_{22}^I) + \sqrt{(a_{11}^I + a_{22}^I)^2 - 4(a_{11}^I a_{22}^I - a_{12}^I a_{21}^I)}\right)$$
$$\lambda_2^I = \frac{1}{2}\left((a_{11}^I + a_{22}^I) - \sqrt{(a_{11}^I + a_{22}^I)^2 - 4(a_{11}^I a_{22}^I - a_{12}^I a_{21}^I)}\right)$$

of A^I (same for A^{II}) and a corresponding matrix of eigenvectors V^I are determined, and the equality (5) is written as:

$$\hat{x}_2 = V^I \begin{bmatrix} e^{\lambda_1^I T_1} & 0 \\ 0 & e^{\lambda_2^I T_1} \end{bmatrix} (V^I)^{-1}\hat{x}_1 + V^I \begin{bmatrix} \frac{e^{\lambda_1^I T_1}-1}{\lambda_1^I} & 0 \\ 0 & \frac{e^{\lambda_2^I T_1}-1}{\lambda_2^I} \end{bmatrix} (V^I)^{-1} B^I$$

(likewise for (6)).

B. Flexible Switching Points and Period Length

For solution of the optimization problem (32) - (34), the switching points $\hat{x}_1$ and $\hat{x}_2$ as well as the semi-periods T_1 and T_2 could be treated as fixed, if determined a-priori from the given limit cycle ϕ_c. This option, however, may lead to a poor approximation of ϕ_c with regard to the distance to the other sample points $x_s^{[i]}(t_j^{[i]})$, or even to infeasibilities with respect to the constraints in (33) - (34). Alternatively, $\hat{x}_1$, $\hat{x}_2$, T_1, and T_2 can be treated as additional degrees of freedom of the optimization, to let them take values within a bounded distance to their counterparts in ϕ_c.

C. Continuity of the Gradient at Switching Points

The question of whether $\dot{x}$ is permitted to be discontinuous at the two switching points of the limit cycle of (1), or whether it is enforced to be continuous, is treated differently in literature, and leads to different properties with respect to uniqueness and stability, see [27]. The work [24] presenting Theorem 1 does not require the gradient to be continuous. As a result, the gradient of the limit cycle obtained from solving (32) - (34) may show sudden jumps at the switching points. This outcome, can be undesired if the oscillating behavior of the underlying system changes smoothly over time. To avoid this situation, the difference of the gradients at the switching points:

$$||A^I\hat{x}_1 + B^I - A^{II}\hat{x}_1 - B^{II}||_2, \quad ||A^I\hat{x}_2 + B^I - A^{II}\hat{x}_2 - B^{II}||_2 \tag{35}$$

can be minimized in addition by extension of the cost function J, or by introducing a new constraint to ensure (35) to be bounded by an appropriate value.

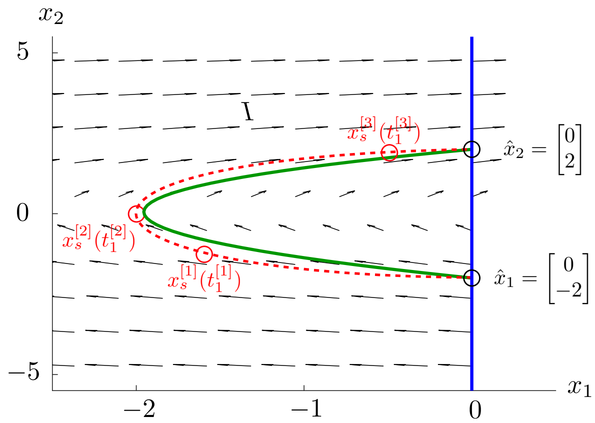

Fig. 4. $\dot{x} = A^I x + B^I$ for $x_1 < 0$ with (36) shown in solid green, determined from the data points shown as red circles, while $\hat{x}_1$, $\hat{x}_2$ are marked by black circles. The arrows in black represent the phase portrait of (36).

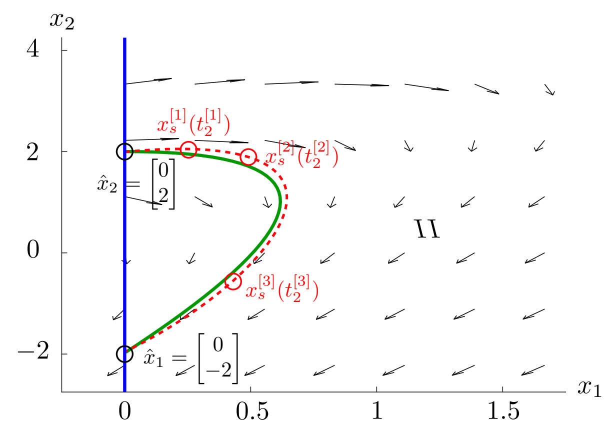

Fig. 5. $\dot{x} = A^{II} x + B^{II}$ for $x_1 > 0$ with (37) shown in solid green. Other data is marked as in Fig. 4.

IV. Numeric Example

To illustrate and evaluate the performance of the proposed method, the limit cycle ϕ_c as shown in Fig. 4 and 5 (marked by the red-dashed line) is approximated by a system of type (1). Note that due to the asymmetrical form of ϕ_c on both sides of the switching line, the use of standard oscillators, such as the van-der Pol one, would result in poor approximation performance. Given the switching points $\hat{x}_1 = [0, -2]^T$ and $\hat{x}_2 = [0, 2]^T$ of ϕ_c, as well as the semi-periods $T_1 = 3.125$ and $T_2 = 0.8090$, a set of $N = 3$ points $x_s^{[i]}(t_j^{[i]})$ are taken into account for each side of the switching line in the cost function J, namely:

Sample point	Time $t_1^{[i]}$	Value
$x_s^{[1]}(t_1^{[1]})$	0.8925	$[-1.5945, -1.2636]^T$
$x_s^{[2]}(t_1^{[2]})$	1.5175	$[-1.9985, -0.0088]^T$
$x_s^{[3]}(t_1^{[3]})$	2.8925	$[-0.4910, 1.8931]^T$
Sample point	Time $t_2^{[i]}$	Value
$x_s^{[1]}(t_2^{[1]})$	0.0893	$[0.2543, 2.0370]^T$
$x_s^{[2]}(t_2^{[2]})$	0.2097	$[0.4910, 1.8930]^T$
$x_s^{[3]}(t_2^{[3]})$	0.6797	$[0.4306, -0.5664]^T$

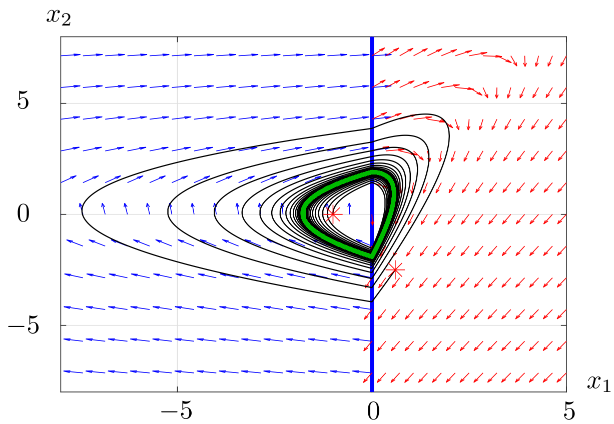

Fig. 6. Limit cycle (solid green) of system (1) parameterized with (36),(37) for different initial values shown as red stars. The solid black lines represent the transient behavior, together with the phase portrait of (36) (blue arrows) and (37) (red arrows); stability and uniqueness of the limit cycle is evident.

The system (1) obtained from solving the optimization problem (32) - (34) is parametrized by:

$$A^I = \begin{bmatrix} 0.0261 & 1.2443 \\ -0.0008 & -0.0380 \end{bmatrix}, B^I = \begin{bmatrix} -0.0154 \\ 1.2733 \end{bmatrix}, \quad (36)$$

$$A^{II} = \begin{bmatrix} -2.8338 & 1.5499 \\ -4.3628 & 2.3678 \end{bmatrix}, B^{II} = \begin{bmatrix} 0.1748 \\ -4.6737 \end{bmatrix}, \quad (37)$$

and the resulting limit cycle is marked in green in Fig. 4 - 6. It can be noticed from the first two figures that ϕ_c is well approximated on both sides of the switching line. To further evaluate the uniqueness and the stability of the obtained limit cycle, the above system can also be examined by use of Theorem 1 with $m = 1.5838 > 0$, $n = -6.1478 < 0$, $\min\{D^I = 1 \cdot 10^{-5}, D^{II} = 0.0518\} > 0$, $T^I = -0.0118$, $T^{II} = -0.4661$, $p = 0.1558 \neq 0$, as well as a sliding set Σ^s obtained to the interval $[-0.113, 0.012]$ on the x_2-axis. All conditions in the latter theorem are satisfied, which implies that the resulting limit cycle is unique and asymptotically stable. The convergence to the limit cycle is also illustrated for different initial states in Fig. 6.

V. CONCLUSIONS

In this paper, a novel method to approximate limit cycles obtained from experimental data by using PSAS is proposed. A set of sufficient conditions are also proposed to ensure the resulting PSAS has a unique and stable limit cycle. These conditions are later used in an optimization-based approach to compute the model parameters for the best approximation (wrt. a selected cost function). Future work aims at extending the method to higher-order systems and to more switching lines.

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