Partial stabilization of nonlinear systems along a given trajectory

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Abstract— In this paper, the problem of partial stabilization of nonlinear systems along a given trajectory is considered. This problem is treated within the framework of stability of a family of sets. Sufficient conditions for the asymptotic stability of a one-parameter family of sets, using time-varying feedback control with trigonometric functions, are derived. The obtained results are applied to a model mechanical system.

I. INTRODUCTION

Trajectory tracking is one of the fundamental control problems which has numerous applications in robotics and process engineering. A theoretical justification of tracking properties of control algorithms requires the stability analysis of the tracking error dynamics in a neighborhood of the reference curve. The stability proof can be straightforwardly achieved, e.g., if the linearized error dynamics is completely controllable. Tracking algorithms, based on the feedback linearization and flatness techniques, are shown to be highly efficient for various engineering models.

For kinematically redundant manipulating robots, the tracking problem can be effectively formulated in terms of a part of the state variables that characterize the control objective. This analogy creates a connection between pragmatically driven issues in the field of robotics and the notion of partial stability, a concept that was rigorously defined by A.M. Lyapunov and has been extensively explored by many researchers (see, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18] and references therein). Specifically, the paper [2] explored the connection between partial stability and full-variable stability in nonholonomic mechanical systems, offering conditions for achieving partial stability. Issues of partial stabilization in the context of Lagrangian systems were investigated in [19], [20]. Controllers that utilize passivity-based approaches for partial stabilization have been suggested in [21], [22], [23]. The issue of achieving partial stabilization in stochastic dynamical systems is addressed, e.g., in [24], [25], [26].

The issue of achieving partial stabilization within a finite time frame for systems in chained-form and cascade configurations has been investigated, as illustrated in [27], [28], [29]. An analysis of more extensive categories of nonlinear control

systems can be found in [15], [30]. The suggested adequate conditions for partial stability are based on the premise that the system allows for a Lyapunov function, with the time derivative that is negatively definite concerning a specific subset of variables.

In the paper [31], the motion planning problem for autonomous vehicles is considered within the framework of manoeuvre automata. In order to ensure the safety of paths in complex environments, it is required to estimate the reachable set of each manoeuvre. The proposed motion planning scenario is illustrated by a unicycle mode with two controls corresponding to the angular velocity and the translational acceleration. The analogy between the equations of motion of nonholonomic systems and underwater vehicles has been pointed out in [32], where driftless control-affine systems have been used to model the kinematics of an autonomous submarine. These equations have been analyzed in the paper [33] in the context of trajectory tracking problem with oscillating inputs. A survey of recent advances in the motion planning of autonomous underwater vehicles (AUV) is presented in [34]. A mathematical model of an unmanned surface vehicle (USV) in the form of a nonlinear control-affine system with 6-dimensinal state and 3-dimensional force input is considered in [35]. For this dynamical model, a tracking controller is constructed under the assumption that the planar reference trajectory is regular enough and C^2-bounded. The stability proof of the tracking algorithm is based on Lyapunov's direct method.

An important application of partial stability theory originates from planning the motion of robotic systems in task-spaces. The goal of the latter problem is to steer the output of a nonholonomic system to a neighborhood of the target point. As the number of output variables (which characterize the task space) is usually less than the dimension of the state space, this task fits into the framework of partial stabilization problems. An approach for solving the motion planning problem in task-space is proposed in [36] based on the Campbell–Baker–Hausdorff–Dynkin formula. The efficiency of this approach has been tested by the unicycle and car models with kinematic control. Fundamental solutions of the Laplace equation are exploited in [37] to generate obstacle-free motion of a disk robot in a bounded connected workspace. The control function, corresponding to the robot velocity, is obtained by an appropriate rescaling of the gradient of the potential function. The control scheme is implemented sequentially, and the convergence of the trajectories to the goal is proved. Computational complexity of the proposed control algorithm is estimated by numerical experiments.

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While the field of partial stability theory has advanced substantially, contributions to the partial stabilization of underactuated nonlinear control systems remain relatively scarce. The challenge of partial stabilization persists for general nonholonomic systems due to the difficulty in formulating an appropriate Lyapunov-like function. In [38], practical conditions for partial asymptotic stability were introduced for control-affine systems that exhibit a partially asymptotically stable equilibrium in their averaged form.

This paper presents a novel approach to partial stabilization that significantly advances the state of the art by addressing the challenge of stabilizing along non-feasible curves – a task not previously tackled. By conceptualizing this problem through the lens of the stability of sets, we establish a unified framework that allows for the stabilization of system behaviors in the vicinity of a given trajectory rather than at a fixed point. The introduction of time-varying feedback laws is a crucial point in our construction, ensuring exponential stability across a family of sets proximal to the non-feasible curve.

The rest of this paper is organized as follows. The partial stabilization problem is formulated in Section II within the framework of a family of sets. The main result (Theorem 1) is presented in Section III, and its proof is given in the Appendix. Section IV illustrates our control design scheme for an autonomous underwater vehicle model.

II. PRELIMINARIES

A. Notations and definitions

Consider a nonlinear system of the form

$$\dot{x} = f_0(t,x) + \sum_{k=1}^{m} f_k(x)u_k, \qquad (1)$$

where $x = (x_1,...,x_n)^\top \in D \subset \mathbb{R}^n$ is the state vector, $u = (u_1,...,u_m)^\top \in \mathbb{R}^m$ is the control, $m < n$, $f_0 : \mathbb{R}^+ \times D \to \mathbb{R}^n$, and $f_1,\ldots,f_m : D \to \mathbb{R}^n$. We represent the state vector as $x = (y^\top, z^\top)^\top$ with $y = (y_1,...,y_{n_1})^\top$ and $z = (z_1,...,z_{n_2})^\top$, $n_1 + n_2 = n$, and assume that $D = D_y \times \mathbb{R}^{n_2}$, where $D_y \subset \mathbb{R}^{n_1}$ is a domain containing the point $y = 0 \in \mathbb{R}^{n_1}$.

We will consider the problem of stabilization of system (1) with respect to its y-variables. For this purpose, we introduce some necessary notations and definitions which will be used throughout the paper.

For vector fields $f, g \in C^1(D;\mathbb{R}^n)$ and a point $x^* \in D$, we define the directional derivative $\mathscr{L}_g f(x^*) = \frac{\partial f(x)}{\partial x} g(x)\big|_{x=x^*}$ and the Lie bracket $[f,g](x^*) = \mathscr{L}_f g(x^*) - \mathscr{L}_g f(x^*)$. For a time dependent $f \in C^1(\mathbb{R}^+ \times D;\mathbb{R}^n)$, we denote $\mathscr{L}_g f(t^*,x^*) = \frac{\partial f(t^*,x)}{\partial x} g(x)\big|_{t=t^*,x=x^*}$. We say that an $f : \mathbb{R}^+ \times D \to \mathbb{R}^n$, is:

- *Lipschitz continuous with respect to x uniformly in t* in a set $\tilde{D} \subseteq D$, if there exists an $L > 0$ such that $\|f(t,x) - f(t,\tilde{x})\| \leq L\|x - \tilde{x}\|$ for all $x,\tilde{x} \in \tilde{D}$, $t \geq 0$;
- *bounded uniformly in t* in a set $\tilde{D} \subseteq D$, if there exists an $M > 0$ such that $\|f(t,x)\| \leq M$ for all $x \in \tilde{D}$, $t \geq 0$.

Definition 1: Given a time-varying feedback law $u^\varepsilon : \mathbb{R}^+ \times D \times \mathbb{R}^{n_1} \to \mathbb{R}^m$ depending on a parameter $\varepsilon > 0$ and a vector function $y^* : \mathbb{R}^+ \to D_y$, the π_ε-*solution* of (1) corresponding to the initial condition $x^0 \in D$ at $t = t_0 \geq 0$ and the control $u = u^\varepsilon(t,x,y^*(t))$ is an absolutely continuous function $x(t) \in D$, defined for $t \in [t_0,+\infty)$, such that $x(t_0) = x^0$ and $\dot{x} = f_0(t,x(t)) + \sum_{k=1}^{m} u_k^\varepsilon(t,x(t_j),y^*(t_j))f_k(x(t)), t \in I_j = [t_j,t_{j+1}), t_j = t_0 + \varepsilon j$ for each $j = 0,1,2,\ldots$.

The concept of π_ε-solutions has been used, e.g., in [39], [40], and its extension to the case of time-varying control parameters is introduced in [38].

Definition 2: A one-parametric family of non-empty sets $\{Y_t\}_{t \geq 0}$ with $Y_t \subset \mathbb{R}^n$ is called *asymptotically stable* for system (1) with a feedback control of the form $u = u^\varepsilon(t,x,y^*(t))$, if it is stable and attractive, i.e.:

- for every $\Delta > 0$, there exists a $\delta > 0$ such that, for every $t_0 \geq 0$ and $x^0 \in B_\delta(Y_{t_0})$, the corresponding π_ε-solution $x(t)$ with $x(t_0) = x^0$ is uniquely defined for $t \geq t_0$ and $x(t) \in B_\Delta(Y_t)$ for all $t \in [t_0;+\infty)$;
- for some $\delta > 0$ and for every $\Delta > 0$, there exists a $t_1 \geq 0$ such that, for every $t_0 \geq 0$ and $x^0 \in B_\delta(Y_{t_0})$, the corresponding π_ε-solution $x(t)$ with $x(t_0) = x^0$ satisfies the property $x(t) \in B_\Delta(Y_t)$ for all $t \in [t_0 + t_1,\infty)$.

The stability concept for families of sets is described, e.g., in [41], [42], [38].

B. Problem statement

Let $D_y \subset \mathbb{R}^{n_1}$ be a non-empty domain, $y^*(t)$ be a curve in D_y, $y^* \in C(\mathbb{R}^+;D_y)$. In this paper, we consider the problem of stabilizing the y-variables of (1) along the curve $y^*(t)$:

Problem 1. *Given a curve $y^* \in C(\mathbb{R}^+;D_y)$ and a number $p > 0$, the goal is to find a control $u^\varepsilon(t,x,y^*)$ such that the family of sets $\{Y_t^p\}_{t \geq 0}$ with*

$$Y_t^p = \{x = (y^\top, z^\top)^\top \in \mathbb{R}^n : \|y^*(t) - y\| < p, z \in \mathbb{R}^{n_2}\} \quad (2)$$

is asymptotically stable for the closed-loop system (1) with $u = u^\varepsilon(t,x,y^*)$ in the sense of Definitions 1 and 2.

In the sequel, by a neighborhood of a set Y_t^p, $t \geq 0$, we mean the set $B_\delta(Y_t^p) = \{x \in \mathbb{R}^n : \|y - y^*\| \leq p + \delta, z \in \mathbb{R}^{n_2}\}$. We assume that p is small enough to guarantee that $B_p(y^*(t)) \subset D_y$ for all $t \geq 0$.

Note that the partial stabilization problem for the case of static $y^*(t) \equiv const$ is considered in [43] under appropriate controlability rank condition. In [38], the problem of stabilizing the trajectories of a nonholonomic system to a reference curve in $\mathbb{R}^n$ is considered. Up to our best knowledge, the problem of *partial stabilization to a curve* is considered here for the first time for underactuated nonlinear systems.

For clarity of presentation, we rewrite system (1) as

$$\dot{y} = g_0(t,x) + \sum_{k=1}^{m} g_k(x)u_k, \quad \dot{z} = h_0(t,x) + \sum_{k=1}^{m} h_k(x)u_k, \quad (3)$$

where $g_k : \mathbb{R}^n \to \mathbb{R}^{n_1}$, $h_k : \mathbb{R}^n \to \mathbb{R}^{n_2}$, $g_0 : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n_1}$ and $h_0 : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n_2}$ are such that the vector fields of system (1) are represented as

$$f_0(t,x) = \begin{pmatrix} g_0(t,x) \\ h_0(t,x) \end{pmatrix}, f_k(x) = \begin{pmatrix} g_k(x) \\ h_k(x) \end{pmatrix}, k = \overline{1,m}.$$

III. Main result

In this section, we consider the class of systems (3), whose control vector fields $g_k \in C^1(\mathbb{R}^n;\mathbb{R}^{n_1})$ satisfy the following rank condition for all $x \in D$:

$$\text{span}\left\{ (g_i(x))_{i\in S_1}, (I_{n_1\times n}[f_{i_1},f_{i_2}](x))_{(i_1,i_2)\in S_2} \right\} = \mathbb{R}^{n_1}, \quad (4)$$

where S_1, S_2 are some sets of indices $S_1 \subseteq \{1,2,...,m\}$, $S_2 \subseteq \{1,2,...,m\}^2$ such that $|S_1| + |S_2| = n_1$.

This assumption represents a relaxation of the controllability rank condition that the vector fields of system (1) with their Lie brackets span the whole tangent space:

$$\text{span}\left\{ (f_i(x))_{i\in\widetilde{S}_1}, ([f_{i_1},f_{i_2}](x))_{(i_1,i_2)\in\widetilde{S}_2} \right\} = \mathbb{R}^n$$

at each $x \in D$ with some $\widetilde{S}_1 \subseteq \{1,2,...,m\}$, $\widetilde{S}_2 \subseteq \{1,2,...,m\}^2$, $|\widetilde{S}_1| + |\widetilde{S}_2| = n$. For the partial stabilization problems, the latter requirement can be replaced with relaxed condition (4). Thus, we take into account only the first n_1 coordinates of $f_i(x)$ and $[f_{i_1},f_{i_2}](x)$, i.e. we exploit the vector fields $g_i(x)$ and $I_{n_1\times n}[f_{i_1},f_{i_2}](x) = \mathscr{L}_{f_{i_1}}(x)g_{i_2}(x) - \mathscr{L}_{f_{i_2}}(x)g_{i_1}(x)$. Consequently, a smaller set of vector fields is needed to satisfy the stabilizability condition, which simplifies the control design. Condition (4) has been proposed in [43] for the case $y^*(t) \equiv const$, and we exploit it here for solving Problem 1. In order to stabilize the y-variables of system (1) along a curve $y^*(t)$, we will use a time-varying feedback control of the form

$$u_k^\varepsilon(t,x,y^*) = \sum_{i\in S_1} \phi_i^k(t,x,y^*)$$
$$+ \varepsilon^{-1/2} \sum_{(i_1 i_2)\in S_2} \phi_{i_1,i_2}^k(t,x,y^*), k = \overline{1,m}, \quad (5)$$

where $\phi_i^k(t,x,y^*) = \delta_{ki}a_i(x,y^*)$,

$$\phi_{i_1,i_2}^k(t,x,y^*) = 2\sqrt{\pi\kappa_{i_1 i_2}|a_{i_1 i_2}(x,y^*)|}\left(\delta_{ki_1}\cos\left(2\pi\kappa_{i_1 i_2}t\varepsilon^{-1}\right)\right.$$
$$\left. + \delta_{ki_2}\text{sign}(a_{i_1 i_2}(x,y^*))\sin\left(2\pi\kappa_{i_1 i_2}t\varepsilon^{-1}\right)\right).$$

Here, $\varepsilon > 0$ is a small parameter, $\kappa_{i_1 i_2} \in \mathbb{N}$ are pairwise distinct numbers, δ_{ij} is the Kronecker delta, and $\left((a_i(x,y^*))_{i\in S_1}, (a_{i_1 i_2}(x,y^*))_{(i_1 i_2)\in S_2}\right)^\top = a(x,y^*)$, where

$$a(x,y^*) = -\alpha\mathscr{F}^{-1}(x)(y - y^*), \quad \alpha > 0, \quad (6)$$

with $\mathscr{F}^{-1}(x)$ denoting the inverse for $n_1 \times n_1$ matrix

$$\mathscr{F}(x) = \left((g_i(x))_{i\in S_1}, (I_{n_1\times n}[f_{i_1},f_{i_2}](x))_{(i_1 i_2)\in S_2}\right).$$

Obviously, $\mathscr{F}(x)$ is nonsingular in D because of (4).

Let us mention that controllers of the form (5)-(6) has been used, e.g. in [40], [38], [43]. In this paper, we adopt the control design from the above mentioned papers to solve Problem 1. Before formulating the main result of this section, we introduce several assumptions on the vector field of system (3) and the curve $y^*(t)$.

Assumption 1: We suppose that the following properties hold in $D = D_y \times \mathbb{R}^{n_2}$.

A1) The functions $f_k \in C^1(D;\mathbb{R}^n)$, $k = \overline{1,m}$, satisfy the rank condition (4). Moreover, $g_k \in C^2(D;\mathbb{R}^{n_1})$, $g_0 \in C^1(\mathbb{R}^+ \times D;\mathbb{R}^{n_1})$, and $h_0 \in C(\mathbb{R}^+ \times D;\mathbb{R}^{n_1})$.

A2) For every compact set $\widetilde{D}_y \subset D_y$ and all $k_1,k_2,k_3 \in \overline{1,m}$, $j_1,j_2 \in \overline{0,m}$, the functions f_{k_1}, $\mathscr{L}_{f_{k_2}}g_{k_1}$, $\mathscr{L}_{f_{k_1}}\mathscr{L}_{f_{k_2}}g_{k_1}$ are bounded in $\widetilde{D} = \widetilde{D}_y \times \mathbb{R}^{n_2}$; the functions f_{k_1} are Lipschits continuous in $\widetilde{D}$; the functions f_0, $\mathscr{L}_{f_{j_1}}g_{j_2}$, $\mathscr{L}_{f_0}L_{f_{k_2}}g_{k_1}$ and $\frac{\partial g_0}{\partial t}$ are bounded uniformly in t in $\widetilde{D}$; the function f_0 is Lipschits continuous w. r. t. x uniformly in t in $\widetilde{D}$.

A3) The function $y^*: \mathbb{R}^+ \to D_y$ is Lipschitz continuous.

The following result shows that the family of controls (5)-(6) solves Problem 1 for system (3) under Assumption 1.

Theorem 1. *Let Assumption 1 be satisfied for system* (3) *and a curve* $y^* \in C(\mathbb{R}^+;D_y)$, *and let* $p,\delta > 0$ *be arbitrary numbers such that* $B_\delta(Y_t^p) \subset D$ *for all* $t \geq 0$, *where the sets* Y_t^p *are defined in* (2).

Then there exists an $\overline{\varepsilon} > 0$ *such that, for each* $\varepsilon \in (0,\overline{\varepsilon}]$, *the family of sets* $\{Y_t^p\}_{t\geq 0}$ *is asymptotically (and even exponentially) stable for system* (3) *with the controls* $u_k = u_k^\varepsilon(t,x,y^*)$ *defined by* (5) *and the initial conditions* $x(0) = x^0 \in B_\delta(Y_t^p)$.

The proof of this theorem is presented in the Appendix.

Remark 1: Unlike the paper [43], we do not require the z-extendability of solutions to system (3), which is instead guaranteed by Assumptions A1)–A2). However, if it holds that $z(t)$-variables of the solutions of system (3) belong to some set $D_2 \subset \mathbb{R}^{n_2}$ whenever the corresponding part $y(t)$ is in D_1, than we can take $\widetilde{D} = D_y \times D_2$ in A2). If, additionally, the functions g_0,g_k are bounded uniformly in t in $\widetilde{D}$, then the boundedness and Lipschitz continuity properties of the functions h_0,h_k are not required. This can be easily seen from the proof of Theorem 1.

Remark 2: A possible approach for stabilizing the y-component of system (3) along a given curve $y = y^*(t)$ can be described by introducing the deviation $\tilde{y} = y - y^*(t)$ and the transformed state vector $\tilde{x} = (\tilde{y}^\top, z^\top)^\top$. Consequently, Problem 1 is related to the stabilization of the set $\{\tilde{x} \in \mathbb{R}^n : \tilde{y} = 0\}$. The latter problem was previously addressed in the paper [43] for systems characterized by time-independent control vector fields $\tilde{f}_k(\tilde{x})$. However, the problem under consideration here differs because the corresponding vector fields $\tilde{f}_k$ depend on time, and the approach of [43] is not directly applicable in our case.

Remark 3: By using control formulas from [42], the obtained result can be extended to systems whose vector fields satisfy the controllability rank condition with iterated Lie brackets of up to length 3.

IV. Case study: an autonomous underwater vehicle model

Consider the equations of motion of an autonomous underwater vehicle with four independent controls:

$$\begin{aligned}
&\dot{x}_1 = \cos(x_5)\cos(x_6)v, \ \dot{x}_2 = \cos(x_5)\sin(x_6)v, \\
&\dot{x}_3 = -\sin(x_5)v, \\
&\dot{x}_4 = \omega_1 + \omega_2\sin(x_4)\tan(x_5) + \omega_3\cos(x_4)\tan(x_5), \quad (7) \\
&\dot{x}_5 = \omega_2\cos(x_4) - \omega_3\sin(x_4), \\
&\dot{x}_6 = \omega_2\sin(x_4)\sec(x_5) + \omega_3\cos(x_4)\sec(x_5).
\end{aligned}$$

Here, (x_1,x_2,x_3) denote the position of the center of mass, (x_4,x_5,x_6) describe the vehicle orientation (Euler
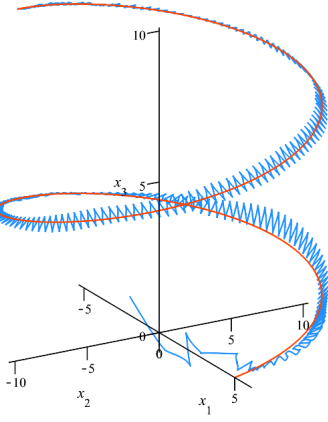
Fig. 1. The blue graph illustrates the behavior of (x_1, x_2, x_3)-components of the solution of system (7)–(8), and the red curve is $y^*(t)$.

angles), v is the translational velocity along the Ox_1 axis, and $\omega_1, \omega_2, \omega_3$ are the angular velocity components. Such equations of motion have been presented, e.g., in [32]. The stabilization problem for system (7) by means of oscillating control is considered in [38]. In this section, we consider the problem of stabilizing the (x_1, x_2, x_3) coordinates of system (7) by three controls v, ω_2, and ω_3, so we assume that the first component of the angular velocity cannot be controlled. Let us denote $v = u_1, \omega_2 = u_2, \omega_3 = u_3$, $y = (x_1, x_2, x_3)^\top$, $z = (x_4, x_5, x_6)^\top$, $x = (y^\top, z^\top)^\top$, and rewrite system (7) in the form (3) with $g_0(t,x) = (0,0,0)^\top$, $h_0(t,x) = (\omega_1(t),0,0)^\top$, $g_1(x) = (\cos x_5 \cos x_6, \cos x_5 \sin x_6, -\sin x_5)^\top$, $g_2(x) = g_3(x) = h_1(x) = (0,0,0)^\top$, $h_2(x) = (\sin x_4 \tan x_5, \cos x_4, \sin x_4 \sec x_5)^\top$, $h_3(x) = (\cos x_4 \tan x_5, -\sin x_4, \cos x_4 \sec x_5)^\top$. The rank condition (4) is satisfied in $D = \{x \in \mathbb{R}^6 : -\frac{\pi}{2} < x_5 < \frac{\pi}{2}\}$ with $S_1 = \{1\}$, $S_2 = \{(1,2),(1,3)\}$. Indeed, it is easy to check that $\mathscr{F}(x) = (g_1(x) \quad I_{3\times 6}[f_1,f_2](x) \quad I_{3\times 6}[f_1,f_3](x))$ is nonsingular in D with $\det \mathscr{F}(x) \equiv 1$. According to (5), we define the controls as $u_k = u_k^\varepsilon(t,x,y^*)$, so that

$$u_1 = a_1(x,y^*) + \sqrt{4\pi \kappa_{12}|a_{12}(x,y^*)|\varepsilon^{-1}} \cos 2\pi \kappa_{12} t \varepsilon^{-1}$$
$$+ \sqrt{4\pi \kappa_{13}|a_{13}(x,y^*)|\varepsilon^{-1}} \cos 2\pi \kappa_{13} t \varepsilon^{-1},$$
$$u_j = \text{sign}(a_{1j}(x,y^*))\sqrt{4\pi \kappa_{1j}|a_{1j}(x,y^*)|\varepsilon^{-1}} \sin 2\pi \kappa_{1j} t \varepsilon^{-1},$$
(8)

where $j = 2,3$, $a(x,y^*) = \alpha \mathscr{F}^{-1}(x)(y-y^*)$.

For numerical simulations, we choose $y^*(t) = (5\cos(0.1t), 10\sin(0.1t), 0.1t)^\top$, $\omega_1(t) = 0.1\sin(t)$ and put $\varepsilon = 0.5$, $\alpha = 5$. Fig. 1 illustrates the behavior of system (7) with control (8) and the initial condition $x^0 = (-1,-1,1,\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{4})^\top$.

V. Conclusion

The presented case study demonstrates that our approach is applicable to the class of underactuated control-affine systems adhering to a specific Lie algebra rank condition, thereby encompassing essentially nonlinear dynamical behavior. On one hand, our method extends the paradigm of

partial stabilization to encompass curve-following behaviors; on the other hand, it generalizes our earlier results by encompassing systems with non-zero drift and situations where the reference curve is defined in a lower dimensional subspace. The outcome of this work is oriented towards robotics, where there is a compelling need to stabilize only a portion of a system's states, enhancing the control and maneuverability of robotic platforms across diverse and potentially unpredictable environments.

Appendix

The proof of Theorem 1 combines and extends the techniques introduced in the papers [38], [44], [43], [45]. Note that the results of those papers cannot be directly applied because of more general assumptions. In particular, we do not require the z-extendability of solutions. For the simplicity and without loss of generality, we put $t_0 = 0$.

Proof of Theorem 1. Given a $p > 0$, let us fix δ, δ' such that $0 < \delta < \delta'$ and $B_{\delta'}(y^*(t)) \subset D_y$ for all $t \geq 0$. Denote $D' = B_{\delta'}(y^*(t)) \times \mathbb{R}^{n_2}$. From assumptions A1) and A2), there exist positive constants $\mu, M_g, M_h, M_{g_0}, M_{h_0}, L_g, L_h, L_{g_0}, L_{h_0}$, such that for all $x,\tilde{x} \in D'$, $t,\tilde{t} \geq 0$, and $k_1,k_2,k_3 \in \{1,2,3,...,m\}$, $\|\mathscr{F}^{-1}(x)\| \leq \mu$, $\|y^*(t)-y^*(\tilde{t})\| \leq L^*|t-\tilde{t}|$,
$\|g_0(t,x)\| \leq M_{g_0}$, $\|g_k(x)\| \leq M_g$, $\|g_k(x)-g_k(\tilde{x})\| \leq L_g\|x-\tilde{x}\|$,
$\|h_0(t,x)\| \leq M_{h_0}$, $\|h_k(x)\| \leq M_h$, $\|h_k(x)-h_k(\tilde{x})\| \leq L_h\|x-\tilde{x}\|$,
$\|g_0(t,x)-g_0(t,\tilde{x})\| \leq L_{g_0}\|x-\tilde{x}\|$,
$\|h_0(t,x)-h_0(t,\tilde{x})\| \leq L_{h_0}\|x-\tilde{x}\|$.
Let $x^0 = (y^{0\top},z^{0\top})^\top \in B_\delta(Y_0^p)$, $U^\varepsilon = \max_{0 \leq t \leq \varepsilon} \sum_{k=1}^m |u_k^\varepsilon(t,x^0,y_0^*)|$. From (6) and Hölder's inequality,

$$U^\varepsilon \leq c_u\sqrt{\varepsilon^{-1}\|y^0-y_0^*\|}, \tag{9}$$

$c_u = \sqrt{|S_1|}\alpha\mu\sqrt{\varepsilon(p+\delta)} + 2\sqrt{2\alpha\pi\mu}\left(\sum_{(j_1 j_2)\in S_2}(\kappa_{j_1 j_2})^{\frac{2}{3}}\right)^{\frac{3}{4}}$.

The first step of the proof is to show that all solutions of system (3) with initial conditions in $B_\delta(Y_0^p)$ are well defined in D' on the time interval $[0,\varepsilon]$ with some small enough $\varepsilon > 0$. Using the integral representation of the y-component of the solutions of system (3) with $x^0 \in B_\delta(Y_0^p)$, we get

$$\|y(t)-y^0\| = \left\| \int_0^t g_0(s,x(s))ds + \sum_{k=1}^m \int_0^t g_k(x(s))u_k(s)ds\right\|$$
$$\leq (L_0+L_g U^\varepsilon)\int_0^t \left(\|y(s)-y^0\| + \|z(s)-z^0\|\right)ds$$
$$+ (M_{g_0}+M_g U^\varepsilon)t,$$
$$\|z(t)-z^0\| \leq (L_{h_0}+U^\varepsilon L_h)\int_0^t \left(\|y(s)-y^0\| + \|z(s)-z^0\|\right)ds$$
$$+ (M_{h_0}+U^\varepsilon M_h)t.$$

Applying Grönwall–Bellman inequality to the both estimates and taking into account $t \in [0,\varepsilon]$, $x^0 \in B_\delta(Y_0^p)$, and estimate (9), we obtain:

$$\|y(t)-y^0\| \leq e^{c_g\sqrt{\varepsilon}}\left(M_{g_0}\varepsilon + M_g c_u\sqrt{\varepsilon\|y^0-y_0^*\|}\right.$$
$$\left. + c_g\varepsilon^{-1/2}\int_0^t \|z(s)-z^0\|ds\right), \tag{10}$$

$$\|z(t)-z^0\| \le e^{c_h\sqrt{\varepsilon}}\Big(M_{h_0}\varepsilon + M_h c_u \sqrt{\varepsilon\|y^0-y_0^*\|}$$
$$+ c_h \varepsilon^{-1/2}\int_0^t \|y(s)-y^0\|ds\Big),$$
(11)

where $c_g = L_{g_0}\sqrt{\varepsilon}+L_g c_u\sqrt{p+\delta}$, $c_h = L_{h_0}\sqrt{\varepsilon}+L_h c_u\sqrt{p+\delta}$. Substituting (11) into (10) and integrating by parts, we get:

$$\|y(t)-y^0\| \le e^{c_g\sqrt{\varepsilon}}\Big(\varepsilon(M_{g_0}+c_g e^{c_h\sqrt{\varepsilon}}M_{h_0}\sqrt{\varepsilon})$$
$$+c_u\sqrt{\varepsilon\|y^0-y_0^*\|}(M_g+c_g e^{c_h\sqrt{\varepsilon}}M_h\sqrt{\varepsilon})\Big)$$
$$+c_g c_h e^{(c_g+c_h)\sqrt{\varepsilon}}\int_0^t \|y(s)-y^0\|ds.$$

Applying again Grönwall–Bellman inequality, we conclude that, for every $\varepsilon > 0$ and for all $t \in [0,\varepsilon]$,

$$\|y(t)-y^0\| \le c_{y_1}\sqrt{\varepsilon\|y^0-y_0^*\|}+c_{y_2}\varepsilon,$$
(12)

where $c_{y_1} = c_u(M_g+c_g e^{c_h\sqrt{\varepsilon}}M_h\sqrt{\varepsilon})c_y$, $c_{y_2} = (M_{g_0}+c_g e^{c_h\sqrt{\varepsilon}}M_{h_0}\sqrt{\varepsilon})c_y$, $c_y = e^{\sqrt{\varepsilon}c_g(1+\sqrt{\varepsilon}c_h e^{(c_g+c_h)\sqrt{\varepsilon}})}$

With the obtained estimate, inequality (11) reads as

$$\|z(t)-z^0\| \le c_{z_1}\sqrt{\varepsilon\|y^0-y_0^*\|}+\varepsilon c_{z_2},$$
(13)

where $c_{z_1} = e^{c_h\sqrt{\varepsilon}}(M_h c_u + \sqrt{\varepsilon}c_h c_{y_1})$, $c_{z_2} = e^{c_h\sqrt{\varepsilon}}(M_{h_0}+\sqrt{\varepsilon}c_h c_{y_2})$. Estimates (12) and (13) ensure the well-definiteness of the solutions of system (3) on the time interval $[0,\varepsilon]$. Indeed, estimate (13) means that there is no blow-up of the z-component of solutions of system (3) with initial condition $x^0 \in \overline{B_\delta(Y_0^p)}$. To show that $y(t) \in B_\delta'(y^*(t))$ for all $t \in [0,\varepsilon]$, we exploit the estimate (12):

$$\|y(t)-y^*(t)\| \le c_{y_1}\sqrt{\varepsilon\|y^0-y_0^*\|}+\varepsilon(c_{y_2}+L^*)+p+\delta.$$
(14)

Taking ε_0 as the positive root of the equation $c_{y_1}\sqrt{\varepsilon(p+\delta)}+\varepsilon(c_{y_2}+L^*)=\delta'-\delta$, we ensure the well-definiteness of the solutions in D' for $t \in [0,\varepsilon]$.

The next step of the proof is to show that the distance between $y(t)$ and $y^*(t)$ does not increase after the time $t = \varepsilon$, i.e. $\|y(\varepsilon)-y^*(\varepsilon)\| \le \|y^0-y_0^*\|$. For this purpose, note that any solution of system (3) with initial data $x^0 \in B_\delta(Y_0^p)$ and controls (5) can be represented by means of the Chen–Fliess type series [46], [40], [38], [43]. Similarly to as it is shown [43], the y-component of the series expansion can be represented in the form

$$y(\varepsilon) = y^0 - \varepsilon\alpha(y^0-y_0^*) + \varepsilon g_0(0,x^0) + R(\varepsilon,x^0),$$
(15)

where $\|R(\varepsilon,x^0)\| \le c_1\varepsilon^{3/2}\|y_0^*-y^0\|^{3/2}+c_2\varepsilon^{\frac{3}{2}}\|y^0-y_0^*\|+c_3\varepsilon^{\frac{3}{2}}$, with some $c_1,c_2,c_3 \ge 0$. Let us analyse the value $\|y(\varepsilon)-y^*(\varepsilon)\|$. From (15),

$$y(\varepsilon)-y^*(\varepsilon) = (1-\varepsilon\alpha)(y^0-y_0^*)-(y^*(\varepsilon)-y_0^*)$$
$$+\varepsilon g_0(0,x^0)+R(\varepsilon,x^0).$$

From A2), A3), and the estimate for $\|R(\varepsilon,x^0)\|$,

$$\|y(\varepsilon)-y^*(\varepsilon)\| \le (1-\varepsilon(\alpha-\sqrt{\varepsilon}q))\|y_0-y_0^*\|$$
$$+\varepsilon(L^*+M_{g_0}+\sqrt{\varepsilon}c_3),$$

provided that $\varepsilon < \varepsilon_1 = \frac{1}{\alpha}$ and $x^0 \in B_\delta(Y_0^p)$. Here $q = c_1\sqrt{p+\delta}+c_2$.

Our next goal is to show the attraction of the y-components of the solution to the p-neighborhood of the curve $y^*(t)$. Assume that $\alpha > \nu(L^*+M_{g_0})/p$ with some $\nu > 1$. Using estimate (14), we may ensure the following property: if $x^0 \in Y_0^{\frac{p}{\nu}}$ then $x(t) \in Y_t^p$ for all $t \in [0,\varepsilon]$ with a small enough ε. Indeed, let us define ε_2 as the positive root of the equation $c_{y_1}\sqrt{\frac{\varepsilon p}{\nu}}+\varepsilon(c_{y_2}+L^*)=\frac{p(\nu-1)}{\nu}$. Then estimate (14) yields

$$\|y(t)-y^*(t)\| \le c_{y_1}\sqrt{\varepsilon\|y^0-y_0^*\|}+\varepsilon c_{y_2}+\varepsilon L^* + p/\nu \le p,$$

provided that $\|y^0-y_0^*\| \le \frac{p}{\nu}$ and $\varepsilon \le \varepsilon_2$.

Consider two possibilities:

1) If $x^0 \in Y_0^{\frac{p}{\nu}}$, then, as discussed above, $\|y(t)-y^*(t)\| \le p$ for all $t \in [0,\varepsilon]$, that is $x(t) \in Y_t^p$ for all $t \in [0,\varepsilon]$.

2) If $x^0 \in B_\delta(Y_0^p)\backslash Y_0^{\frac{p}{\nu}}$, i.e. $\|y^0-y_0^*\| \ge \frac{p}{\nu}$, then

$$\|y(\varepsilon)-y^*(\varepsilon)\| \le \|y_0-y_0^*\|\big(1-\varepsilon(\alpha-\nu(L^*+M_{g_0})/p$$
$$-\sqrt{\varepsilon}(q+\nu c_3/p))\big)$$

For an arbitrary $\lambda \in (0,\alpha-\nu(L^*+M_{g_0})/p)$, let us define $\varepsilon_3 = \left(\frac{(\alpha-\lambda)p-\nu(L^*+M_{g_0})}{pq+\nu c_3}\right)^2$. Then, for every $\varepsilon \in (0,\varepsilon_3)$, $\|y(\varepsilon)-y^*(\varepsilon)\| \le (1-\varepsilon\lambda)\|y_0-y_0^*\|$.

Thus, $\|y(\varepsilon)-y^*(\varepsilon)\| \le \|y^0-y_0^*\| \le p+\delta$ and $x(\varepsilon) \in B_\delta(Y_\varepsilon^p) \subset D'$.

So we may repeat all the above argumentation for the solutions with the initial condition $x(\varepsilon)$ with the same choice of $\lambda \in (0,\alpha-\nu(L^*+M_{g_0})/p)$ and $\varepsilon \in \min\{\varepsilon_0,\varepsilon_1,\varepsilon_3\}$. This proofs the well-definiteness in D' of the solutions of system (3) with $x^0 \in B_\delta(Y_0^p)$ for $t \in [0,2\varepsilon]$. Besides, we can consider again two cases:

1) if $\|y(\varepsilon)-y^*(\varepsilon)\| \le p/\nu$ then $\|y(t)-y^*(t)\| \le p$ for $t \in [\varepsilon,2\varepsilon]$;

2) if $\|y(\varepsilon)-y^*(\varepsilon)\| > p/\nu$ then $\|y(2\varepsilon)-y^*(2\varepsilon)\| \le (1-\varepsilon\lambda)\|y(\varepsilon)-y^*(\varepsilon)\|$.

Iterating all above-described steps, we may conclude that the solutions of system (3) with control (5) and initial conditions $x^0 \in B_\delta(Y_0^p)$ are well defined in D' for all $t \ge 0$. Furthermore, if $x^0 \in Y_0^{\frac{p}{\nu}}$ then $x(t) \in Y_0^p$ for all $t \ge 0$. If $x^0 \in B_\delta(Y_0^p)\backslash Y_0^{\frac{p}{\nu}}$, then there exists an $N \in \mathbb{N}$ such that $\|y(t)-y^*(t)\| > p$ for each $t = 0,\varepsilon,2\varepsilon,...,(N-1)\varepsilon$ and $\|y(t)-y^*(t)\| \le \frac{p}{\nu}$ for all $t \in [0,+\infty)$. It remains to describe the behavior of $y(t)$ for an arbitrary $t \in [0,N\varepsilon]$.

As follows from the previous argumentation, the following estimate holds for $t = 0,\varepsilon,2\varepsilon,...,(N-1)\varepsilon$:

$$y(j\varepsilon)-y^*(j\varepsilon) \le (1-\varepsilon\lambda)^j\|y^0-y_0^*\| \le e^{-\lambda j\varepsilon}\|y^0-y_0^*\|,$$

for $j = 0,1,...,N-1$. For an arbitrary $t \in [0,N\varepsilon]$, denote by $t_{in} = \left[\frac{t}{\varepsilon}\right]$ the integer part of $\frac{t}{\varepsilon}$. Notice that $t-t_{in}\varepsilon < \varepsilon$, then

$$\|y(t)-y^*(t)\| \le \|y(t_{in}\varepsilon)-y^*(t_{in}\varepsilon)\| + \|y(t)-y(t_{in}\varepsilon)\|$$
$$+\|y^*(t)-y^*(t_{in}\varepsilon)\| \le \gamma_1 e^{-\frac{\lambda t}{2}}+\varepsilon\gamma_2,$$

where $\gamma_1 = e^{\frac{\varepsilon\lambda}{2}}\sqrt{\|y^0-y_0^*\|}(c_{y_1}\sqrt{\varepsilon}+e^{\frac{\varepsilon\lambda}{2}}\sqrt{\|y^0-y_0^*\|})$, $\gamma_2 = c_{y_2}+L^*$. This completes the proof.

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