On unifying control barrier and Lyapunov functions using QP and Sontag's formula with an application to tumor dynamics

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Abstract— A common tool in system theory for formulating control laws that achieve local asymptotic stability are Control Lyapunov functions (CLFs), while Control Barrier functions (CBFs) are typically employed to enforce safety constraints. Combining these two types of functions is of interest, because it leads to stabilizing controllers with safety guarantees. A common approach to merge CLFs and CBFs is to solve an optimization problem where both CLF and CBF inequalities are imposed as constraints. In this paper, we show via an example from the literature that this approach can lead to undesirable behavior (i.e., slow convergence and oscillating inputs). Then, we propose a novel cost function that penalizes the deviation from Sontag's formula by using a state-dependent weighting matrix. We show that by minimizing the developed cost function subject to a CBF constraint, local asymptotic stability is obtained with an explicit domain of attraction, without using a CLF constraint. To deal with vanishing properties of the weight matrix as the state approaches the equilibrium, we introduce a hybrid continuous control law that recovers Sontag's formula locally. The effectiveness of the developed hybrid stabilizing control law based on CLFs and CBFs is illustrated in stabilization of a 3D tumor model, subject to physiological constraints (i.e., all states must be positive), which yields useful insights into optimal cancer treatment design.

I. INTRODUCTION

Control Lyapunov functions (CLFs), introduced in [1], have been commonly used to develop stabilizing control laws for nonlinear systems. Based on the CLF theorem, analytically derived "universal" control laws can be developed such as Sontag's "universal" control law introduced in [2]. Another way to use CLFs is by constructing a point-wise minimum norm (PMN) controller based on a quadratic programming (QP) problem, such as the one introduced in [3]. Closely related to CLFs are Control Barrier functions (CBFs). The latter are safety-related functions used to enforce safety constraints for controlled systems. CBFs define a set of safe states for the system and act as a barrier to prevent the system from entering undesirable or unsafe regions. Safety of trajectories can be enforced by solving a QP problem which uses the CBF condition as an explicit constraint, as introduced in [4], [5]. By also adding the CLF decrease condition to such a QP problem, one can combine CLFs and CBFs and achieve safety together with local asymptotic stability (LAS), which is desirable for most real-life systems. For example, for biological models, such as tumor models, physiological constraints require that the state trajectories must remain positive at all times, [6].

In the original CBF-QP problem formulation introduced in [4], [5], [7] a cost function was minimized that penalizes the deviation from the computed control action and a desired nominal controller, e.g., such as Sontag's unconstrained stabilizing control law. However, this approach does not necessarily guarantee asymptotic stability. Subsequently, the CLF-CBF-QP problem was introduced, see e.g., [4], which utilizes an explicit CLF decrease constraint in order to guarantee stability and minimizes the norm of the computed control action. This method required the introduction of an additional parameter, i.e., a slack variable, δ, to ensure feasibility of the combined CLF and CBF constraints. The introduction of δ, presented another challenge; the states are not guaranteed to converge towards the desired equilibrium. To address this challenge, different methodologies were proposed in recent studies. In [8], the original CLF-CBF-QP problem was modified such that the CLF condition is satisfied if the CBF constraint is inactive, which yields LAS without explicitly constructing a domain of attraction (DOA). However, the utilized cost function therein still makes a trade-off between minimizing the norm of the computed control action and the weighted norm of δ. Similarly, in [9], it was proposed to combine the CLF-CBF-QP formulation with a cost function that penalizes the deviation of the computed control action from a nominal, known stabilizing controller. In this approach LAS can be guaranteed if the weight of δ is properly tuned, but in principle, this optimal tuning should be done for every initial state, which is impractical.

In this paper we show for an example from [9] that although both methods above provide safety and LAS, the tuning of the weight on δ can result in undesired behaviour, such as getting stuck into an extremely slow convergence behavior or generating highly oscillating inputs. Motivated by this, in this paper we derive an alternative solution to unify CLFs and CBFs based on the original *CBF-QP* formulation [4]. As proposed therein we only use a CBF constraint, we minimize a cost function that penalizes the deviation of the computed control action from Sontag's unconstrained stabilizing control law (i.e., a nominal controller) and we introduce a novel, state-dependent weighting matrix which implicitly guarantees that the CLF condition holds, whenever this is feasible. In this way, we remove the need to explicitly incorporate a relaxed CLF constraint and as such, we remove the need to use an additional variable δ and the problems coming with it. However, the proposed approach suffers from a different challenge, i.e., the state dependent weighting matrix vanishes when the state approaches the equilibrium. To solve this issue, we introduce a hybrid stabilization

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formula which switches from solving the developed CBF-QP problem to Sontag's formula when the CBF constraint becomes inactive. We prove that the designed hybrid Sontag-CBF-QP solution (or S-CBF-QP for short) yields a continuous control law and an enlarged domain of attraction compared to the CLF-CBF-QP solutions in the literature.

To showcase the benefits of the developed S-CBF-QP stabilizing solution, we consider the challenging problem of stabilizing a 3D nonlinear tumor model [6].

II. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote the field of real numbers, the set of non-negative reals and the set of positive reals, respectively. For a vector $x \in \mathbb{R}^n$, x_i denotes the i-th element of x, $\|x\|_2 = \sqrt{x_1^2 + ... + x_n^2}$ denotes the two-norm of x and $\|x\|_2^2 = x_1^2 + ... + x_n^2 = x^\top x$ the squared two-norm of x. For any suitable function $f \in \mathbb{R}^{n \times n}$ we introduce the notation $\|x\|_{f(x)}^2$ to denote $= x^\top f(x)x$. For a matrix $A \in \mathbb{R}^{n \times n}$, $A^\top$ denotes its transpose and $A \succ 0$ indicates that "A is positive definite", i.e. for all $x \in \mathbb{R}^n \setminus \{0\}$ it holds that $x^\top A x > 0$ and $A = A^\top$. A set $\mathcal{S} \subseteq \mathbb{R}^n$ is called a proper set if it has a non-empty interior. Moreover, it is called ξ-proper for some $\xi \in \mathbb{R}^n$ if it contains ξ in its interior. Let $\partial \mathcal{S}$ denote the boundary of $\mathcal{S}$ and $int(\mathcal{S})$ the interior of $\mathcal{S}$. The distance from a point $x \notin \mathcal{S}$ to the set $\mathcal{S}$ is given by $\|x\|_{\mathcal{S}} := \inf_{y \in \mathcal{S}} \|x - y\|$. For a closed set $\mathcal{S}$, the Boulingand's tangent cone is defined by

$$\mathcal{T}_{\mathcal{S}}(x) = \left\{ y : \liminf_{\tau \to 0} \frac{\|x + \tau y\|_{\mathcal{S}}}{\tau} = 0 \right\}, \tag{1}$$

following Definition 4.6 of [10]. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. Additionally, if $\lim_{s \to \infty} \alpha(s) = \infty$, then α belongs to class $\mathcal{K}_\infty$. A function $\alpha : \mathbb{R} \to \mathbb{R}$ belongs to *extended class $\mathcal{K}$* [11] if it is strictly increasing, $\alpha(0) = 0$ and $\lim_{s \to a} \alpha(s) = \infty$, $\lim_{s \to -b} \alpha(s) = -\infty$ for some $a, b \in \mathbb{R}_{>0}$.

Consider the continuous-time, time-invariant, nonlinear system

$$\dot{x}(t) = \bar{f}(x(t), u(t)), \quad t \in \mathbb{R}_{\geq 0}, \tag{2}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the continuous-time state and input trajectories and where $\bar{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a globally Lipschitz continuous function. For this paper, only the class of control systems which are affine with respect to the input are considered, i.e.,

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad t \in \mathbb{R}_{\geq 0}, \tag{3}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the continuous-time state and input trajectories and where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ are globally Lipschitz continuous functions. The solution of system (3) with initial condition $x(0) = x_0$ and input u at time t is denoted by $x(t, u, x_0)$. For brevity, any such solution is denoted by $x(t)$ or x if x_0, time t and u are understood. A pair (x_e, u_e), denotes an equilibrium point of system (3), if $f(x_e) + g(x_e)u_e = 0$.

Definition II.1 ([12]) *(i)* An equilibrium pair (x_e, u_e) of system (3), with known control input $u(t)$, is *Lyapunov stable* if for any $\epsilon > 0$ there exists $\delta > 0$ such that for all corresponding state trajectories of (3) it holds that $\|x(0) - x_e\| \leq \delta \Rightarrow \|x(t) - x_e\| \leq \epsilon$ for all $t \geq 0$. *(ii)* We call an equilibrium pair (x_e, u_e) of system (3), with control input $u(t)$, *attractive in* $\mathcal{S} \subseteq \mathbb{R}^n$ if for each $x(0) \in \mathcal{S}$ it holds that all corresponding state trajectories of (3) satisfy $\lim_{t \to \infty} \|x(t) - x_e\| = 0$. *(iii)* We call an equilibrium pair (x_e, u_e) of system (3), with control input $u(t)$, asymptotically stable (AS) in $\mathcal{S}$ if it is Lyapunov stable and attractive in $\mathcal{S}$.

Definition II.2 ([12]) A set $\mathcal{S} \subseteq \mathbb{R}^n$ is called a controlled invariant set for system (3), if there exists a state-feedback law $u(t) := \kappa(x(t))$, such that for each $x(0) \in \mathcal{S}$, $x(t) \in \mathcal{S}$ for all $t \geq 0$.

Next, we define a safe set as follows:

$$\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \geq 0\}, \tag{4}$$

with boundary and interior defined as

$$\partial \mathcal{C} = \{x \in \mathbb{R}^n : h(x) = 0\}, \tag{5}$$
$$int(\mathcal{C}) = \{x \in \mathbb{R}^n : h(x) > 0\}, \tag{6}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function referred to as a barrier function. For this paper we consider *concave* functions for h, which result in *convex* safe sets.

Definition II.3 ([7]) Let a safe set $\mathcal{C}$ be defined by (4) with respect to a barrier function h. Then h is called a Control Barrier function (CBF) for system (3), if there exists an extended class $\mathcal{K}$ function α_h and a input $u \in \mathbb{R}^m$ such that

$$\nabla h(x)^\top (f(x) + g(x)u) \geq -\alpha_h(h(x)), \quad \forall x \in \mathcal{C}. \tag{7}$$

Proposition II.4 ([4]) *If h is a Control Barrier function for system (3), then $\mathcal{C}$ is a controlled invariant set.*

Next, we provide a control Lyapunov function definition.

Definition II.5 ([12]) Let $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuously differentiable function that satisfies

$$\alpha_1(\|x - x_e\|) \leq W(x) \leq \alpha_2(\|x - x_e\|), \quad \forall x \in \mathcal{S} \tag{8}$$

where $\mathcal{S} \subseteq \mathbb{R}^n$ for some functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Then, if there exists a state-feedback control law $u := \kappa(x)$ with $\kappa(x_e) = u_e$ such that

$$\nabla W(x)^\top (f(x) + g(x)\kappa(x)) < 0, \quad \forall x \in \mathcal{S} \setminus \{x_e\}, \tag{9}$$

the function W is called a Control Lyapunov function (CLF) for system (3) for all $x \in \mathcal{S}$.

Note that the above definition corresponds to a regional CLF, i.e., a CLF in $\mathcal{S}$. W is a global CLF if $\mathcal{S} = \mathbb{R}^n$. Following standard Lyapunov arguments, the set $\mathcal{S}$ and any sub-level set of W contained in $\mathcal{S}$ is rendered invariant by

the feedback law $u = \kappa(x)$ for system (3) [12]. For this paper, only *convex* functions are considered for W, which result in *convex* level sets of W.

Definition II.6 ([13]) The CLF W is said to have the small control property if there exists a continuous state-feedback control law $u = \kappa(x)$ where $\kappa(x_e) = u_e$, that satisfies

$$\nabla W(x)^\top (f(x) + g(x)\kappa(x)) < 0, \qquad (10)$$

for all x in the neighborhood of the equilibrium point x_e with $x \neq x_e$.

Next, we present a slightly adjusted version of Sontag's formula, valid for systems with multiple inputs, non-zero equilibria and which allows for tuning the rate of decrease of the Lyapunov function. Let (x_e, u_e) be an equilibrium pair of system (3), let W be a CLF for system (3). Then define Sontag's formula as:

$$u_{son}(x, u_e) = \kappa(x, u_e) + u_e, \qquad (11)$$

where

$$\kappa(x, u_e) :=$$
$$\begin{cases} -\frac{a(x,u_e)+\gamma\sqrt{a(x,u_e)^2+\|b(x)^\top\|_2^4}}{\|b(x)^\top\|_2^2} b(x)^\top & \text{if } \|b(x)^\top\|_2^2 > 0 \\ 0 & \text{if } \|b(x)^\top\|_2^2 = 0 \end{cases},$$

and where $a(x, u_e) := \nabla W(x)^\top (f(x) + g(x)u_e)$, $b(x) := \nabla W(x)^\top g(x)$ and $\gamma \in \mathbb{R}_{>0}$ influences the convergence rate.

Lemma II.7 *Consider system (3) with input $u = u_{son}(x, u_e)$ as given in (11). Assume there exists a CLF W in a x_e-proper set $\mathcal{S} \subseteq \mathbb{R}^n$ for system (3). Then the state feedback $u = u_{son}(x, u_e)$ is smooth and continuous at x_e and, if $\mathcal{S}$ is rendered invariant by u, then the equilibrium point (x_e, u_e) of system (3) is asymptotically stable in $\mathcal{S}$.*

Proof: Given W is a CLF for system (3) for equilibrium pair (x_e, u_e), we can use the CLF condition (8), to establish that $\nabla W(x_e) = 0$ and thus

$$a(x_e, u_e) = b(x_e) = \kappa(x_e, u_e) = 0, \qquad (12)$$

which results in $u_{son}(x_e, u_e) = u_e$. Let us prove that u_{son} asymptotically stabilizes the equilibrium point (x_e, u_e) of system (3) for all $x \in \mathbb{R}^n$. For simplicity, $a(x, u_e)$ is denoted by a, $b(x)$ is denoted by b and $u_{son}(x, u_e) = \kappa(x, u_e) + u_e$ is denoted by $u_{son} = \kappa + u_e$. Then we have that

$$\nabla W(x)^\top (f(x) + g(x)u_{son})$$
$$= \nabla W(x)^\top (f(x) + g(x)(\kappa + u_e))$$
$$= \nabla W(x)^\top (f(x) + g(x)u_e) + \nabla W(x)^\top g(x)\kappa(x)$$
$$= a + b\kappa$$
$$= a + bb^\top \frac{-a - \gamma\sqrt{a^2 + \|b^\top\|_2^4}}{\|b^\top\|_2^2}$$
$$= a + \|b^\top\|_2^2 \frac{-a - \gamma\sqrt{a^2 + \|b^\top\|_2^4}}{\|b^\top\|_2^2}$$
$$= -\gamma\sqrt{a^2 + \|b^\top\|_2^4} < 0, \quad \forall x \in \mathcal{S} \setminus \{x_e\}.$$

Thus u_{son} asymptotically stabilizes equilibrium point (x_e, u_e) of system (3) for all $x \in \mathcal{S} \subseteq \mathbb{R}^n$. The claim follows from standard Lyapunov arguments, see, for example [12].

Additionally, the CLF W satisfies the small control property as $u_{son}(x_e, u_e) = x_e$ and it satisfies (10). Following Theorem 1 of [2], we have that $u_{son}(x, u_e)$ is smooth on $\mathcal{S} \subseteq \mathbb{R}^n \setminus \{x_e\}$ and continuous at x_e. ∎

A. CBF-QP

Although Sontag's control law can globally stabilize equilibria, it cannot guarantee satisfaction of constraints on controlled state trajectories, which is a substantial limitation for many real-life systems. As a result, researchers explored the approach of combining CLFs with CBFs to obtain stability and safety guarantees.

Within this context, two commonly used quadratic programming (QP) problems have been formulated in order to compute a pointwise input while satisfying the safety constraints defined by a CBF. Given a desired control input $u_{nom}(x)$, originally a CBF-QP was defined as follows in [4].

CBF-QP problem:

$$\min_{u} \quad \|u - u_{nom}(x)\|_2^2$$
$$\text{s.t.} \quad \nabla h(x)^\top f(x) + \nabla h(x)^\top g(x)u \geq -\alpha_h(h(x)). \qquad (13)$$

Although this standard CBF-QP problem guarantees the forward invariance of the safe set $\mathcal{C}$, it does not have any (local) asymptotically stability guarantees, [14]. Therefore, an alternative problem was introduced in [4] as follows.

CLF-CBF-QP problem:

$$\min_{u,\delta} \quad \|u\|_2^2 + p\delta^2$$
$$\text{s.t.} \quad \nabla W(x)^\top f(x) + \nabla W(x)^\top g(x)u \leq -\alpha_W(W(x)) + \delta$$
$$\nabla h(x)^\top f(x) + \nabla h(x)^\top g(x)u \geq -\alpha_h(h(x)), \qquad (14)$$

where $\alpha_{\mathcal{W}}(W(x)) \in \mathcal{K}_\infty$, $p > 0$ and δ is an additional optimization parameter to guarantee feasibility. This baseline CLF-CBF-QP problem was later modified/analyzed in [8], [9] in order to guarantee local asymptotic stability (LAS).

B. Motivating example and problem statement

Next, we analyze the sensitivity of the CLF-CBF-QP problem defined in [9], to the weight p of the parameter δ, based on a linear system example taken from [9], i.e. consider system (3) with

$$f(x) = -\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad (15)$$

with the desired equilibrium point given by $(x_e, u_e) = \left((0 \quad 0)^\top, 0 \right)$. The nominal controller is taken as $u_{son}(x)$ as defined in (11). The technique of [15] was used to compute a quadratic CLF $W(x) = x^\top P x$ where

$$P = \begin{bmatrix} 3.4142 & -2.4142 \\ -2.4142 & 2.4142 \end{bmatrix}.$$

The CBF is given by $h(x) = -0.1x_1^2 - 0.15x_1x_2 - 0.1x_2^2$ [9], with the corresponding convex safe set depicted as the white area in Figure 1.
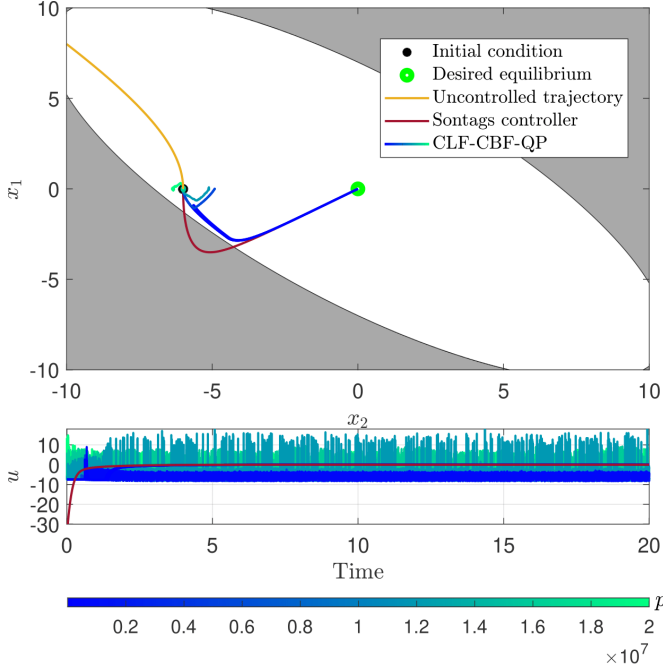

Fig. 1. Example with the standard CLF-CBF-QP problem, for different p with the values of p defined in the color bar.

As it can be seen in Figure 1, for some initial condition and different values of p, the CLF-CBF-QP problem yields a highly oscillating input. Additionally, for higher values of p, trajectories get stuck into an extremely slow converging behavior. This is caused by the fact that the high penalty on δ makes the CLF constraint active, which conflicts with the CBF constraint. It is worth to point out that we obtained a similar behavior using the CLF-CBF-QP problem variations in both [8] and [9].

It is also important to note that for CLF-CBF-QP problems, in current research, the domain of attraction (DOA), - that is, the region where the closed-loop equilibrium is asymptotically stable - has not been explicitly characterized; except for the trivial DOA estimate provided by the largest level set of the CLF contained inside the safe set, which can be conservative.

Problem statement: Given the above analysis of CLF-CBF-QP problems, our objective is to develop a stabilizing, continuous control law based on the original CBF-QP formulation [4], which guarantees LAS without imposing the relaxed CLF constraint and therefore removing the need of the additional parameter δ. Moreover, we would like to guarantee LAS for an explicitly defined DOA estimate, which is less conservative than the largest level set of the CLF contained inside the safe set.

III. Main results

Consider the nonlinear control affine system (3) with a CLF W and a CBF h. As typically done, we assume that

there exists an equilibrium pair (x_e, u_e) for system (3), where the state variable x_e belongs to interior of the safe set $\mathcal{C}$.

Next we introduce an alternative CBF-QP formulation.

S-CBF-QP problem

$$
\begin{aligned}
\min_u \quad & \|u - u_{son}(x)\|^2_{g(x)^\top \nabla W(x) \nabla W(x)^\top g(x)} \\
\text{s.t.} \quad & \nabla h(x)^\top f(x) + \nabla h(x)^\top g(x)u \geq -\alpha_h(h(x)),
\end{aligned}
\tag{16}
$$

where the nominal controller is defined as the modified version of Sontag's "universal" control law defined in (11). For simplicity of exposition define

$$
\begin{aligned}
Q(x) &:= g(x)^\top \nabla W(x) \nabla W(x)^\top g(x), \\
L_f h(x) &:= \nabla h(x)^\top f(x), \\
L_g h(x) &:= \nabla h(x)^\top g(x).
\end{aligned}
$$

The optimal control input resulting from the optimization problem is given by

$$
\begin{aligned}
u^*(x) = \ & \arg\min_u \|u - u_{son}(x)\|^2_{Q(x)} \\
& \text{s.t.} \quad L_f h(x) + L_g h(x)u \geq -\alpha_h(h(x)).
\end{aligned}
\tag{17}
$$

Above, recall that

$$
\|u - u_{son}(x)\|^2_{Q(x)} = (u - u_{son}(x))^\top Q(x)(u - u_{son}(x)).
$$

A property of the CLF W, is that its gradient satisfies $\nabla W(x_e) = 0$. Consequently, the cost function of the S-CBF-QP problem is not well defined as x approaches x_e, i.e., it vanishes asymptotically. Therefore, we introduce a hybrid stabilization formula which switches from solving the developed CBF-QP problem to Sontag's formula when the CBF constraint becomes inactive. To that end let us define two regions

$$
\begin{aligned}
\mathcal{R}_1 &:= \{x \in \mathcal{C} \mid L_f h(x) + \alpha_h(h(x)) \tag{18} \\
& \qquad\qquad + L_g h(x)u_{son}(x) \geq 0\}, \\
\mathcal{R}_2 &:= \{x \in \mathcal{C} \mid L_f h(x) + \alpha_h(h(x)) \tag{19} \\
& \qquad\qquad + L_g h(x)u_{son}(x) \leq 0\}.
\end{aligned}
$$

We assume that $\mathcal{R}_1$ is a x_e-proper set. The hybrid S-CBF-QP control law is given as follows:

$$
\phi(x) := \begin{cases} u_{son}(x) & \text{if } x \in \mathcal{R}_1 \\ u^*(x) & \text{if } x \in \mathcal{R}_2 \end{cases}.
\tag{20}
$$

Before presenting the main result, we first present some properties of the S-CBF-QP problem in (17) and of the hybrid S-CBF-QP control law in (20).

Lemma III.1 *The S-CBF-QP problem in* (17) *is feasible and has an unique solution for all* $x \in \mathcal{R}_2$. *Additionally, the resulting control input* $u^*(x)$ *is Lipschitz continuous for all* $x \in \mathcal{R}_2$.

For the proof we refer to [16] due to space limitations.

Remark III.2 The S-CBF-QP problem is utilized as part of the hybrid S-CBF-QP control law in (20), and thus it is only employed for $x \in \mathcal{R}_2$ to compute a control input. For

$x \in \mathcal{R}_1$, Sontag's formula is used instead. Therefore, it is not necessary to analyze Lipschitz continuity and the uniqueness of the solution of the S-CBF-QP problem for $x \in \mathcal{R}_1$.

Corollary III.3 *Assuming that $\mathcal{R}_1$ is a x_e-proper set, the hybrid control law $\phi(x)$ in (20) is continuous for all $x \in \mathcal{C}$.*

The above result directly follows from:
- *(i)* The switching condition: for all $x \in \partial \mathcal{R}_1 \cup \partial \mathcal{R}_2$, i.e. when $L_f h(x) + L_g h(x) u_{son}(x) = -\alpha_h(h(x))$, $\phi(x) = u^*(x) = u_{son}(x)$;
- *(ii)* For any $x \in \mathcal{R}_1$, $\phi(x) = u_{son}(x)$;
- *(iii)* The continuity of $u_{son}(x)$ and of $u^*(x)$, which were shown in Lemma II.7 and Lemma III.1, respectively.

Definition III.4 ([17]) A CLF W and a CBF h for system (3) have the control sharing property, if for some x_e-proper set $\mathcal{A} \subseteq \mathbb{R}^n$, there exists a control input, such that the following inequalities are simultaneously satisfied:

$$\nabla h(x)^\top (f(x) + g(x)u) \geq -\alpha_h(h(x)) \quad \forall x \in \mathcal{A}, \quad (21)$$
$$\nabla W(x)^\top (f(x) + g(x)u) < 0 \quad \forall x \in \mathcal{A} \setminus \{x_e\}. \quad (22)$$
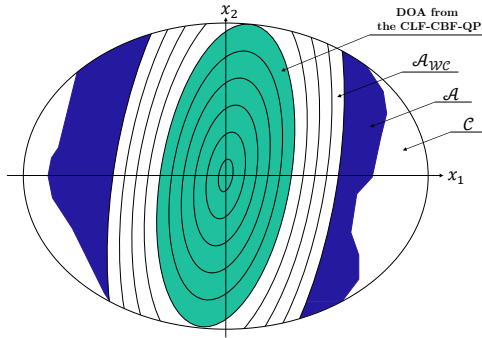

Fig. 2. Graphical illustration of the sets $\mathcal{W}^*$, $\mathcal{C}$ and $\mathcal{A}_{\mathcal{WC}}$.

Note that $\mathcal{A} \subseteq \mathcal{C}$. A set $\mathcal{W}$ is defined as the sub-level set of the CLF W, i.e. $\mathcal{W} := \{x \in \mathbb{R}^n \mid W(x) \leq c\}$ where $c > 0$. Let c^* be defined such that

$$c^* = \arg\max_c \; c$$
$$\text{s.t.} \quad c > 0 \quad\quad (23)$$
$$\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{A},$$

let $\mathcal{W}^* := \{x \in \mathbb{R}^n \mid W(x) \leq c^*\}$ and define $\mathcal{A}_{\mathcal{WC}} := \mathcal{W}^* \cap \mathcal{C}$. For a visual example of $\mathcal{A}$ and $\mathcal{A}_{\mathcal{WC}}$, see Figure 2. The main result is stated next.

Theorem III.5 *Consider system (3) with the equilibrium pair (x_e, u_e), a CLF W and a CBF h with the associated safe set $\mathcal{C}$ for system (3). Assume W and h have the control sharing property for all $x \in \mathcal{A}$ as in Definition III.4 and $\mathcal{A}, \mathcal{R}_1, \mathcal{A}_{\mathcal{WC}} \neq \emptyset$. Then, the equilibrium x_e of system (3) in closed-loop with $u = \phi(x)$ as defined in (20) is asymptotically stable for all $x \in \mathcal{A}_{\mathcal{WC}}$. Additionally, the hybrid control law $\phi(x)$ renders the set $\mathcal{A}_{\mathcal{WC}}$ controlled invariant for the closed-loop system (3)-(20).*

For the proof we refer to [16] due to space limitations.

To demonstrate the effectiveness of the developed hybrid control law (20), we follow the example shown in Section II-B. The same CLF W and CBF h are used. As can be seen in Figure 3, the states converge to the desired equilibrium while still maintaining the safety guarantees. Additionally, by tuning γ, the tuning parameter that dictates the convergence rate introduced in Lemma II.7, the decrease rate can be tuned to allow for a more conservative input. Note that in Figure 3, the input makes a sharp increase around 2 seconds. This abrupt change of u, corresponds to the moment the CBF constraints go from active to inactive and occurs for all mentioned CBF-QP and CLF-CBF-QP problems. I.e., the abrupt change does not originate from the switching in our hybrid S-CBF-QP control law.
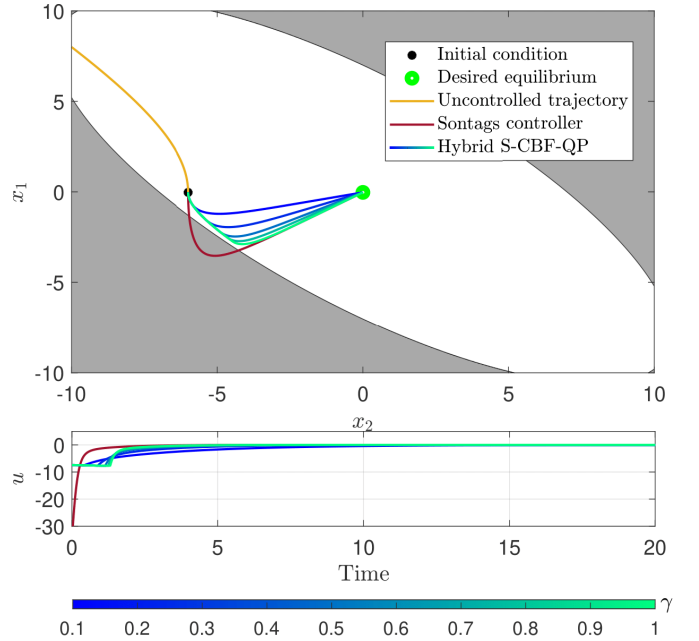

Fig. 3. Example (15) with the introduced hybrid S-CBF-QP control law for different γ with the values of γ defined in the color bar.

IV. REALISTIC STABILIZATION OF TUMOR DYNAMICS

In this section, we apply the developed S-CBF-QP to the stabilization of tumor dynamics using the effects of immunotherapy with the developed hybrid control law (20).

Consider the nonlinear tumor model as system (3) with

$$f(x) = \begin{bmatrix} R_T x_1 - \frac{R_T}{K_T} x_1^2 - \frac{\alpha_{TN} R_T}{K_T} x_1 x_2 \\ -\alpha_{NT} x_2 x_1 + \beta x_2 x_3 \\ R_R x_3 - \frac{R_R}{K_R} x_3^2 - \frac{\beta R_R}{K_R} x_2 x_3 \end{bmatrix},$$
$$g(x) = \begin{bmatrix} -\frac{R_T}{K_T} x_1 x_2 \\ 0 \\ 0 \end{bmatrix} \quad\quad (24)$$

which describes the regression and progression of normal and tumor cells as proposed in [6]. The tumor cell population is denoted by x_1 and the resting and hunting immune cell populations are denoted by x_2 and x_3, respectively. The parameters are chosen as follows, $\alpha_{NT} = 0.5$, $\alpha_{TN} = 0.9$,

$\beta = 0.9$, $K_R = K_T = 10$ and $R_R = R_T = 0.9$ as suggested in [6].

Next, consider the desired equilibrium point $(x_e, u_e) = \left(\begin{pmatrix} 6.4286 & 7.1429 & 3.5714 \end{pmatrix}^\top, -0.4 \right)$ representing tumor dormancy [6]. By linearizing system (24) around the desired equilibrium point, the technique of [15] can be used to compute a quadratic CLF $W(x) = x^\top P x$ where

$$P = \begin{bmatrix} 0.3564 & -0.2472 & -0.4017 \\ -0.2472 & 0.4597 & 0.3699 \\ -0.4017 & 0.3699 & 4.6665 \end{bmatrix}.$$

For this system we design and implement three CBFs which are given by $h_1(x) = -(x_1 - 5)^2 + 25$, $h_2(x) = 1 - e^{-x_2}$ and $h_3(x) = 1 - e^{-x_3}$. All three CBFs are chosen to ensure the positivity of the controlled state trajectories. Additionally, h_1 is defined to limit the amount of tumor cells in the body. All three CBFs are implemented as separate CBF constraints.
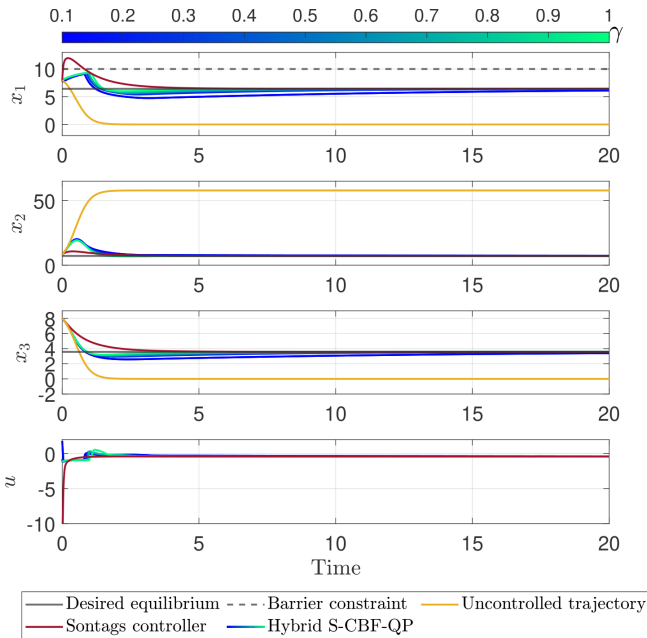

Fig. 4. Stabilization of equilibrium representing tumor dormancy with the introduced hybrid S-CBF-QP control law for different γ with the values of γ defined in the color bar.

As can be seen in Figure 4, the trajectories remain positive and the tumor cells within the bound defined by h_1, while converging towards the desired equilibrium. Multiple trajectories are shown for different values of γ. As can be seen, γ has a direct effect on the rate with which the states converge towards the equilibrium. For invasive tumors, γ can be tuned such that the states converge faster towards the equilibrium. However, for cases with non-invasive tumors, γ can be tuned such that the treatment is less invasive, i.e. with less drug usage.

V. CONCLUSIONS

In this paper, we proposed an alternative CBF-QP problem formulation that uses a state-dependent weighting matrix to implicitly enforce the satisfaction of a CLF constraints, when feasible. Compared to previous CLF-CBF-QP problems, the derived formulation does not require an additional paramter δ, which eliminates the need to tune the penalty of this parameter for every initial state. Based on the developed CBF-QP problem we derive a hybrid control law that switches locally to Sontag's formula and we provide an explicit domain of attraction for the resulting closed-loop system, which is less conservative than previous DOA estimates in the literature. The proposed hybrid control law was used to stabilize a desired equilibrium presenting a state of tumor dormancy. Via this example, we have shown that this method is computationally effective and also adjustable depending on the preference of treatment.

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