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Unconstrained Parameterization of Stable LPV Input-Output Models: with Application to System Identification

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***Abstract*— Ensuring stability of discrete-time (DT) linear parameter-varying (LPV) input-output (IO) models estimated via system identification methods is a challenging problem as known stability constraints can only be numerically verified, e.g., through solving Linear Matrix Inequalities. In this paper, an unconstrained DT-LPV-IO parameterization is developed which gives a stable model for any choice of model parameters. To achieve this, it is shown that *all* quadratically stable DT-LPV-IO models can be generated by a mapping of transformed coefficient functions that are constrained to the unit ball, i.e., a small-gain condition. The unit ball is then reparameterized through a Cayley transformation, resulting in an unconstrained parameterization of all quadratically stable DT-LPV-IO models. As a special case, an unconstrained parameterization of all stable DT linear time-invariant transfer functions is obtained. Identification using the stable DT-LPV-IO model with neural network coefficient functions is demonstrated on a simulation example of a parameter-varying mass-damper-spring system.**

I. Introduction

Ever stringent performance requirements from practice necessitate to also model and identify the nonlinear behavior of systems [1]. These nonlinear characteristics make modeling these systems based on first-principles increasingly difficult and time-consuming, such that it becomes necessary to adopt data-driven system modeling tools, i.e., system identification.

Linear parameter-varying (LPV) systems [2] are a powerful surrogate system class for capturing nonlinear and time-varying behaviour. In LPV systems, the signal relations are linear, but the coefficients describing these relations are functions of a time-varying scheduling signal ρ that is assumed to be measurable online. The resulting parameter-varying behavior can embed certain nonlinear characteristics under the correct choice of ρ [3]. Data-driven system identification methods for LPV systems have been thoroughly developed in the last decades, both for input-output (IO) [4]–[6] as well as state-space (SS) representations [7], [8].

Even though many systems to be modeled in an LPV form are known to be stable, ensuring stability of the identified model is a challenging problem for two reasons. First, the parameter estimates are sensitive to modeling errors, finite data effects, and measurement noise, such that the identified model can be unstable even if the underlying system is stable [9], [10]. Second and most importantly, for LPV models there is no analytic explicit constraint on the model parameters that ensures stability. Instead, stability can only be verified by optimization, e.g., by testing the feasibility of Linear Matrix Inequalities (LMIs) representing a stability condition [11].

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To ensure stability of the identified model, recently unconstrained *state-space* (SS) models have been developed that are stable for any choice of model parameters [12]–[14]. This is achieved by reparameterizing the stability constraint in the form of an LMI through new unconstrained parameters in a necessary and sufficient manner, such for any choice in these new parameters, the LMI is satisfied, i.e., stability is guaranteed. In [12], such an unconstrained parameterization is developed for discrete-time (DT) Lur'e models with neural network nonlinearities. This technique has subsequently been applied to continuous-time Lur'e models [13] and discrete-time LPV SS models [14].

In contrast, to guarantee stability of an identified *LPV input-output* (LPV-IO) model, current methods either explicitly enforce a stability constraint during identification [15]–[17] or restrict the model class such that it is a priori guaranteed to be stable [18], [19]. However, enforcing a stability constraint - usually in the form of an LMI or sum of squares condition - during identification severely increasing the computational complexity of the optimization [9], [20]. Alternatively, restricting the model class is usually done according to a simple but often conservative approximation of all stable models, limiting the representation capability.

To overcome the computational complexity and conservatism of the previous approaches, in this paper it is shown that *all* quadratically stable (QS) DT-LPV-IO models can be generated by a mapping of unconstrained transformed coefficient functions. The approach is based on reparameterizating the LPV coefficient functions such that the LMI representing the QS condition is satisfied for any choice of model parameters, similar to the state-space case [12]–[14]. A consequence of this reparameterization is that still only QS systems can be represented. Thus, an LPV system that is stable but not quadratically cannot be exactly represented. However, quadratic stability already represents a significant improvement over the current conservative approximations, and constrained approaches usually also consider QS.

The main contribution of this paper is an unconstrained parameterization of all quadratically stable DT-LPV-IO models, allowing for unconstrained system identification with a priori stability guarantees. The set of stable DT transfer functions is obtained as a special case. The main contribution consists of the following sub-contributions.

C1) A criterion in the form of a matrix inequality to test stability of LPV-IO models (Section III), and a corresponding graphical interpretation of the allowed coefficient function value sets in which stability is guaranteed (Section V).

C2) A reparameterization of the coefficient functions of LPV-IO models such that above criterion is satisfied for any choice of the new model parameters (Section IV).

C3) A simulation example demonstrating the applicability of the developed method (Section V).

Notation: $\|\cdot\|_2$ represents the Euclidean vector norm. $\mathbb{Z}_{\geq 0}$ represents the set of non-negative integers. A *symmetric* matrix $P \in \mathbb{R}^{n\times n}$ is said to be positive definite if $x^\top P x > 0$ $\forall x \in \mathbb{R}^n \setminus \{0\}$, also denoted by $P \succ 0$ and $P \in \mathbb{S}_{\succ 0}$. Similarly, $P \prec 0$ and $P \in \mathbb{S}^n_{\prec 0}$ denote negative-definiteness.

II. Problem Formulation

Consider the LPV-IO model with input $u_k \in \mathbb{R}$ and output $y_k \in \mathbb{R}$ represented by the difference equation

$$y_k = -\sum_{i=1}^{n_a} a_i(\rho_k) y_{k-i} + \sum_{i=0}^{n_b-1} b_i(\rho_k) u_{k-i}, \tag{1}$$

with time index $k \in \mathbb{Z}_{\geq 0}$, model order $n_a \geq 0$, $n_b \geq 1$, and coefficient functions $a_i(\rho), b_i(\rho) : \mathbb{P} \to \mathbb{R}$ describing the dependence of the difference equation on the scheduling signal $\rho_k \in \mathbb{P} \subseteq \mathbb{R}^{n_\rho}$.

The coefficient functions are parameterized by a function $g_\phi(\rho)$ depending on model parameters $\phi \in \mathbb{R}^{n_\phi}$, i.e.,

$$g_\phi(\rho) = \begin{bmatrix} a_1(\rho) & a_2(\rho) & \cdots & b_{n_b-1}(\rho) \end{bmatrix}^\top. \tag{2}$$

Examples include affine coefficient functions, e.g., $g_\phi^{\text{aff}}(\rho) = E\rho + c$ with $\phi = \text{vec}(E, c)$, polynomial basis function expansions [6], e.g., $g_\phi^{\text{pol}}(\rho) = c + E_1\rho + E_2\rho^2 + \ldots + E_d\rho^d$ with $\phi = \text{vec}(E_1, \ldots, E_d, c)$, or a neural network [21], [22]

$$g_\phi^{\text{NN}} = E_{\text{L}}\sigma(E_{\text{L}-1}\sigma(\cdots(E_0\rho + c_0)\cdots) + c_{\text{L}-1}) + c_L,$$

with nonlinear activation function σ, e.g., $\sigma(\cdot) = \tanh(\cdot)$, and parameters $\phi = \text{vec}(E_0, c_0, \ldots E_{\text{L}}, c_{\text{L}})$.

Remark 1 *For ease of notation, $u_k, y_k \in \mathbb{R}$ is considered. However, all results immediately extend to the setting in which $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$.*

Given model class (1) and a parameterization of $g_\phi(\rho)$, a natural question is if the model is stable for the chosen model parameters ϕ and the relevant range of ρ. Here, stability is defined as the model output y_k asymptotically going to zero for zero input, as defined next.

Definition 2 *Given a parameterization $g_\phi : \mathbb{P} \to \mathbb{R}^{n_a+n_b}$ with parameters $\phi \in \mathbb{R}^{n_\phi}$, the LPV-IO model* (1) *is said to be uniformly asymptotically stable if for any time $\bar{k}$, any scheduling signal ρ with $\rho_k \in \mathbb{P}$, and any input u with $u_k = 0$ $\forall k > \bar{k}$, the response y_k of* (1) *satisfies* $\lim_{k\to\infty} y_k = 0$.

The goal of this paper then is to parameterize $g_\phi(\rho)$ in (2) in such a way that LPV-IO model (1) is stable for any choice of ϕ, i.e., guaranteeing stability without constraints, for any parameterization of g_ϕ (e.g., NN). Of course for stability such a g_ϕ should necessarily result in finite $a_i(\rho), b_i(\rho)$ for all possible ρ, i.e., $\|g_\phi(\rho)\|_2 < \infty$ $\forall \rho \in \mathbb{P}$, which is taken as a precondition in the remainder of the paper.

III. Stable LPV-IO models

To obtain an unconstrained parameterization of (1), first a condition for determining stability of (1) as in Defintion 2 in terms of $a_i(\rho), b_i(\rho)$ is required. This section derives such a stability condition in the form of a matrix inequality, constituting contribution C1.

A. Maximum State-space Representation

To derive a stability condition, first it is required to define a state for (1) such that standard Lyapunov techniques can be used. To avoid issues around minimum realizations in absence of any structure in the coefficient functions [23], a non-minimum state-space representation for (1) is used.

More specifically, (1) can be equivalently represented as (4) with state x storing the previous inputs and outputs as

$$\begin{aligned} x_k &= \begin{bmatrix} \bar{y}_k^\top & \bar{u}_k^\top \end{bmatrix}^\top \in \mathbb{R}^{n_a+n_b} \\ \bar{y}_k &= \begin{bmatrix} y_{k-1} & \ldots & y_{k-n_a} \end{bmatrix}^\top \in \mathbb{R}^{n_a} \\ \bar{u}_k &= \begin{bmatrix} u_{k-1} & \ldots & u_{k-n_b+1} \end{bmatrix}^\top \in \mathbb{R}^{n_b-1}. \end{aligned} \tag{3}$$

Representation (4) is compactly written as

$$\begin{aligned} x_{k+1} &= A(\rho_k)x_k + B(\rho_k)u_k \\ y_k &= C(\rho_k)x_k + D(\rho_k)u_k, \end{aligned} \tag{5}$$

where

$$A(\rho) = \left[\begin{array}{c|c} F - GK(\rho) & GL(\rho) \\ \hline 0 & F_b \end{array}\right] \quad B(\rho) = \left[\begin{array}{c} Gb_0(\rho) \\ \hline G_b \end{array}\right]$$
$$C(\rho) = \left[\begin{array}{c|c} -K(\rho) & L(\rho) \end{array}\right] \qquad D(\rho) = b_0(\rho). \tag{6}$$

Matrices F, F_b and G, G_b correspond to a discrete-time buffer system that collects past samples of y_k and u_k, i.e.,

$$\begin{aligned} F &= \begin{bmatrix} 0 & 0 \\ I_{n_a-1} & 0 \end{bmatrix} \in \mathbb{R}^{n_a\times n_a} & G &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^{n_a}, \\ F_b &= \begin{bmatrix} 0 & 0 \\ I_{n_b-2} & 0 \end{bmatrix} \in \mathbb{R}^{n_b-1\times n_b-1} & G_b &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^{n_b-1}, \end{aligned} \tag{7}$$

with 0 entries of appropriate dimensions. $K(\rho), L(\rho)$ collect coefficient functions $a_i(\rho), b_i(\rho)$ as

$$\begin{aligned} K(\rho) &= \begin{bmatrix} a_1(\rho) & \ldots & a_{n_a-1}(\rho) & a_{n_a}(\rho) \end{bmatrix} \in \mathbb{R}^{n_a} \\ L(\rho) &= \begin{bmatrix} b_1(\rho) & \ldots & b_{n_b-2}(\rho) & b_{n_b-1}(\rho) \end{bmatrix} \in \mathbb{R}^{n_b-1}. \end{aligned} \tag{8}$$

Although representation (4) is not a minimal representation of (1), if (1) is controllable and observable, (4) is controllable and detectable, hence it can be used to directly characterize stability of (1).

B. Quadratically Stable LPV-IO Models

Representation (5) allows for using standard Lyapunov techniques to determine stability. Specifically, stability can be analyzed by considering a common quadratic Lyapunov function, as formalized next.

Lemma 3 *Given parameters $\phi \in \mathbb{R}^{n_\phi}$, the LPV-IO model* (1) *is stable as in Definition 2 if there exists a $\mathcal{P} \succ 0$ such that*

$$\mathcal{P} - A^\top(\rho)\mathcal{P}A(\rho) \succ 0 \quad \forall \rho \in \mathbb{P}. \tag{9}$$

The proof follows by standard Lyapunov arguments. All LPV-IO models that satisfy Lemma 3 are called quadratically

$$x_{k+1} = \left[\begin{array}{ccccc|ccccc} -a_1(\rho_k) & -a_2(\rho_k) & \dots & -a_{n_a-1}(\rho_k) & -a_{n_a}(\rho_k) & b_1(\rho_k) & b_2(\rho_k) & \dots & b_{n_b-2}(\rho_k) & b_{n_b-1}(\rho_k) \\ 1 & & & \emptyset & 0 & 0 & 0 & \dots & 0 & 0 \\ & 1 & & & 0 & 0 & 0 & \dots & 0 & 0 \\ & & \ddots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \emptyset & & & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & & & \emptyset & 0 \\ 0 & 0 & \dots & 0 & 0 & & 1 & & & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & \emptyset & & & 1 & 0 \end{array}\right] x_k + \left[\begin{array}{c} b_0(\rho_k) \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{array}\right] u_k \tag{4a}$$

$$y_k = \left[\begin{array}{ccccc|ccccc} -a_1(\rho_k) & -a_2(\rho_k) & \dots & -a_{n_a-1}(\rho_k) & -a_{n_a}(\rho_k) & b_1(\rho_k) & b_2(\rho_k) & \dots & b_{n_b-2}(\rho_k) & b_{n_b-1}(\rho_k) \end{array}\right] x_k + b_0(\rho_k) u_k \tag{4b}$$

stable (QS). Condition (9) provides a computational test for determining QS of the LPV-IO model class (1), showing that determining stability of IO parameterizations can be addressed using state-space methods.

IV. Unconstrained Parameterization of Stable LPV-IO models

In this section, it is shown that there exists $\mathcal{P}, K(\rho), L(\rho)$ that satisfy (9) if and only if there exist unconstrained variables $X(\rho), Z(\rho), W$ related to $\mathcal{P}, K(\rho)$ in a one-to-one way and that $L(\rho)$ is irrelevant for satisfying (9). These relations then allow for an unconstrained reparameterization of the coefficient functions such that (9) is always satisfied, constituting Contribution C2. Consequently, any choice of model parameters ϕ of this unconstrained reparameterization results in coefficient functions for which (1) is stable.

A. Eliminating the Influence of $L(\rho)$

First it is shown that only $F - GK(\rho)$ has to be considered to satisfy the stability condition (9), i.e., $L(\rho)$ is already unconstrained. Intuitively, since $A(\rho)$ is upper block diagonal, only its diagonal blocks $F - GK(\rho)$ and F_b determine stability. However, F_b represents a simple LTI chain of delays and is trivially stable, meaning that the stability of $F - GK(\rho)$ determines the stability of (1). These claims are formalized by the following lemma.

Lemma 4 *For a given $A(\rho)$ as in* (6)*, there exists a $\mathcal{P} \succ 0$ satisfying* (9) *if and only if there exists a $P \succ 0$ such that*

$$P - (F - GK(\rho))^\top P (F - GK(\rho)) \succ 0 \quad \forall \rho \in \mathbb{P}, \tag{10}$$

and $\|L(\rho)\|_2 < \infty \ \forall \rho \in \mathbb{P}$.

Thus, in the remainder, only (10) needs to be considered, parameterizing all quadratic Lyapunov functions for $F - GK(\rho)$, i.e., for the recurrence $y_k = -\sum_{i=1}^{n_a} a_i(\rho) y_{k-i}$.

B. Convex Reparameterization of $K(\rho)$ for Stability

As a step towards an unconstrained reparameterization, this subsection reparameterizes all $P, K(\rho)$ that satisfy (10) in terms of a convex set of variables W and $M(\rho)$, turning constraint (10) that is non-convex in $P, K(\rho)$ into a convex one in $W, M(\rho)$. Condition (10) is rewritten as follows.

Lemma 5 *For a given $K(\rho)$ as in* (6)*, there exists a $P \succ 0$ satisfying stability condition* (9) *if and only if P, $K(\rho)$ satisfy*

$$\begin{aligned} &P - F^\top P F + F^\top P G (G^\top P G)^{-1} G^\top P F \\ &\quad - H^\top(\rho) G^\top P G H(\rho) \succ 0 \quad \forall \rho \in \mathbb{P}, \end{aligned} \tag{11}$$

with

$$H(\rho) = K(\rho) - (G^\top P G)^{-1} G^\top P F. \tag{12}$$

Proof. The proof is based on completing the squares of the quadratic term $(K^\top(\rho) G^\top P G K(\rho))$ and linear term $(F^\top P G K(\rho))$ in (10). First, note that $H(\rho)$ is well-defined: by the structure of G, $G^\top P G$ is the (1,1) entry of P, which is positive as $P \succ 0$, and thus $G^\top P G \in \mathbb{R}_{>0}$ and its inverse are well-defined. Second, with $H(\rho)$ in (12), it holds that

$$\begin{aligned} &- H^\top(\rho) G^\top P G H(\rho) = K^\top(\rho) G^\top P F + F^\top P G K(\rho) \\ &\quad - K^\top(\rho) G^\top P G K(\rho) - F^\top P G (G^\top P G)^{-1} G^\top P F. \end{aligned} \tag{13}$$

Then expanding (10) and substituting (13) gives (11). □

While Lemma 5 is just a reformulation of (10), it is useful for characterizing all candidate P that should be considered for (10), as formalized by the following corollary.

Corollary 6 *Any $P \succ 0$ satisfying* (10) *has to satisfy*

$$P - F^\top P F + F^\top P G (G^\top P G)^{-1} G^\top P F \succ 0. \tag{14}$$

This corollary immediately follows from (11) since $H^\top(\rho) G^\top P G H(\rho) \succ 0$. It illustrates that not any $P \succ 0$ can be a Lyapunov function because $F - GK(\rho)$ is structured: P needs to additionally satisfy the Riccati inequality (14). Given this specification of all P that should be considered, all solutions $K(\rho), P$ that satisfy (10) can be reparameterized in terms of a convex set of matrices, as formalized next.

Theorem 7 *The LPV-IO model* (1) *satisfies* (10) *with a $P \succ 0$ if and only if there exists an $M(\rho) : \mathbb{P} \to \mathbb{R}^{1 \times n_a}$ with $M^\top(\rho) M(\rho) \prec I \ \forall \rho \in \mathbb{P}$ and $W \in \mathbb{S}^{n_a}_{\succ 0}$ such that*

$$P - F^\top P F + F^\top P G (G^\top P G)^{-1} G^\top P F = W \tag{15}$$

$$K(\rho) = (G^\top P G)^{-1} G^\top P F + X_Q^{-1} M(\rho) X_W \tag{16}$$

where X_W, X_Q are matrix factorizations given by $W = X_W^\top X_W$, $G^\top P G = X_Q^\top X_Q$.

Proof. (10) $\Rightarrow$ (15)-(16): If (10) is satisfied with a $P \succ 0$, then, by Corollary 6, P satisfies (14). Thus there exists a $W \succ 0$ such that (15) is satisfied, and W can be factorized as $W = X_W^\top X_W$ with X_W full rank. Then (11) reads as

$$X_W^\top X_W - H^\top(\rho) G^\top P G H(\rho) \succ 0 \quad \forall \rho \in \mathbb{P}, \tag{17}$$

with $H(\rho)$ as defined in (12). Since $P \succ 0$, also $G^\top P G \succ 0$, such that it can be factorized as $G^\top P G = X_Q^\top X_Q$.

Multiplying (17) on the left with $X_W^{-\top}$ and on the right with X_W^{-1}, and substituting $G^\top PG = X_Q^\top X_Q$ results in

$$I - X_W^{-\top} H^\top(\rho) X_Q^\top X_Q H(\rho) X_W^{-1} \succ 0 \quad \forall \rho \in \mathbb{P}. \tag{18}$$

Now define $M(\rho) = X_Q H(\rho) X_W^{-1}$ such that $M(\rho)$ satisfies $I \succ M^\top(\rho)M(\rho)\ \forall\rho \in \mathbb{P}$. Then $M(\rho)$ is related to $H(\rho)$ by $H(\rho) = X_Q^{-1} M(\rho) X_W$ and thus to $K(\rho)$ by (12) giving

$$K(\rho) = (G^\top PG)^{-1} G^\top PF + X_Q M(\rho) X_W^{-1}. \tag{19}$$

(10) $\Leftarrow$ (15)-(16): Given any $W \succ 0$, by controllability of F, G, Riccati equation (15) has a unique positive definite solution $P \succ 0$ [24]. Then, given this $P \succ 0$ and $M(\rho)$, construct $K(\rho)$ as in (16) and substitute it in (10) to obtain

$$\begin{aligned} &P - (F - GK(\rho))^\top P(F - GK(\rho)) = P - F^\top PF \\ &+ F^\top PG\left((G^\top PG)^{-1} G^\top PF + X_Q^{-1} M(\rho) X_W\right) \\ &+ \left(F^\top PG(G^\top PG)^{-1} + X_W^\top M^\top(\rho) X_Q^{-\top}\right) G^\top PF \\ &- F^\top PG(G^\top PG)^{-1} G^\top PG(G^\top PG)^{-1} G^\top PF \\ &- X_W^\top M^\top(\rho) X_Q^{-\top} G^\top PF - F^\top PG X_Q^{-1} M(\rho) X_W \\ &- X_W^\top M^\top(\rho) X_Q^{-\top} G^\top PG X_Q^{-1} M(\rho) X_W, \end{aligned} \tag{20}$$

which can be simplified to

$$\begin{aligned} &P - F^\top PF + F^\top PG(G^\top PG)^{-1} G^\top PF \\ &- X_W^\top M^\top(\rho) X_Q^{-\top} G^\top PG X_Q^{-1} M(\rho) X_W. \end{aligned} \tag{21}$$

Using (15) and $G^\top PG = X_Q^\top X_Q$, this is equivalent to

$$W - X_W^\top M^\top(\rho) M(\rho) X_W = X_W^\top (I - M^\top(\rho) M(\rho)) X_W \succ 0,$$

where positive-definiteness follows by $M^\top(\rho)M(\rho) \prec I\ \forall\rho \in \mathbb{P}$, i.e., the constructed $K(\rho), P$ satisfy (9). $\square$

Theorem 7 can be interpreted as a small-gain result, stating that $M(\rho)$ should be norm-bounded by 1 for all ρ. Theorem 7 then states that all coefficient functions $K(\rho)$ that result in a quadratically stable LPV-IO model can be generated from transformed coefficient functions $M(\rho)$ constrained to the unit ball, and a rotation, scaling and translation of this unit ball based on P through (16). The set of allowable P is described by the image of the positive-definite cone $W \succ 0$ under Riccati equation (15). Last, since $\{W \mid W \succ 0\}$ and $\{M(\rho) \mid M^\top(\rho)M(\rho) \prec I\}$ are convex sets, Theorem 7 shows that the set of all quadratically stable LPV-IO models is convex in W and transformed coefficient functions $M(\rho)$.

C. Unconstrained Reparameterization of $K(\rho)$ for Stability

To obtain an unconstrained parameterization, the convex constraints on variables $W, M(\rho)$ of Theorem 7 are reparameterized in terms of unconstrained variables X_W related to W and $X_M(\rho), Z_M(\rho)$ related to $M(\rho)$. Specifically, the convex set of norm-bounded matrices is generated from free matrices through a Cayley transformation, as is defined next.

Lemma 8 *Given $M \in \mathbb{R}^{n\times m}$ with $n \geq m$. Then $M^\top M \prec I$ if and only if there exist $X_M, Y_M \in \mathbb{R}^{m\times m}$, $Z_M \in \mathbb{R}^{n-m\times m}$ with X_M full rank such that*

$$M = \begin{bmatrix} (I-N)(I+N)^{-1} \\ -2Z_M(I+N)^{-1} \end{bmatrix} \tag{22}$$

with $N = X_M^\top X_M + Y_M - Y_M^\top + Z_M^\top Z_M$.

Remark 9 *The rank condition on X_M is not restrictive: without loss of generality, X_M can be chosen upper triangular with its diagonal equal to e^d with $d \in \mathbb{R}^m$ a free vector.*

For a proof, see [14, Lemma 1]. Lemma 8 states that any matrix that is norm-bounded by 1 can be formed through a Cayley transformation of a passive matrix N, that is formed as a positive definite term $X^\top X + Z^\top Z$ and a skew-symmetric term $Y - Y^\top$. Given Lemma 8, the convex constraint $M^\top(\rho)M(\rho) \prec I$ of Theorem 7 is satisfied if and only if there exist unconstrained matrix functions related to $M(\rho)$ through this Cayley transformation, as shown next.

Theorem 10 *The LPV-IO model* (1) *satisfies* (10) *with $P \succ 0$ if and only if there exist $X_M(\rho): \mathbb{P} \to \mathbb{R}$, $Z_M(\rho): \mathbb{P} \to \mathbb{R}^{n_a-1\times 1}$, $X_W \in \mathbb{R}^{n_a\times n_a}$ with $X_M(\rho) \neq 0\ \forall\rho \in \mathbb{P}$ and X_W full rank such that*

$$W = X_W^\top X_W \quad M(\rho) = \begin{bmatrix} \mathrm{Cayley}(N(\rho)) \\ -2Z(\rho)(I+N(\rho))^{-1} \end{bmatrix}^\top, \tag{23}$$

with $N(\rho) = X_M^\top(\rho) X_M(\rho) + Z_M^\top(\rho) Z_M(\rho)$ and W, $M(\rho)$ satisfying (15)-(16).

The proof of Theorem 10 follows by combining Theorem 7 with Lemma 8. In the above theorem, $X_M(\rho)$ and $Z_M(\rho)$ can be any bounded function of ρ and the transformations (23),(15),(16) guarantee that the resulting LPV-IO model with $K(\rho)$ is stable, with $P \succ 0$ a Lyapunov function for $F - GK(\rho)$ as in (10). Thus, an unconstrained parameterization of all quadratically stable DT-LPV-IO models is obtained.

Remark 11 *Note that for a non-varying ρ, Theorem 10 gives an unconstrained reparameterization of all stable linear time-invariant DT transfer functions.*

Zooming out to model class (1) with coefficient functions $a_i(\rho), b_i(\rho)$ described by g_ϕ as in (2), Theorem 10 states that if a quadratically stable DT-LPV-IO model is desired, it can be represented in an unconstrained way by a g_ϕ that consists two elements: matrix functions $X_M(\rho), Z_M(\rho), L(\rho)$ that describe the coefficient functions in a transformed space, and a transformation T_{X_W} that maps $X_M(\rho), Z_M(\rho)$ to $K(\rho)$ dependent on X_W. Matrix functions $X_M(\rho), Z_M(\rho), L(\rho)$ can be any parameterized function, e.g., an affine map or neural network, see also Section II, and T_{X_W} ensures that the DT-LPV-IO model with the resulting $a_i(\rho), b_i(\rho)$ is QS. The QS DT-LPV-IO model is visualized in Fig. 1.

V. Application to System Identification

In this section, the unconstrained parameterization of quadratically stable DT-LPV-IO models of Theorem 10 is applied in a system identification setup[1].

A. LPV Output-Error System Identification Setup

The considered data-generating system with process component $\mathcal{G}: u_k, \rho_k \to \tilde{y}_k$ with $u_k, \rho_k, \tilde{y}_k \in \mathbb{R}$ is given by

$$u_k = m\delta^2 \tilde{y}_k + c\delta\tilde{y}_k + k(\rho_k)\tilde{y}_k \qquad y_k = \tilde{y}_k + v_k, \tag{24}$$

[1]The code used to generate the example can be found at `https://gitlab.tue.nl/kon/stable-lpv-io-estimation`

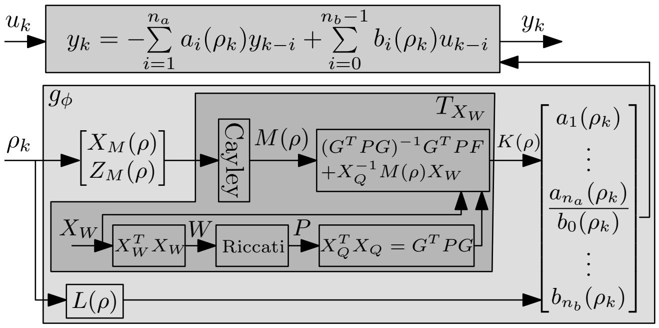

Fig. 1. Diagram of the stable DT-LPV-IO model class. For any value of X_W, and for any parameterized functions $X_M(\rho), Z_M(\rho), L(\rho)$, the coefficient functions $a_i(\rho), b_i(\rho)$ result in a stable DT-LPV-IO model.

with $y_k \in \mathbb{R}$ the measurement of the true output $\tilde{y}_k$ corrupted by i.i.d. white noise v with $\mathbb{E}(v^2) = \sigma_v^2$, resulting in an LPV output-error (OE) identification setup[2]. Here $\delta = (1 - q^{-1})/T_s$ is the backward difference operator, q is the forward-time shift, e.g., $qy_k = y_{k+1}$, and T_s is the sampling time. Consequently, $\mathcal{G}$ can be recognized as the Euler discretization of a mass-damper-spring system with parameter-varying stiffness $k(\rho)$ and fixed mass and damping m, d. By expanding δ, $\mathcal{G}$ can be written as (1) with $n_a = 2, n_b = 1$. Lastly, an LMI check shows that $\mathcal{G}$ is QS, such that it can be represented by a stable DT-LPV-IO model.

In this simulation example, $\rho \in \mathbb{P} = [0, 1]$, $T_s = 1$ and the true coefficient functions are given by

$$k(\rho) = 1 - (1 + e^{-7\rho+7})^{-1}, \tag{25}$$

and $m = 1$, $d = 0.1$, where $k(\rho)$ represents, e.g., spring softening as a function of temperature. These coefficient functions result in frozen LTI behaviours of $\mathcal{G}$ whose frequency responses are visualized in Fig. 2.

A dataset $\mathcal{D} = \{u_k, y_k, \rho_k\}_{k=1}^N$ of length $N = 1000$ samples is generated by $\mathcal{G}$ with $u_k = \sum_{i=1}^{10} \sin(2\pi f_i k/T_s)$, a multisine with frequencies f_i linearly spaced between 0.01 and 0.1 Hz, and $\rho_k = 1 - kN^{-1}$ a linear scheduling trajectory. The noise variance is set as $\sigma_v^2 = 0.1$ for signal-to-noise ratio $10 \log_{10}(\|\tilde{y}\|^2/\|v\|^2) = 19.5$ dB.

B. Model Parameterization and Identification Criterion

A stable DT-LPV-IO model with $n_a = 2, n_b = 1$ is chosen as a model for $\mathcal{G}$, i.e., the model has the same order as $\mathcal{G}$. Then any parameterization for $L(\rho), X_M(\rho_k), Z_M(\rho_k)$ can be considered, and the model can be optimized using prediction-error minimization based on unconstrained gradient-based optimization [6], [22].

Specifically, in this paper, the transformed coefficient functions $X_M(\rho), Z_M(\rho), L(\rho)$ are parameterized as

$$\begin{bmatrix} X_M & Z_M & L \end{bmatrix}(\rho) = E_2\sigma(E_1\sigma(E_0\rho + c_0)) + c_1) + c_2, \tag{26}$$

with $\sigma = \tanh$, $E_0 \in \mathbb{R}^{5\times1}, E_1 \in \mathbb{R}^{5\times5}, E_2 \in \mathbb{R}^{3\times5}, c_0 \in \mathbb{R}^5, c_1 \in \mathbb{R}^5, c_2 \in \mathbb{R}^3$, i.e., a neural network with 2 hidden layers of 5 nodes each and 3 outputs since $L(\rho_k), X_M(\rho_k), Z_M(\rho_k) \in \mathbb{R}$ for $n_a = 2, n_b = 1$. Consequently, the model parameters are $\phi = \mathrm{vec}(E_0, c_0, \ldots E_2, c_2, X_W) \in \mathbb{R}^{n_\phi}$ with $X_W \in \mathbb{R}^{2\times2}$ upper triangular such that $n_\phi = 58 + 3$.

[2]More generic noise model structures can easily be incorporated, but for ease of notation, an LPV-OE setting is considered.

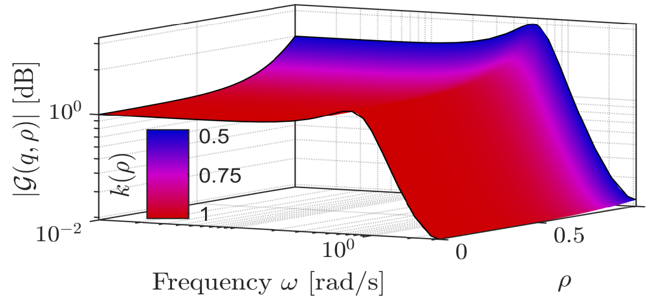

Fig. 2. Bode plot of frozen dynamics of $\mathcal{G}(\delta, \rho)$, displaying a resonance with scheduling-dependent stiffness.

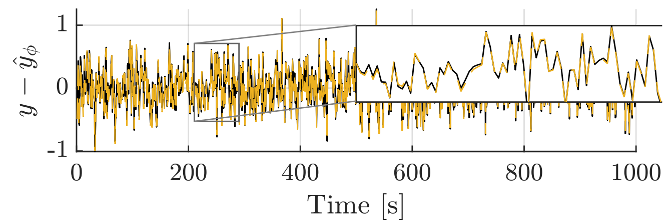

Fig. 3. Prediction residuals $y - \hat{y}_\phi$ (—) and noise realization $y - \tilde{y}$ (—) for the validation dataset. The prediction residuals coincide with the noise, i.e., the LPV-IO model (1) with neural network coefficient functions is able to learn all dynamics while simultaneously guaranteeing stability.

In the above OE setting with noiseless ρ_k, the model parameters ϕ are found by minimizing the ℓ_2 loss of the prediction error $V_N(\phi)$ as $\phi^* = \arg\min_\phi V_N(\phi)$ with

$$V_N(\phi) = \sqrt{\frac{1}{N} \sum_{k=1}^{N} (y_k - \hat{y}_{k,\phi})^2}, \tag{27}$$

where $\hat{y}_{k,\phi}$ is the simulated model response.

Criterion (27) is optimized using the Levenberg-Marquardt optimization algorithm, resulting in a training time of 37 seconds on an Z-book G5 using a Intel Core i7-8750H CPU.

C. Identification with Stability Guarantees

For the training dataset, the estimated parameter vector ϕ^* achieves $V_N(\phi^*) = 0.313$, corresponding to the noise level with $\sigma_v = 0.316$, illustrating that the only remaining contribution to $V_N(\phi^*)$ is noise that cannot be predicted. For a similar but different validation dataset, $V_N(\phi^*) = 0.320$, indicating that the model can generalize well. The residuals $y - \hat{y}_\phi$ for this validation dataset are shown in Fig. 3.

D. Visualization of Stability Sets

In this section, the evolution of the coefficient set that can be represented by the DT-LPV-IO model during the iterations of the optimization is visualized, i.e., the set K_P in which $a_i(\rho), b_i(\rho)$ can take values $\forall\rho$ by construction, see Theorem 10. Specifically, given X_W during optimization, all coefficients $K(\rho) = \begin{bmatrix} a_1(\rho) & a_2(\rho) \end{bmatrix}$ corresponding to this X_W can be constructed using (15),(16),(23), resulting in the

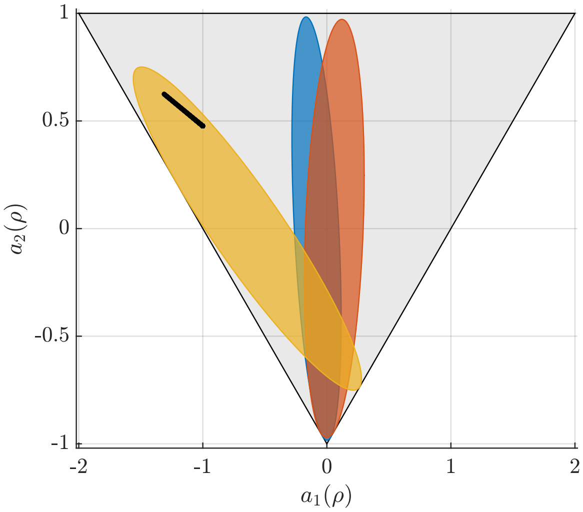

Fig. 4. Coefficient set $K_P = \{K(\rho) = [a_1(\rho), a_2(\rho)] \mid P \succ 0, (F - GK(\rho))^\top P(F - GK(\rho)) - P \prec 0 \; \forall \rho \in \mathbb{P}\}$, i.e., the set of all possible values $K(\rho)$ such that LPV IO-model (5) is stable with Lyapunov certificate P at iteration 1 (▭), 10 (▭), and 100 (▭) of the optimization. P is optimized such that the true coefficient functions (▬) are contained within the stable coefficient set. Optimizing P thus corresponds to rotating, scaling and translating this set. All stable coefficient sets are included in the stability triangle that describes the stable coefficients for the LTI case (▭), and the full triangle can be represented by varying P.

coefficient sets K_P of Fig. 4. The following observations are made.

- Mapping (16) corresponds to scaling, rotating and translating the unit ball $\|M(\rho)\|_2^2 < 1$, resulting in the ellipsoidal shape of K_P. Thus, P can be thought of as describing all possible rotations, translations and scalings of $K(\rho)$ such that LPV-IO model (1) is stable.
- Graphically, each K_P visualizes a set in which the function $K(\rho)$ can generate outputs for the LPV-IO model to be stable. Thus, the true coefficient functions necessarily have to be fully contained in a K_P for $K(\rho)$ to be able to describe them. Thus, optimizing X_W is equivalent to transforming the ellipsoid such that it encapsulates the true coefficient functions.
- For an LPV-IO model to be quadratically stable, a K_P, corresponding to some P, must exist which fully encapsulates the true coefficient functions. Fig. 4 thus provides a graphical tool for accessing stability properties of an LPV-IO model.

VI. Conclusion

In this paper, the class of all quadratically stable DT-LPV-IO models is reparameterized in terms of unconstrained model parameters. This unconstrained parameterization is achieved through reparameterizing the quadratic stability condition in a necessary and sufficient way through a Riccati equation and a Cayley transformation. The parameterization allows for using arbitrary dependency of the scheduling coefficients on the scheduling signal ρ, e.g., a polynomial or neural network dependency.

The resulting stable DT-LPV-IO model class enables system identification with a priori stability guarantees on the identified model in the presence of modeling errors and measurement noise. Since it does not require enforcing an LMI condition during estimation, models within this class can be identified using standard unconstrained optimization routines, significantly decreasing the computational complexity.

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