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Data-Driven Reduced-Order Unknown-Input Observers

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Abstract— **In this paper we propose a data-driven approach to the design of reduced-order unknown-input observers (rUIOs). We first recall the model-based solution, by assuming a problem set-up slightly different from those traditionally adopted in the literature, in order to be able to easily adapt it to the data-driven scenario. Necessary and sufficient conditions for the existence of a reduced-order unknown-input observer, whose matrices can be derived from a sufficiently rich set of collected historical data, are first derived and then proved to be equivalent to the ones obtained in the model-based framework. Finally, a numerical example is presented, to validate the effectiveness of the proposed scheme.**

I. INTRODUCTION

Since the seventies, the control community has put lots of efforts in finding solutions to the state estimation problem in the presence of unknown inputs acting on the system. Several methods have been employed, ranging from algebraic [12], [13], [19] and geometric [2] methods to generalized inverse approaches [14], [15]. The majority of the existing solutions requires the perfect knowledge of the system, namely that the matrices involved in the process description are available. However, in practical situations this is not always the case and very often one has to deal with black box models, relying only on the information provided by the inputs and the outputs of the system. On the other hand, nowadays, in the big data era, large amounts of data can be collected and used to get insights into the process that has generated them. Hence data-driven techniques in the field of control theory have gained increasing attention. Data-driven methods have been proposed, in particular, to tackle the state estimation problem [4], [8], [16], and more specifically the unknown-input state estimation problem. In particular, in [17], a novel data-driven unknown-input observer (UIO), based on behavioral system theory and the result known as Willems' Fundamental Lemma [20], has been proposed. Necessary and sufficient conditions for the existence of a UIO that makes the state estimation error converge asymptotically to zero, regardless of the unknown inputs, have been derived, based on data. In [9] the design of full-order UIOs has been further explored, by providing weaker conditions for problem solvability, and a complete parametrization of the UIOs one can derive from a given set of historical data. Moreover, it has been shown that the data-driven approach provides a problem solution under the same conditions under which a UIO can be derived from the complete knowledge of the system matrices. The algorithms proposed in [9], [17] are purely data-driven, namely they do not require any preliminary identification step.

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When the system whose state we want to estimate is extremely complex, implementing a full order UIO may be particularly demanding. Indeed, reduced-order observers have been widely investigated, from a model-based perspective, due to their parsimonious nature that is always a desirable characteristic in engineering applications. In [5], [10], a reduced-order unknown-input state estimator (whose dimension is equal to the difference between the dimension of the state and the dimension of the unknown input) has been proposed, by first eliminating the effect of the unknown input on part of the state variables, and then designing a conventional Luenberger observer for the subsystem driven by known inputs only. A uniform design procedure for constructing reduced-order unknown-input observers (rUIOs) of order either equal to the difference between the dimension of the state and the dimension of the unknown input, or to the difference between the dimension of the state and the dimension of the output, has been proposed in [11]. The second type of reduced-order unknown-input observers has been investigated also in [3], [13]. However, to the best of the authors' knowledge, rUIOs have never been addressed from a data-driven perspective.

In this paper we propose a data-driven approach to the design of reduced-order unknown-input observers, by adopting a hybrid solution, since we first identify from data the output matrix of the data-generating system and then we leverage solely the collected data to design the rUIO. The result is not only an algorithm for state estimation of lower complexity with respect to the full-order ones, but also a less demanding procedure to generate the observer matrices from the collected data, due to their lower dimensions.

The results proposed in this paper clearly bear similarities with those derived in [9], [17] where a data-driven approach to the design of full-order UIOs is proposed. However, adapting the traditional model-based methods for rUIO design to the data-driven context is not immediate. Indeed, classic model-based techniques have either introduced the restrictive hypothesis that the output variables are a subset of the state variables (see, e.g., [13]), or have resorted to a generic change of basis in the state space [11], which would be difficult to extend to the data-driven approach. So, the first step has been to revise the model-based solution to the problem, in such a way that its extension to, and comparison with, the one we propose based on collected data is possible. The necessary and sufficient conditions for the existence of a reduced-order unknown-input observer provided, e.g., in [11], [13], hold also in our setting, but need to be particularized to our specific description of the system. Based on them, we propose a data-driven algorithm to solve the problem. More in detail, we provide necessary and sufficient conditions for the existence of a reduced-order data-driven UIO and we show that they are actually equivalent to the ones obtained in the model-based approach. This means that the data-driven implementation does not impose additional assumptions that would be unnecessary if we knew the system matrices. On the other hand, the possibility to effectively design an rUIO from data, by avoiding redundancy and minimizing the computational effort, without affecting the estimation performance, is quite important from a practical point of view.

The paper is organized as follows. Section II introduces the rUIO design problem and presents a revised solution in the model-based framework. Section III proposes the problem solution by using a data-driven approach. Finally, a numerical example illustrates the paper results.

Notation. Given a matrix $M \in \mathbb{R}^{p\times m}$, we denote by $M^\dagger \in \mathbb{R}^{m\times p}$ its *Moore-Penrose inverse* [1]. Note that if M is of full row rank, then $M^\dagger = M^\top(MM^\top)^{-1}$. The null and column spaces of M are denoted by $\ker(M)$ and $\mathrm{Im}(M)$, respectively. Given a vector sequence $v(t) \in \mathbb{R}^n$, where $t \in \mathbb{Z}_+$, we use the notation $\{v(t)\}_{t=0}^N$, $N \in \mathbb{Z}_+$, to indicate the sequence of vectors $v(0), \dots, v(N)$.

II. Problem Formulation

Consider a discrete-time linear time-invariant (LTI) system Σ, described by the following equations:

$$x(t+1) = Ax(t) + Bu(t) + Ed(t), \tag{1a}$$
$$y(t) = Cx(t), \tag{1b}$$

where $t \in \mathbb{Z}_+$, $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the (known) control input, $d(t) \in \mathbb{R}^q$ is the unknown input or disturbance, and $y(t) \in \mathbb{R}^p$ is the output. The dimensions of the system matrices are omitted, as they can be deduced from the dimensions of the system variables. The analysis carried out in this paper would still hold, with minor changes, if we replaced the output equation in (1b) with $y(t) = Cx(t) + Du(t)$. However, in order not to make the subsequent calculations unnecessarily involved, in the following we assume that $y(t)$ only depends on $x(t)$. Without loss of generality (w.l.o.g.), we assume that $E \in \mathbb{R}^{n\times q}$ is of full column rank, and that $C \in \mathbb{R}^{p\times n}$ is of full row rank, i.e., $\mathrm{rank}(E) = q$ and $\mathrm{rank}(C) = p$. If E is not of full column rank, we can redefine the disturbance vector. On the other hand, if C does not have full row rank, we can neglect redundant measurements. We assume w.l.o.g. that $C = [\ C_1 \mid C_2\ ]$, with $C_1 \in \mathbb{R}^{p\times(n-p)}$ and $C_2 \in \mathbb{R}^{p\times p}$ nonsingular, so that after partitioning the state vector as $x(t) = [\ x_1(t)^\top \ x_2(t)^\top]^\top$, where $x_1(t) \in \mathbb{R}^{n-p}$ and $x_2(t) \in \mathbb{R}^p$, the output equation in (1b) becomes

$$y(t) = C_1x_1(t) + C_2x_2(t). \tag{2}$$

By premultiplying both sides of (2) by C_2^{-1}, we obtain

$$x_2(t) = C_2^{-1}y(t) - C_2^{-1}C_1x_1(t). \tag{3}$$

Therefore, if we can estimate the first part of the state vector, namely $x_1(t)$, we can easily recover also the remaining state variables by making use of (3).

Definition 1. *An LTI system $\hat{\Sigma}$ of order $n-p$, described by the equations*

$$z(t+1) = A_{UIO}z(t) + B^u_{UIO}u(t) + B^y_{UIO}y(t), \tag{4a}$$
$$\hat{x}_1(t) = z(t) + D_{UIO}y(t), \tag{4b}$$
$$\hat{x}_2(t) = -C_2^{-1}C_1z(t) + (C_2^{-1} - C_2^{-1}C_1D_{UIO})y(t), \tag{4c}$$

where $t \in \mathbb{Z}_+$, $z(t) \in \mathbb{R}^{n-p}$ is the state and $\hat{x}(t) = [\hat{x}_1(t)^\top \ \hat{x}_2(t)^\top]^\top \in \mathbb{R}^n$ is the output, is a reduced-order unknown-input observer (rUIO) *for system Σ in (1) if $e(t) \triangleq x(t) - \hat{x}(t)$ asymptotically converges to zero for every choice of $z(0)$ and every input/output pair $(\{u(t)\}_{t\in\mathbb{Z}_+}, \{y(t)\}_{t\in\mathbb{Z}_+})$ of the system (1).*

In other words, a reduced-order unknown-input observer is an LTI system of dimension lower than the dimension of the system Σ that, when fed by the input/output trajectories of Σ, generated corresponding to an arbitrary $x(0)$ and an arbitrary disturbance $d(t)$, provides as its output an asymptotic estimate of the state of Σ, independently of its initial condition $z(0)$.

Clearly, by the way we have defined it, a reduced-order UIO $\hat{\Sigma}$ exists if and only if a full-order UIO for $x_1(t)$ alone, described by (4a) and (4b) (see [6], [7]), exists. The "only if" part is obvious. Conversely, if the observer (4a) - (4b) ensures that $e_1(t) \triangleq x_1(t) - \hat{x}_1(t)$ converges to zero asymptotically, then by making use of (3) and (4c) we can ensure that

$$e_2(t) \triangleq x_2(t) - \hat{x}_2(t) = -C_2^{-1}C_1e_1(t) \tag{5}$$

converges to zero, in turn. So, from now on we will focus on the UIO (4a)-(4b). To design it, it is convenient to partition all the matrices of the system in (1) conformably with the block partition of C, namely as

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22}\end{array}\right], \quad B = \left[\begin{array}{c} B_1 \\ \hline B_2\end{array}\right], \quad E = \left[\begin{array}{c} E_1 \\ \hline E_2\end{array}\right].$$

By splitting the dynamics of the two parts of the state vector, we can rewrite equation (1a) as

$$x_1(t+1) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) + E_1d(t) \tag{6a}$$
$$x_2(t+1) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) + E_2d(t). \tag{6b}$$

If we now substitute equation (3) in (6a), we get

$$\begin{aligned} x_1(t+1) &= [A_{11} - A_{12}C_2^{-1}C_1]x_1(t) + A_{12}C_2^{-1}y(t) \\ &+ B_1u(t) + E_1d(t). \end{aligned} \tag{7}$$

We can now provide necessary and sufficient conditions for a state-space model described as in (4) to represent an rUIO for system Σ.

Proposition 2. *The system in (4) is an rUIO for system Σ*

if and only if the following conditions hold:

$$A_{UIO} = (I - D_{UIO}C_1)(A_{11} - A_{12}C_2^{-1}C_1) - D_{UIO}C_2(A_{21} - A_{22}C_2^{-1}C_1) \tag{8a}$$

$$B^u_{UIO} = (I - D_{UIO}C_1)B_1 - D_{UIO}C_2B_2 \tag{8b}$$

$$B^y_{UIO} = A_{UIO}D_{UIO} + (I - D_{UIO}C_1)A_{12}C_2^{-1} - D_{UIO}C_2A_{22}C_2^{-1} \tag{8c}$$

$$(I - D_{UIO}C_1)E_1 - D_{UIO}C_2E_2 = 0 \tag{8d}$$

$$A_{UIO} \text{ is Schur stable.} \tag{8e}$$

When so, the state estimation error on $x_1(t)$ obeys the autonomous asymptotically stable dynamics

$$e_1(t+1) = A_{UIO}e_1(t). \tag{9}$$

Proof. By making use of equations (3), (6b) and (7), and the UIO description in (4a)-(4b), we derive the dynamics of $e_1(t) = x_1(t) - \hat{x}_1(t)$, i.e.,

$$\begin{aligned}
e_1(t+1) &= x_1(t+1) - \hat{x}_1(t+1) \\
&= x_1(t+1) - z(t+1) - D_{UIO}y(t+1) \\
&= x_1(t+1) - A_{UIO}z(t) - B^u_{UIO}u(t) - B^y_{UIO}y(t) \\
&\quad - D_{UIO}C_1x_1(t+1) - D_{UIO}C_2x_2(t+1) \\
&= A_{UIO}e_1(t) + \big[(I - D_{UIO}C_1)(A_{11} - A_{12}C_2^{-1}C_1) \\
&\quad - A_{UIO} - D_{UIO}C_2A_{21} + D_{UIO}C_2A_{22}C_2^{-1}C_1\big]x_1(t) \\
&\quad + \big[(I - D_{UIO}C_1)A_{12}C_2^{-1} + A_{UIO}D_{UIO} - B^y_{UIO} \\
&\quad - D_{UIO}C_2A_{22}C_2^{-1}\big]y(t) \\
&\quad + \big[(I - D_{UIO}C_1)B_1 - D_{UIO}C_2B_2 - B^u_{UIO}\big]u(t) \\
&\quad + \big[(I - D_{UIO}C_1)E_1 - D_{UIO}C_2E_2\big]d(t).
\end{aligned}$$

Therefore, $e_1(t)$ is independent of the disturbance $d(t)$ and asymptotically convergent to zero, for every choice of $u(t), t \in \mathbb{Z}_+$, $x(0)$ and $z(0)$, if and only if conditions (8a)÷(8e) hold. The second statement is obvious. □

We now introduce the concept of *acceptor*, previously adopted in the context of behavior theory [18]. Roughly speaking, a system $\tilde{\Sigma}$ is an acceptor for system Σ if it receives as its input trajectories the input/output trajectories generated by Σ and admits among its possible outputs the state trajectory that Σ generates corresponding to that specific input/output pair.

Definition 3. *Given system Σ, described by the equations (1), we say that an LTI system $\tilde{\Sigma}$ described by*

$$\begin{aligned}
z(t+1) &= \tilde{A}z(t) + \tilde{B}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \\
\tilde{x}(t) &= \tilde{C}z(t) + \tilde{D}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix},
\end{aligned}$$

where $t \in \mathbb{Z}_+$, $z(t) \in \mathbb{R}^{n_z}$ is the state of the system, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the system inputs, and $\tilde{x}(t) \in \mathbb{R}^n$ is the output, is an acceptor *for Σ if for every input/output/state trajectory $(\{u(t)\}_{t\in\mathbb{Z}_+}, \{y(t)\}_{t\in\mathbb{Z}_+}, \{x(t)\}_{t\in\mathbb{Z}_+})$ generated by Σ, there exists an initial condition $z(0)$ for $\tilde{\Sigma}$ such that $\{\tilde{x}(t)\}_{t\in\mathbb{Z}_+} = \{x(t)\}_{t\in\mathbb{Z}_+}$ is the output of $\tilde{\Sigma}$ corresponding to the input pair $(\{u(t)\}_{t\in\mathbb{Z}_+}, \{y(t)\}_{t\in\mathbb{Z}_+})$ and the initial condition $z(0)$.*

The following result, that will be used later in the paper, formalizes the fact that a system described as in (4) is an acceptor for Σ if and only if conditions (8a)÷(8d) hold.

Lemma 4. *Given system Σ, described by the equations (1), an LTI system described as in (4) is an acceptor for Σ if and only if its matrices satisfy (8a)÷(8d).*

Proof. If conditions (8a)÷(8d) hold, the dynamics of the estimation error is described as in (9). Let $(\{u(t)\}_{t\in\mathbb{Z}_+}, \{y(t)\}_{t\in\mathbb{Z}_+}, \{x(t)\}_{t\in\mathbb{Z}_+})$ be an input/output/state trajectory generated by Σ. If we assume $z(0) = x_1(0) - D_{UIO}y(0)$ then $\hat{x}_1(0) = x_1(0)$ and $e_1(0) = 0$. Therefore $e_1(t) = 0$ for every $t \in \mathbb{Z}_+$. This ensures that e_2 is identically zero, in turn, and hence $\hat{x}(t) = x(t)$ for every $t \in \mathbb{Z}_+$.

Conversely, if at least one of the conditions (8a)÷(8d) does not hold, the dynamics of the estimation error is not that of an autonomous system. So, it is always possible to find an initial condition $x(0)$ and input signals $\{u(t)\}_{t\in\mathbb{Z}_+}$ and $\{d(t)\}_{t\in\mathbb{Z}_+}$ such that for every $z(0)$ the estimation error is not identically zero. □

It is worth remarking that while a UIO is always an acceptor, the converse holds if and only if also condition (8e) holds.

We now provide necessary and sufficient conditions for the solvability of (8) and thus for the existence of a reduced-order UIO (4). In [11] these conditions have been derived under the assumption that $C = [\ 0 \mid I_p\ ]$, a situation we can always reduce ourselves to by resorting to a suitable change of basis (see [13]). Therefore, in the following lemma we only recall these conditions without providing the proof.

Lemma 5. *There exist matrices $A_{UIO}, B^u_{UIO}, B^y_{UIO}$ and D_{UIO} of suitable sizes that satisfy conditions (8a)÷(8a), and hence there exists an rUIO of the form (4), if and only if*

(a) $\operatorname{rank}(CE) = \operatorname{rank}(E) = q$, *and*

(b) $\operatorname{rank}\begin{bmatrix} zI_n - A & -E \\ C & 0 \end{bmatrix} = n + q,\ \forall z \in \mathbb{C}, |z| \geq 1,$

or, equivalently (see Theorem 2 in [6]), the triple (A, C, E) is strong detectable [6].*

The conditions stated in the previous lemma are the same conditions that guarantee the existence of a full-order unknown-input observer [6], [7]. So, there exists a reduced-order UIO if and only if there exists a full-order UIO. We are now ready to formalize the problem we want to solve.

Problem. *Given system Σ described as in (1), with unknown matrices, design (if possible) a data-driven rUIO for Σ, described by equations (4), by making use only of some data collected during an offline finite time experiment.*

III. DATA-DRIVEN REDUCED-ORDER UIO

As in [9], [17], we suppose that the system matrices are unknown and that we have performed an *offline* experiment during which we have collected some input/output/state trajectories in the time-interval $[0, T-1]$, with $T \in \mathbb{Z}_+$ sufficiently large. It has already been highlighted in [9], [17] that assuming to have access to the state, during the offline experiment, is necessary, since it would not be possible to uniquely identify the state of the system, and hence to construct a UIO, without knowing the dimension and the basis of the state-space. The input/output/state trajectories can be represented by the following sequences of vectors, i.e., $u_d = \{u_d(t)\}_{t=0}^{T-2}$, $y_d = \{y_d(t)\}_{t=0}^{T-1}$ and $x_d = \{x_d(t)\}_{t=0}^{T-1}$. Even if we cannot measure the disturbance $d(t)$, it is however convenient to define also the sequence of historical unknown input data, namely $d_d = \{d_d(t)\}_{t=0}^{T-2}$. For the subsequent analysis, we group the data into the following matrices:

$$
\begin{aligned}
U_p &\triangleq \begin{bmatrix} u_d(0) & \dots & u_d(T-2) \end{bmatrix} \in \mathbb{R}^{m\times(T-1)},\\
X_p &\triangleq \begin{bmatrix} x_d(0) & \dots & x_d(T-2) \end{bmatrix} \in \mathbb{R}^{n\times(T-1)},\\
X_f &\triangleq \begin{bmatrix} x_d(1) & \dots & x_d(T-1) \end{bmatrix} \in \mathbb{R}^{n\times(T-1)},\\
Y_p &\triangleq \begin{bmatrix} y_d(0) & \dots & y_d(T-2) \end{bmatrix} \in \mathbb{R}^{p\times(T-1)},\\
Y_f &\triangleq \begin{bmatrix} y_d(1) & \dots & y_d(T-1) \end{bmatrix} \in \mathbb{R}^{p\times(T-1)},\\
D_p &\triangleq \begin{bmatrix} d_d(0) & \dots & d_d(T-2) \end{bmatrix} \in \mathbb{R}^{q\times(T-1)}.
\end{aligned}
$$

where the subscripts p and f stand for past and future, respectively. In the following, we will make use of the following matrices

$$
\Phi \triangleq \begin{bmatrix} U_p \\ Y_p \\ Y_f \\ X_p \end{bmatrix}, \quad \text{and} \quad \Phi_1 \triangleq \begin{bmatrix} U_p \\ Y_p \\ Y_f \\ X_{p,1} \end{bmatrix}. \tag{10}
$$

Before providing the data-driven formulation of the reduced-order UIO, we introduce the following assumption (the same we adopted in [9] for full-order UIOs, and that follows from more restrictive assumptions of persistence of excitation of the input sequences u_d and d_d):

Assumption: The matrix $\begin{bmatrix} U_p^\top & D_p^\top & X_p^\top \end{bmatrix}^\top$ is of full row rank, i.e., $m+q+n$.

Since the historical data have been generated by the system Σ, they have to satisfy (1) and in particular it must hold

$$
Y_p = C X_p. \tag{11}
$$

Under the previous Assumption, the matrix X_p is of full row rank and thus admits a right inverse. Therefore,

$$
C = Y_p X_p^\dagger = Y_p X_p^\top (X_p X_p^\top)^{-1}.
$$

Once we have recovered the matrix C from the output/state data, we can also check if it has full row rank. If not, we can discard the measurements that are linearly dependent on the others. Again, there is no loss of generality in assuming that C can be block-partitioned as $C = [\, C_1 \mid C_2 \,]$, where C_2 is nonsingular square. Now that we have the matrix C along with its partition, we can split the generic state vector $x_d(t)$, belonging to the sequence of historical data x_d, into two blocks $x_d(t) = \begin{bmatrix} x_{d,1}(t)^\top & x_{d,2}(t)^\top \end{bmatrix}^\top$, conformably with the block partition of C. Consequently, the matrices of the state data split into two parts, namely for $i = 1, 2$,

$$
\begin{aligned}
X_{p,i} &\triangleq \begin{bmatrix} x_{d,i}(0) & \dots & x_{d,i}(T-2) \end{bmatrix} \in \mathbb{R}^{n_i\times(T-1)},\\
X_{f,i} &\triangleq \begin{bmatrix} x_{d,i}(1) & \dots & x_{d,i}(T-1) \end{bmatrix} \in \mathbb{R}^{n_i\times(T-1)},
\end{aligned}
$$

where $n_1 = n - p$ and $n_2 = p$, and the following identities hold

$$
\begin{aligned}
X_{p,2} &= C_2^{-1} Y_p - C_2^{-1} C_1 X_{p,1}\\
X_{f,2} &= C_2^{-1} Y_f - C_2^{-1} C_1 X_{f,1}.
\end{aligned}
$$

When dealing with data-driven techniques, it is important that the collected data are representative of the underlying system. The following definition captures this concept.

Definition 6. *[9], [17] The set of (input/output/state) trajectories $(\{u(t)\}_{t\in\mathbb{Z}_+}, \{y(t)\}_{t\in\mathbb{Z}_+}, \{x(t)\}_{t\in\mathbb{Z}_+})$ is said to be* compatible with the historical data (u_d, y_d, x_d) *if*

$$
\begin{bmatrix} u(t) \\ y(t) \\ x(t) \\ x(t+1) \end{bmatrix} \in \operatorname{Im}\left(\begin{bmatrix} U_p \\ Y_p \\ X_p \\ X_f \end{bmatrix}\right), \quad \forall t \in \mathbb{Z}_+. \tag{12}
$$

Under the Assumption, it has been proved in [9] (see, also, [17]) that the trajectories generated by the system Σ in (1) are all and only those compatible with the given historical data. In [17, Lemma 2], it has also been shown that there exists an acceptor of order n for Σ described by

$$
\begin{aligned}
z(t+1) &= A_{UIO} z(t) + B_{UIO}^u u(t) + B_{UIO}^y y(t),\\
\hat{x}(t) &= z(t) + D_{UIO} y(t),
\end{aligned}
$$

if and only if

$$
\ker(X_f) \supseteq \ker(\Phi). \tag{13}
$$

This result, applied to the reduced-order scenario, leads to the following proposition (whose proof can be obtained by suitably adjusting that of Lemma 9 in [9]).

Proposition 7. *There exists an acceptor described as in (4) for Σ (equivalently, an acceptor described by (4a)-(4b) for the trajectories $(\{x_1(t)\}_{t\in\mathbb{Z}_+}, \{u(t)\}_{t\in\mathbb{Z}_+}, \{y(t)\}_{t\in\mathbb{Z}_+})$ of Σ), whose matrices are built using the collected data $U_p, Y_p, Y_f, X_{p,1}$, if and only if*

$$
\ker(X_{f,1}) \supseteq \ker(\Phi_1). \tag{14}
$$

If so, for every $[\, S_1 \mid S_2 \mid S_3 \mid S_4 \,] \in \mathbb{R}^{n\times(m+2p+n-p)}$ satisfying

$$
X_{f,1} = [\, S_1 \mid S_2 \mid S_3 \mid S_4 \,]\, \Phi_1 \tag{15}
$$

the matrices of an acceptor can be expressed in terms of the matrices S_1, S_2, S_3 and S_4 as

$$
\begin{aligned}
A_{UIO} &\triangleq S_4, & B_{UIO}^u &\triangleq S_1,\\
B_{UIO}^y &\triangleq S_2 + S_4 S_3, & D_{UIO} &\triangleq S_3.
\end{aligned}
$$

Conversely, for every acceptor described by the matrices A_{UIO}, B^u_{UIO}, B^y_{UIO} and D_{UIO}, we can obtain a solution of (15) by assuming

$$S_1 \triangleq B^u_{UIO}, \quad S_2 \triangleq B^y_{UIO} - A_{UIO}D_{UIO},$$
$$S_3 \triangleq D_{UIO}, \quad S_4 \triangleq A_{UIO}.$$

Remark 8. *From Lemma 4 and Proposition 7, it follows that the kernels inclusion in (14) corresponds exactly to conditions (8a)÷(8d) derived in the model-based approach, by imposing the decoupling from all the exogenous variables in the estimation error dynamics. Indeed, from Proposition 7 it follows that (14) is equivalent to the existence of matrices A_{UIO}, B^u_{UIO}, B^y_{UIO} and D_{UIO} such that*

$$X_{f,1} = \left[\, B^u_{UIO} \,\middle|\, B^y_{UIO} - A_{UIO}D_{UIO} \,\middle|\, D_{UIO} \,\middle|\, A_{UIO} \,\right] \Phi_1$$

If we now exploit the fact that the data have been generated by the system Σ, we can substitute Y_f in the previous equation with the following expression $Y_f = C_1X_{f,1} + C_2X_{f,2} = C_1(A_{11}X_{p,1} + A_{12}X_{p,2} + B_1U_p + E_1D_p) + C_2(A_{21}X_{p,1} + A_{22}X_{p,2} + B_2U_p + E_2D_p) = [C_1(A_{11} - A_{12}C_2^{-1}C_1) + C_2(A_{21} - A_{22}C_2^{-1}C_1)]X_{p,1} + CBU_p + (C_1A_{12}C_2^{-1} + C_2A_{22}C_2^{-1})Y_p + CED_p$ and obtain

$$\begin{aligned} X_{f,1} = \; & [B^u_{UIO} + D_{UIO}CB \mid D_{UIO}CE \mid B^y_{UIO} \\ & -A_{UIO}D_{UIO} + D_{UIO}(C_1A_{12} + C_2A_{22})C_2^{-1} \mid \\ & A_{UIO} + D_{UIO}[C_1(A_{11} - A_{12}C_2^{-1}C_1) \\ & + C_2(A_{21} - A_{22}C_2^{-1}C_1)]] \begin{bmatrix} U_p \\ D_p \\ Y_p \\ X_{p,1} \end{bmatrix}. \end{aligned} \tag{16}$$

At the same time, the historical data have to satisfy the equations of system Σ (see (7)) and thus

$$X_{f,1} = \left[B_1 | E_1 | A_{12}C_2^{-1} | A_{11} - A_{12}C_2^{-1}C_1\right] \begin{bmatrix} U_p \\ D_p \\ Y_p \\ X_{p,1} \end{bmatrix}. \tag{17}$$

Since the matrix $\begin{bmatrix} U_p^\top & D_p^\top & Y_p^\top & X_{p,1}^\top \end{bmatrix}^\top$ is of full row rank [1]*, by equating the right hand side of (16) and (17) we obtain exactly the conditions in (8a)÷(8d).*

Next, we show that the two conditions in (13) and (14) are actually equivalent, namely we can build a reduced-order acceptor for the input/output/state trajectories of system Σ if and only if we can build a full order acceptor for the same system.

Proposition 9. *Under the Assumption on the data and the hypothesis that C_2 is nonsingular, conditions (13) and (14) are equivalent.*

Proof. Condition (13) implies that

$$X_f = \left[\, T_1 \mid T_2 \mid T_3 \mid T_4 \,\right] \Phi \tag{18}$$

[1] The full row rank property of the matrix $\begin{bmatrix} U_p^\top & D_p^\top & Y_p^\top & X_{p,1}^\top \end{bmatrix}^\top$ follows directly from the Assumption on the data and the full row rank property of the matrix C.

holds for some $\left[\, T_1 \mid T_2 \mid T_3 \mid T_4 \,\right] \in \mathbb{R}^{n\times(m+2p+n)}$. We partition the matrix T_4 conformably with the partition of the vector x, namely $T_4 = \left[\, T_{41} \mid T_{42} \,\right]$, with $T_{41} \in \mathbb{R}^{n\times(n-p)}$ and $T_{42} \in \mathbb{R}^{n\times p}$, and we observe that $X_{p,2} = C_2^{-1}Y_p - C_2^{-1}C_1X_{p,1}$. This implies that

$$\begin{aligned} T_4X_p &= \left[\, T_{41} \mid T_{42} \,\right] \begin{bmatrix} X_{p,1} \\ C_2^{-1}Y_p - C_2^{-1}C_1X_{p,1} \end{bmatrix} \\ &= (T_{41} - T_{42}C_2^{-1}C_1)X_{p,1} + T_{42}C_2^{-1}Y_p, \end{aligned}$$

yielding

$$X_{f,1} = [I_{n-p} \;\; 0]\left[T_1 | T_2 + T_{42}C_2^{-1} | T_3 | T_{41} - T_{42}C_2^{-1}C_1\right]\Phi_1$$

and hence (14) holds.

Conversely, condition (14) implies that there exists $\left[\, S_1 \mid S_2 \mid S_3 \mid S_4 \,\right] \in \mathbb{R}^{(n-p)\times(m+2p+n-p)}$ s.t.

$$X_{f,1} = \left[\, S_1 \mid S_2 \mid S_3 \mid S_4 \,\right] \Phi_1.$$

Moreover, we have

$$\begin{aligned} X_{f,2} &= C_2^{-1}Y_f - C_2^{-1}C_1X_{f,1} \\ &= C_2^{-1}Y_f - C_2^{-1}C_1 \left[\, S_1 \mid S_2 \mid S_3 \mid S_4 \,\right] \Phi_1 \\ &= \left[-C_2^{-1}C_1S_1 \mid -C_2^{-1}C_1S_2 \mid \right. \\ &\quad \left. C_2^{-1} - C_2^{-1}C_1S_3 \mid -C_2^{-1}C_1S_4\right] \Phi_1 \end{aligned}$$

which leads to

$$X_f = \begin{bmatrix} X_{f,1} \\ X_{f,2} \end{bmatrix} = \begin{bmatrix} S_1 & S_2 & S_3 & S4 & 0 \\ -C_2^{-1}C_1S_1 & -C_2^{-1}C_1S_2 & C_2^{-1} - C_2^{-1}C_1S_3 & -C_2^{-1}C_1S_4 & 0 \end{bmatrix} \begin{bmatrix} U_p \\ Y_p \\ Y_f \\ X_{p,1} \\ X_{p,2} \end{bmatrix}$$

which implies (13). □

So far, we have designed only a data-driven acceptor of the form (4) for the system in (1). To make this acceptor an rUIO, we need to impose a further requirement, namely we have to guarantee that the dynamics of the estimation error is not only autonomous, but also Schur stable.

Theorem 10. *Given the historical data (u_d, y_d, x_d), satisfying the Assumption, there exists a reduced-order unknown-input observer for system Σ of the form (4), designed from the historical data, if and only if $\exists \left[\, S_1 \mid S_2 \mid S_3 \mid S_4 \,\right] \in \mathbb{R}^{(n-p)\times(m+2p+n-p)}$, with S_4 Schur stable, such that (15) holds.*

Proof. As discussed in the previous section, a system described as in (4) is a reduced-order unknown-input observer for system Σ if and only if it is an acceptor and the dynamics of $e_1(t) = x_1(t) - \hat{x}_1(t)$ is autonomous and asymptotically stable. By Proposition 7 and the subsequent Remark 8, we know that there exists an acceptor for system Σ of the form (4), designed from the historical data, if and only if $\exists \left[\, S_1 \mid S_2 \mid S_3 \mid S_4 \,\right] \in \mathbb{R}^{(n-p)\times(m+2p+n-p)}$, such that (15) holds. Since we have shown that the estimation error on the first components of the state follows the autonomous dynamics $e_1(t+1) = A_{UIO}e_1(t) = S_4e_1(t)$, such autonomous dynamics is asymptotically stable if and only if $S_4 = A_{UIO}$ is Schur stable. $\square$

Note that the general solution to (15) is given by

$$\left[\begin{array}{c|c|c|c} S_1 & S_2 & S_3 & S_4 \end{array}\right] = X_{f,1}\Phi_1^\dagger + W\left(I - \Phi_1\Phi_1^\dagger\right),$$

where W is an arbitrary matrix of suitable dimensions. So, once condition (14) holds, one has to explore if there is a solution to (15), in the set of solutions parametrized above, with S_4 Schur stable.

We finally provide a numerical example to illustrate the obtained results.

Example 11. *Consider a system Σ of order $n = 5$ described as in (1) for the following choice of matrices:*

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 0 & 3/4 \\ \hline 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -5/4 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right],$$

$$B = \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ 2 & 1 \\ \hline -2 & 1 \\ 0 & 0 \\ 1 & 0 \end{array}\right], \quad E = \left[\begin{array}{c} E_1 \\ \hline E_2 \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \\ 2 & 1 \\ 1 & 0 \end{array}\right],$$

$$C = \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] = \left[\begin{array}{cc|ccc} 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 2 & 0 & -1 \\ 3 & 0 & 2 & -1 & 1 \end{array}\right].$$

Historical (both known and unknown) input data have been randomly generated, uniformly in the interval $(-5, 5)$ for the known input $u(t)$, and in the interval $(-2, 2)$ for the disturbance $d(t)$. The time-interval of the offline experiment has been set to $T = 11$. We have collected the data corresponding to the input/output/state trajectories and then checked that all the assumptions are satisfied and that the kernels inclusion holds. Clearly, from Y_p and X_p we have recovered the exact expression of C. We have then set as matrices of the rUIO in (4) the ones corresponding to the following particular solution of equation (15):

$$\left[B^u_{UIO} | B^y_{UIO} - A_{UIO}D_{UIO} | D_{UIO} | A_{UIO}\right] = X_{f,1}\Phi_1^\dagger,$$

namely

$$A_{UIO} = \begin{bmatrix} 0.1580 & -0.4135 \\ 0.3763 & 0.0029 \end{bmatrix}, B^u_{UIO} = \begin{bmatrix} 0.6797 & -0.8599 \\ 1.8089 & 1.0409 \end{bmatrix},$$

$$B^y_{UIO} = \begin{bmatrix} -0.1618 & 0.0889 & -0.0382 \\ 0.1104 & -0.1670 & 0.3555 \end{bmatrix},$$

$$D_{UIO} = \begin{bmatrix} 0.1200 & -0.0201 & 0.3800 \\ -0.0136 & -0.0546 & 0.0136 \end{bmatrix}.$$

It is easy to verify that the matrix A_{UIO} is Schur stable. Finally, we have tested the performance of the designed rUIO corresponding to the (known) input $u(t) = \begin{bmatrix} 0.8\cos(0.2t+2) & 3t \end{bmatrix}^\top$, $t \in \mathbb{Z}_+$, and a random disturbance $d(t)$ whose first and second components take values uniformly in the interval $(-5, 5)$ and $(-2, 2)$, respectively.

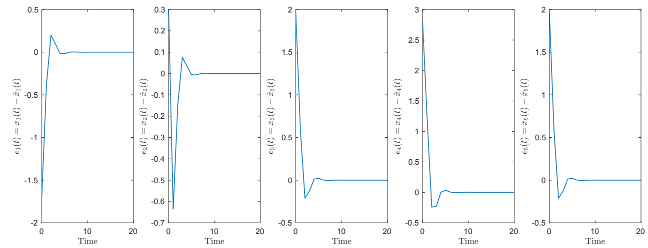

Fig. 1. Dynamics of the state estimation error

Figures 1 illustrates the state estimation error, that asymptotically converges to zero, as expected.

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