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Viabilizability of Control Signals under Control Authority Degradation

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Abstract— **In this work, we solve the problem of mitigating control authority degradation in real time. In particular, we focus on controlled nonlinear affine-in-control evolution equations with finite control input and finite- or infinite-dimensional state. We consider control input degradation parameterized by Lipschitz continuous maps. These degradation modes are encountered in practice due to actuator wear and tear, hard locks on actuator ranges due to over-excitation, as well as more general changes in the control allocation dynamics. In previous work, we have derived sufficient conditions for real-time identifiability of control authority degradation. In this work, we build on these results by introducing the concept of *viabilizability*, which deals with the existence of a *viabilizing map*. Viabilizing maps remap commanded control signals to viabilized signals that produce a minimally disturbed approximation of the commanded control signal after control authority degradation. We develop sufficient conditions on viabilizability for a class of control degradation modes, as well as error bounds on approximate viabilizing maps and methods for viabilizing fixed gain controllers.**

I. Introduction

In control systems, fault detection and mitigation are key in ensuring prolonged safe operation in safety-critical environments [1]. Any physical system undergoes gradual degradation during its operational life cycle, for instance due to interactions with the environment or internally as a result of actuator wear and tear. Gradual degradation or impairment, as the name suggests, often degrades the performance of a system in cases when potential degradation modes were not taken into account during control synthesis. Fault tolerance is a key property of systems that are capable of mitigating or withstanding system faults, including gradual degradation.

In this work, we consider a known nonlinear control-affine system of the form of

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{1}$$

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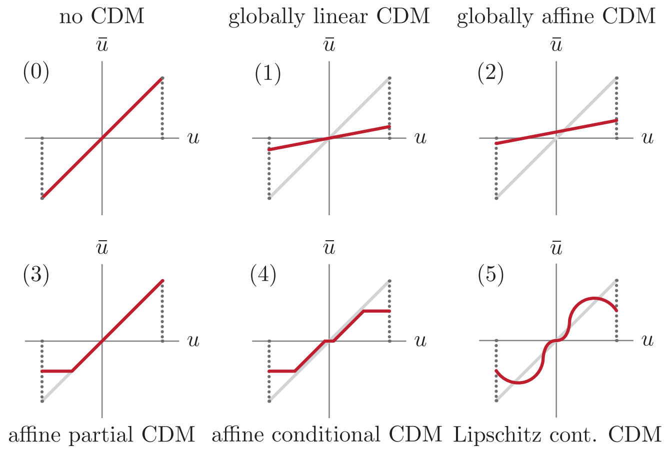

Fig. 1: Comparison between various classes of control authority degradation maps.

where $x \in X$, $u \in U \subseteq \mathcal{U}$, X and $\mathcal{U}$ are Hilbert spaces, and $f : X \to X$ and $g : X \to \mathcal{L}(\mathcal{U}, X)$. In this work, we assume $\mathcal{U} = \mathbb{R}^m$. In addition, we assume that U is a *star-shaped* subset of $\mathbb{R}^m$ such that span $U = \mathbb{R}^m$. Finally, we assume that the full-state of the degraded system,

$$\dot{\bar{x}}(t) = f(\bar{x}(t)) + Rg(\bar{x}(t))Pu(t), \tag{2}$$

is known without error.

In system (2), a control action degradation map R can model changes in the control allocation function g, which may include actuator reconfiguration, such as a change in the trim angle on aircraft control surfaces or misalignment of actuators due to manufacturing imperfections or wear and tear. Since R acts after g, it does not directly remap the control signal $u(t)$, but it changes the action of a control input on the system; we therefore talk about *control effectiveness*, as opposed to control authority in the case of a control authority degradation map (CDM) P, which acts before g. Changes in the drift dynamics $f(x(t))$ will not be treated in this work.

It bears mentioning that CDMs are capable of modeling changes in the actuator's interaction with the physical system. This is particularly relevant to medical biophysics, where the underlying partial differential equations may undergo slight changes in the control input map as a result of changes in the surgical probe contact [2]. In such cases, when a nominal (optimal) control law is available, introducing a viabilizing map obviates the need to synthesize a new controller from scratch.

In addition to identifying or approximating CDM P, which we have studied in past work [3], we are interested in 'undoing' the effects of control authority degradation as much as possible. In particular, we are interested in the set of control signals (1) that can still be replicated in (2) when the CDM is acting; we call this the set of *viable control inputs*, U_{v}. With knowledge of P, we develop here a method to obtain, for $u_{\mathrm{cmd}} \in U_{\mathrm{v}}$, u_{v} such that $Pu_{\mathrm{v}} = u_{\mathrm{cmd}}$; here, u_{cmd} and u_{v} are called commanded and viabilized control inputs, respectively. This approach is closely related to a technique known in the literature as *fault hiding* [4]. Fault hiding is achieved by introducing an output observer based on the output of the degraded system, and augmenting the nominal system model by introducing so-called virtual actuators, which requires a nonlinear reconfiguration block that is strongly dependent on the underlying problem structure and failure modes [4, §3.6, p. 42]. In the setting considered in this work, we show that we can adopt the fault hiding philosophy under much less stringent constraints for a general class of systems and degradation modes.

In this work, we are interested in modeling unknown degraded system dynamics (2) for a time-invariant control authority degradation map (CDM) $P : U \to \bar{U}$, and no control effectiveness degradation (i.e., $R = I$). This problem amounts to reconstructing, or *identifying*, P:

Problem 1 (Identifiability of Control Authority Degradation Maps)**.** For a class of time-invariant CDMs $P \in \mathcal{P}$, for $\mathcal{P}$ known, if possible, identify P based on a finite number of full state, velocity, and control input observations ($\bar{x}(t)$, $\dot{\bar{x}}(t)$, $u(t)$) of the degraded system.

Ideally, we would like to identify general nonlinear CDMs with known bounds on the approximation error. In this work, we deal with the CDMs shown in Fig. 1.

We have addressed this problem in our previous work [3], where we presented a method for approximating CDMs online based on a finite number of full state, velocity, and control input observations ($\bar{x}(t)$, $\dot{\bar{x}}(t)$, $u(t)$) of the degraded system. In this work, we are interested in exploring *viability under CDM P*, i.e., determining whether it is possible to maintain known system properties under control authority degradation mode P. In particular, we are interested in finding a *viabilizing map* s_{v} that takes a commanded input u_{cmd}, and maps it to a value $u_{\mathrm{v}} \in \bar{U}$, such that $Pu_{\mathrm{v}} = u_{\mathrm{cmd}}$, if such a value u_{v} exists. If such a value u_{v} does not exist, we consider the projection of u_{cmd} onto the set of viable control inputs $U_{\mathrm{v}} := \operatorname{range} s_{\mathrm{v}}$, and produce bounds on the error $\|u_{\mathrm{cmd}} - \operatorname{proj}_{U_{\mathrm{v}}} u_{\mathrm{cmd}}\|$. This latter error can be thought of as an input disturbance, which could be accounted for in robust controller synthesis. We refer to this remapping of u_{cmd} given P under s_{v} as a *viabilizing control remapping*, where the goal is to *mitigate as best as possible the effects of control authority degradation*. We formalize this problem as follows:

Problem 2 (Viabilizability of Control Authority Degradation Maps)**.** For a class of (partially) identified time-invariant CDMs $\mathcal{P} \in P$, determine if it is possible to construct a viabilizing map s_{v} such that $Ps_{\mathrm{v}}(u) = u$ for all $u \in U$. If not, is it possible to obtain an approximately viabilizing map $s_{\mathrm{v}} : U \to \bar{U}$ such that $Ps_{\mathrm{v}}(u) \in u + \mathcal{B}_\epsilon$ for some known radius $\epsilon > 0$?

We have omitted proofs in this work in light of space constraints; many proofs are self-evident, and those that are not will be published in future work.

II. Preliminaries

We use $\|\cdot\|$ to denote the Euclidean norm. Given two sets $A, B \subseteq \mathbb{R}^n$, we denote by $A + B$ their Minkowski sum $\{a + b : a \in A, b \in B\}$; the Minkowski difference is defined as $A - B = \{a - b : a \in A, b \in B\}$. By 2^A we refer to the power set of A, i.e., the family of all subsets of A; $\mathscr{C}(A)$ denotes the family of all compact convex subsets of A. We denote a closed ball centered around the origin with radius $r > 0$ as $\mathcal{B}_r$. By $\mathcal{B}(x, r)$ we denote $\{x\} + \mathcal{B}_r$. We denote by $\mathcal{L}(A, B)$ the set of bounded linear operators, and by $C(A, B)$ the set of closed linear operators between A and B. We define $\mathbb{R}_+ := [0, \infty)$. For two points in a Banach space $\mathscr{B} \ni a, b$, let $[a, b]$ denote the convex hull of a and b, i.e., $[a, b] := \operatorname{conv}\{a, b\}$. Given a point $x \in S$ and a set $A \subseteq S$, we denote $d(x, A) := \inf_{y \in A} d(x, y)$. We define the distance between two sets $A, B \subseteq \mathbb{R}^n$ to be

$$d(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|. \tag{3}$$

We denote the Hausdorff distance as

$$d_{\mathrm{H}}(A, B) := \max\{d(A, B), d(B, A)\}, \tag{4}$$

An alternative characterization of the Hausdorff distance reads [5, pp. 280–281]:

$$d_{\mathrm{H}}(A, B) = \inf\{\rho \geq 0 : A \subseteq B_{+\rho}, B \subseteq A_{+\rho}\}, \tag{5}$$

where $X_{+\rho}$ denotes the *ρ-fattening* of X, i.e., $X_{+\rho} := \bigcup_{x \in X} \{y \in \mathbb{R}^n : \|x - y\| \leq \rho\}$.

We denote by ∂A the boundary of A in the topology induced by the Euclidean norm. For a function $g : A \to B$, we denote by g^{-1} the inverse of this function if an inverse exists and otherwise denoting the preimage. By $\operatorname{dom}(g)$ we refer to the domain of the function (in this case A). We denote by $g^\dagger$ the Moore–Penrose pseudo-inverse of a linear function g. We use the Iverson bracket notation $[\![\cdot]\!]$, where the value is 1 if the expression between the brackets is true, and 0 otherwise.

In this work, we shall consider star-shaped sets, which are defined as follows:

Definition II.1 (Star-shaped Set and MGFs [6, §15, p. 128])**.** We call a closed compact set $K \subseteq \mathscr{B}$ *star-shaped* if there exist (i) $\varsigma \in K$, and (ii) a unique function $\varrho : \mathcal{B}_1 \to \mathbb{R}_+$, such that

$$K = \bigcup_{l \in \mathcal{B}_1} [\varsigma, \varsigma + \varrho(l) l],$$

where $\mathcal{B}_1$ denotes the unit ball in $\mathscr{B}$. We call ϱ a *Minkowski gauge function* (MGF), and ς the *star center*.

We now proceed by solving Problem 1 for an unknown multi-mode affine CDMs, which allows for approximating Lipschitz continuous nonlinear CDMs with bounded error.

III. Identifiability of Control Authority Degradation Maps

We now consider Problem 1. Let us assume that for U, the Minkowski gauge function ϱ is known. Let $P : U \to \bar{U}$ be an unknown control authority degradation map (CDM). We assume that $\bar{U}$ is also a star-shaped set, providing conditions on P and U under which this holds. It bears mentioning that star-shaped sets are more general than convex sets; most results presented in this work will apply to star-shaped sets, which include polytopes, polynomial zonotopes, and ellipsoids.

Before we provide any results on the identifiability of control authority degradation modes, we pose the following key assumption on the nominal system dynamics (1). We allow for an *infinite-dimensional* state-space X, that is to say, X is a set of *functions*, but $X = \mathbb{R}^n$ is also captured:

Assumption 1. For system (1), assume that

i. $g(x) \in \mathbb{R}^{n\times m}$ has closed range for all $x \in X$;
ii. $g(x)$ is injective for all $x \in X$, i.e., $\ker(g(x)) = \{0\}$;
iii. $f(x)$ is known at least one $x \in X$.

Remark 1. In the case of finite-dimensional systems, i.e., $X \subseteq \mathbb{R}^n$, the first two conditions of Assumption 1 can be stated as:

i. The system is not overactuated, i.e., $m \le n$;
ii. $g(x)$ is of full-column rank for all $x \in X$.

We shall consider the case of multiple control degradation modes acting throughout the space U. The simplest of the so-called *conditional control authority degradation modes* (c-CDMs) acts only on a compact subset of U; we refer to these c-CDMs as *partial control authority degradation modes* (p-CDMs). Consider two compact star-shaped sets $\check{U}, \hat{U} \subseteq U$, and two p-CDMs

$$P_{\check{U}}(u) := u + [\![u \in \check{U}]\!](P - I)u, \tag{6}$$

$$P_{\hat{U}}(u) := u + [\![u \notin \hat{U}]\!](P - I)u, \tag{7}$$

for some control degradation map P. Here, $P_{\check{U}}$ is an *internally acting* partial CDM (i.e., acting inside $\check{U}$), whereas $P_{\hat{U}}$ is an *externally acting* partial CDM (acting outside $\hat{U}$); when this distinction is immaterial, we use a combined hat and check symbol (e.g., $\hat{\check{U}}$), where $\hat{\check{U}}$ is simply called the *affected set* of control inputs.

In reconstructing an N-mode CDM, we face the problem of discerning which control inputs belong to which conditional degradation mode. To make this problem tractable, we pose the following assumption:

Assumption 2. Let the internally acting N-mode CDM satisfy the following properties:

i. The number of modes N is known;
ii. $\check{\mathcal{U}}$ is a family of convex sets;
iii. $\check{\mathcal{P}}$ is a family of affine maps denoted by $Q_i = p_i + P_i$.
iv. There exists a known $\delta > 0$ such that for all $i \neq j$, $d_{\mathrm{H}}\left((\check{U}_i, \check{P}_i\check{U}_i), (\check{U}_j, \check{P}_j\check{U}_j)\right) \ge \delta$.

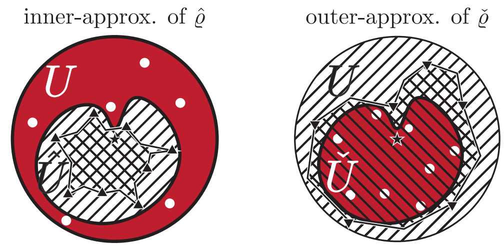

Fig. 2: Comparison between inner- and outer-approximations of $\hat{U}$ and $\check{U}$ respectively, for a 1-mode c-CDM. The region with top-right-pointing hatching indicates the set in which the control input is unaffected; the red-colored region indicates the affected set. The respective approximations of $\hat{\check{U}}$ allow one to find regions in which control inputs are guaranteed to be unaffected. In the left image, the set indicated by top-left-pointing hatching is an inner-approximation of $\hat{U}$, and in the right image this set is an outer-approximation of $\check{U}$.

We are also interested in obtaining outer-approximations of $\check{U}$ and inner-approximations of $\hat{U}$ for each degradation mode, as illustrated in Fig. 2, so that we can restrict control inputs to regions that are guaranteed to be unaffected. Since we only have access to a finite number of control input samples, we pose the following assumption regarding the regularity of the MGF associated with $P\hat{\check{U}}$.

Assumption 3. Assume that $\hat{\check{U}}$ has star center $\hat{\check{\varsigma}} = 0$, and assume that the MGF $\hat{\check{\varrho}}$ associated with $\hat{\check{U}}$ is Lipschitz continuous, i.e., there exists a known $\hat{\check{L}}$ such that

$$|\hat{\check{\varrho}}(l) - \hat{\check{\varrho}}(l')| \le \hat{\check{L}}\|l - l'\|, \tag{8}$$

for all $l, l' \in \mathcal{B}_1$.

We can now pose a key result on the guaranteed approximation of Lipschitz MGFs from a finite set of samples.

Proposition 1. *[[3, Prop. 1]] Assume that Assumption 3 holds for the unknown MGFs $\check{\varrho}$ and $\hat{\varrho}$. Then, for some given $\check{u}, \check{u}' \in \check{U}$ and $\hat{u}, \hat{u}' \notin \hat{U}$, we have for all $\mu \in [0, 1]$:*

$$\check{\varrho}\left(\frac{\mu\check{l} + (1-\mu)\check{l}'}{\|\mu\check{l} + (1-\mu)\check{l}'\|}\right) \le \min\left\{\|\check{u}\| + (1-\mu)\check{L}\|l - l'\|, \|\check{u}'\| + \mu\check{L}\|\check{l} - \check{l}'\|\right\}, \tag{9}$$

and

$$\hat{\varrho}\left(\frac{\mu\hat{l} + (1-\mu)\hat{l}'}{\|\mu\hat{l} + (1-\mu)\hat{l}'\|}\right) \ge \max\left\{0, \|\hat{u}\| - (1-\mu)\hat{L}\|\hat{l} - \hat{l}'\|, \|\hat{u}'\| - \mu\hat{L}\|\hat{l} - \hat{l}'\|\right\}, \tag{10}$$

where $\hat{\check{l}} := \hat{\check{u}}/\|\hat{\check{u}}\|$ and $\hat{\check{l}}' := \hat{\check{u}}'/\|\hat{\check{u}}'\|$.

We now reiterate a theorem on the identifiability of N-mode conditional control authority degradation modes (c-CDMs), where multiple affine CDMs act on disjoint subsets of U.

Theorem 1 (Reconstructing N-mode Affine c-CDMs [3, Thm. 1])**.** *Consider system* (2) *and Assumptions 1–2. Assume that the c-CDM is represented by N unknown internally acting* affine *maps Q_i, each acting on mutually disjoint* *unknown star-shaped sets $\check{U}_i \subseteq U$, giving $Q_{\check{U}}$ as the p-CDM. Let there be a given array of distinct state–input pairs $[(\bar{x}[i], u[i])]_{i=1}^{N}$, and a corresponding array of degraded velocities $[\dot{\bar{x}}[i]]_{i=1}^{N'}$ obtained from system (2), with $N' \geq N(m+1)$. Let there also be a given array of undegraded state–input pairs $[(x_*[i], u_*[i])]_{i=1}^{M}$, with $M \geq m$. Assume that there exist m state–input pairs indexed by $\jmath$ and $\jmath_*$, such that the arrays of input vectors $\{u[\jmath_j]\}_{j=1}^{m}$ and $[u_*[\jmath_{*,j}]]_{j=1}^{m}$ are linearly independent.*

Cluster the array $[(u[i], \tilde{u}[i])]_{i=1}^{N'}$ into N clusters with a Hausdorff distance of at least δ between each pair of clusters. If each cluster i contains at least m vectors $u[i]$ that are linearly independent, then $Q_{\check{U}}$ can then be approximated as follows:

$$\tilde{Q}_{\check{U}} u = \begin{cases} u & u \notin \check{U}_{\text{outer}}, \\ \sum_{i=1}^{N} [\![u \in \check{U}_{i,\text{inner}}]\!] Q_i u & u \in \check{U}_{\text{inner}}, \\ \text{inconclusive} & u \in \check{U}_{\text{outer}} \setminus \check{U}_{\text{inner}}, \end{cases} \tag{11}$$

where $\check{U}_{\text{inner}} := \bigcup_{i=1}^{N} \check{U}_{i,\text{inner}}$ and $\check{U}_{\text{outer}} := \bigcup_{i=1}^{N} \check{U}_{i,\text{outer}}$. Each Q_i are obtained as by considering for each cluster $v_i := g^{\dagger}(\bar{x}[j])(\dot{\bar{x}}[j] - f(\bar{x}[j]))$ where index j is not part of the array of linearly independent inputs indexed by $\jmath$, $\mathbf{u} := [\, u[\jmath_1]-v_i \ \cdots \ u[\jmath_m]-v_i \,]$ and

$$\Delta\mathbf{u} := \left[g^{\dagger}(\bar{x}[\jmath_j])(\dot{\bar{x}}[\jmath_j] - f(\bar{x}[\jmath_j])) - u[\jmath_j] - v_i \right]_{j=1}^{m}.$$

Linear operator P_i is obtained as

$$P_i = (\mathbf{u} + \Delta\mathbf{u})\mathbf{u}^{\mathsf{T}}(\mathbf{u}\mathbf{u}^{\mathsf{T}})^{-1}. \tag{12}$$

The translation p_i is obtained as $p_i = v_j - P_i u[j]$, which yields the i'th mode affine CDM Q_i:

$$Q_i u := p_i + P_i u. \tag{13}$$

Remark 2. This result incorporates p-CDMs that map a set $\check{U}$ to a constant, e.g., $Q_{\check{U}}\check{U} = p$. To highlight the utility of this result, it should be noted that the *hypotheses given here allow for commonly encountered degradation modes such as deadzones and saturation to be modeled* (see Fig. 1(4)). Additionally, Theorem 1 allows for *discontinuous* control authority degradation modes, a property that is often not present in prior work.

We can now consider the case in which P is a Lipschitz continuous CDM. We consider an approximation of P by an N-mode affine c-CDM $\tilde{P}$, for which we derive an explicit error bound given that the Lipschitz constant of P, L_P, is known.

Theorem 2 (Approximating Lipschitz continuous CDMs by N-mode Affine c-CDMs [3, Thm. 2])**.** *Let the hypotheses of Theorem 1 hold, with the exception that $P := Q_{\check{U}}$ is now a Lipschitz continuous CDM with Lipschitz constant L_P and Assumption 2 is now dropped. If N clusters that satisfy the linear independence requirements of Theorem 1 are identified, then the resulting N-mode affine c-CDM approximation $\tilde{P}$ has the following error:*

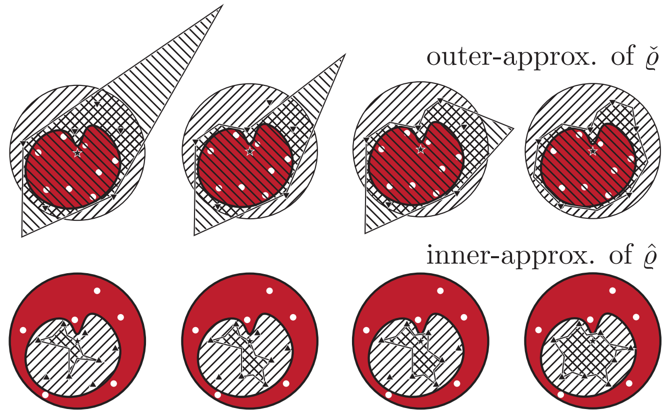

Fig. 3: Comparison between inner- and outer-approximations of $\hat{U}$ and $\check{U}$ respectively, based on Proposition 1 and Theorem 1 for an increasing number of samples for a 1-mode c-CDM. Clearly, for a larger number of points of sufficiently dispersed points, increasingly tight approximations are obtained as formalized in Lemma 1.

For all $u \in \check{U}_{i,\text{inner}}$ and all $i = 1, \ldots, N$,

$$\|Pu - \tilde{P}u\| \leq \| \min_{j=1,\ldots,m} \varepsilon_{i,j} + L_P \|u[i,j] - u\|, \tag{14}$$

where $\varepsilon_{i,j} := \|P\mathbf{u}_i[j] - \check{P}_i\mathbf{u}_i[j]\|$, and $u[i,j] := \mathbf{u}_i[j]$, where $\mathbf{u}_i$ is an array composed of all control inputs in the i'th cluster.

We can now pose a convergence result on the N-mode affine c-CDM approximation $\tilde{P}$ of a Lipschitz continuous CDM P.

Corollary 1. Error bound (14) is monotonically decreasing in the the number of samples N' and the number of c-CDM modes N. In the limit of the $N', N \to \infty$, error bound (14) converges to zero.

It is in general impossible to uniquely determine each $\tilde{\check{U}}$ from finitely many samples. Intuitively, given a greater number of distinct samples inside $\hat{\check{U}}$ and $U \setminus \hat{\check{U}}$, it should be possible to more tightly approximate $\hat{\check{U}}$. This idea is illustrated in Fig. 3. We now state a lemma on the convergence of inner- and outer-approximations of the affected set $\tilde{\check{U}}$.

Lemma 1. *Consider $\epsilon > 0$, such that a given set of $N_\epsilon \geq m$ distinct pairs $(u, P_{\tilde{\check{U}}}u)$ denoted by $\mathcal{G}U_{N,\epsilon}$, satisfies Assumptions 1–3, where $P_{\tilde{\check{U}}}$ is (i) an N-mode affine c-CDM, or (ii) a Lipschitz continuous CDM. Let $\mathcal{G}U_N$ be such that for each u_i in $\mathcal{G}U_{N,\epsilon}$, $\bigcup_{i=1}^{N_\epsilon} \mathcal{B}_\epsilon(u_i) \supseteq \tilde{\check{U}}$; i.e., ϵ-balls centered at each sampled control input form a cover of $\tilde{\check{U}}$. Let $\tilde{\check{U}}_{\text{inner}}^{N_\epsilon}$ and $\tilde{\check{U}}_{\text{outer}}^{N_\epsilon}$ denote the corresponding inner- and outer-approximations of $\tilde{\check{U}}$ using the procedure given in Theorem 1 from $\mathcal{G}U_{N,\epsilon}$. Then, we have $\tilde{\check{U}}_{\text{inner}}^{N_\epsilon} \subseteq \tilde{\check{U}}_{\text{inner}}^{N_{\epsilon'}} \subseteq \tilde{\check{U}}$ and $\tilde{\check{U}} \subseteq \tilde{\check{U}}_{\text{outer}}^{N_{\epsilon'}} \subseteq \tilde{\check{U}}_{\text{outer}}^{N_\epsilon}$ for all $\epsilon' < \epsilon$. In addition, we have*

$$\lim_{\epsilon \to 0} \tilde{\check{U}}_{\text{inner}}^{N_\epsilon} = \lim_{\epsilon \to 0} \tilde{\check{U}}_{\text{outer}}^{N_\epsilon} = \tilde{\check{U}}.$$

Remark 3. In Lemma 1, note that the ϵ-covering argument is required to ensure that the distinct samples are sufficiently dispersed; simply considering $N \to \infty$ does not ensure convergence of the Hausdorff distance between the inner- and outer-approximation to zero. This fact can also be observed when looking at Fig. 3.

Having obtained a generator of the N-mode affine c-CDM approximations to Lipschitz continuous CDMs along with key approximation error convergence results, we can proceed by trying to 'invert' P so as to remap commanded control signals u_{cmd} to *viabilizing* signals u_{v} such that $Pu_{\text{v}} = u_{\text{cmd}}$, if possible. Instead of restricting the control inputs to regions that are guaranteed to be unaffected, we consider the problem of remapping commanded control signals to their closest *viable* counterpart so as to achieve *approximate fault masking*.

IV. Viabilizability under Control Authority Degradation

From Theorem 1 we can define the following set of remapped control inputs that can effectively be applied to system (2), which we refer to as *viable control inputs*:

$$U_{\text{v}} := \left(\tilde{Q}_{\check{\mathcal{V}}}\check{U}_{\text{inner}}\right) \cup \left(U \setminus \check{U}_{\text{outer}}\right). \tag{15}$$

We are interested in taking a commanded control input $u_{\text{cmd}} \in U_{\text{v}}$ and computing $u \in U$ such that $Q_{\check{\mathcal{U}}}u = u_{\text{cmd}}$, where this u is referred to as a *viabilizing control input*; we shall treat the implications of *inviable* $u_{\text{cmd}} \in U \setminus U_{\text{v}}$ later. To simplify notation, we shall refer to $\tilde{Q}^{-1}_{\check{\mathcal{V}}}$ as $\tilde{F} : \tilde{V} \to 2^{\tilde{V}}$, where $\tilde{V} := \text{conv}(\tilde{U}_{\text{v}} \cup U)$, i.e., the convex hull of $\tilde{U}_{\text{v}} \cup U$.

It is guaranteed that commands in U_{v} are viable by (11) in Theorem 1. Since in general, there exist more than one $u \in U$ such that $Q_{\check{\mathcal{U}}}u = u_{\text{cmd}}$ for $u_{\text{cmd}} \in U_{\text{v}}$, we are interested in finding a *continuous selector* s_{v} that maps u_{cmd} to a viabilizing input. This is of particular importance if the actuator does not admit discontinuous control signals, or if there are actuation rate constraints. In the remainder of this work, we assume a (potentially discontinuous) viabilizing map s_{v} is available.

Since U_{v} need not be a connected set, and $U_{\text{v}} \setminus U$ need not be empty, the problem of finding $u \in U$ such that $Q_{\check{\mathcal{U}}}u = u_{\text{cmd}}$ is nontrivial.

Before we pose results on the existence of such a Lipschitz selection, it bears mentioning that such Lipschitz selectors *do not exist* if $\mathcal{U}$ is infinite-dimensional [7, p. 569]; we hence limit our discussion to finite-dimensional $\mathcal{U} \subseteq \mathbb{R}^m$ as before.

Before we consider the approximation of $Q_{\check{\mathcal{V}}}$ given in Theorem 1, we shall consider the true sets and functions, which are indicated by symbols that lack a tilde.

A. Lipschitz Selection of $Q^{-1}_{\check{\mathcal{V}}}$

Our goal is to find a selector $s : 2^V \to V$ that enjoys certain regularity properties, so as to ensure regularity of the resulting remapped control signal. The results that we shall leverage specify that the range of F be on $\mathscr{C}(V)$, the family of closed convex nonempty sets in V. Our main result will provide an explicit construction of a Lipschitz continuous selector s_{v} that *can be evaluated efficiently*. We shall now pose conditions on $Q_{\check{\mathcal{V}}}$ that ensure that its preimage lying in $\mathscr{C}(V)$.

Lemma 2 ($\mathscr{C}(\mathbb{R}^m)$-valued Preimage of $Q_{\check{\mathcal{V}}}$)**.** *For an internally acting N-mode affine c-CDM satisfying the hypotheses of Theorem 1, $\widetilde{\text{CDM}}_N$, suppose that for all $i = 1, \dots, N$, the following properties hold:*

(a) $(\check{P}_i\check{U}_i) \cap \left(U \setminus \bigcup_{j=1}^N \check{U}_j\right) = \emptyset$; and
(b) $(\check{P}_i\check{U}_i) \cap (\check{P}_j\check{U}_j) = \emptyset$ for all $j \neq i$.
(c) $(\check{P}_i\check{U}_i) \supseteq \left(U \setminus \bigcup_{j=1}^N \check{U}_j\right)$; or
(d) $(\check{P}_i\check{U}_i) \subseteq (\check{P}_j\check{U}_j)$ or $(\check{P}_i\check{U}_i) \supseteq (\check{P}_j\check{U}_j)$ for some $j \neq i$.
(e) $\check{P}_i\check{U}_i = \check{P}_j\check{U}_j$ and $\check{U}_i \cup \check{U}_j$ is convex for some $j \neq i$.

Then, we have range $Q^{-1}_{\check{\mathcal{V}}} \subseteq \mathscr{C}(\mathbb{R}^m)$.

Remark 4. The lemma above covers CDMs that map a set $\check{U}$ to a constant, e.g., $Q_{\check{\mathcal{V}}}\check{U} = p$. To highlight the utility of this result, it should be noted that the hypotheses given here allow for commonly encountered degradation modes such as deadzones and saturation to be modeled (see Fig. 1(4)).

We can formulate the following corollary on left-invertibility of $Q_{\check{\mathcal{V}}}$, i.e., its preimage lies in V:

Corollary 2. In addition to the hypotheses of Lemma 2, if for all $i = 1, \dots, N$ we have that $\check{P}_i$ is injective, then $Q^{-1}_{\check{\mathcal{V}}}$ is single-valued, i.e., range $Q^{-1}_{\check{\mathcal{V}}} \subseteq \mathbb{R}^m$.

Having established conditions for F to have values in $\mathscr{C}(V)$, we would now like for F to be Lipschitz continuous. We show that if $Q_{\check{\mathcal{V}}}$ is piecewise linear, such a property indeed holds.

Proposition 2. *Given an internally acting N-mode affine c-CDM satisfying the hypotheses of Theorem 1, $\widetilde{\text{CDM}}_N$, if $Q_{\check{\mathcal{V}}}$ is a piecewise linear map, then $F = Q^{-1}_{\check{\mathcal{V}}}$ is Lipschitz with Lipschitz constant $L_F \leq \max\left\{1, \max_i \left\|\check{P}_i^\dagger\right\|\right\}$.*

Proof. It is clear that $F^{-1} = Q_{\check{\mathcal{V}}}$ is Lipschitz by piecewise linearity; its Lipschitz constant is $L_{F^{-1}} \leq \max\left\{1, \max_i \left\|\check{P}_i\right\|\right\}$, which follows by considering the constituent affine maps, as well as the identity map on the unaffected part of the domain. Then, the inverse of F^{-1}, i.e. F, will consist of the inverse of each of the constituent affine maps acting on convex domains by Lemma 2, giving $L_F \leq \max\left\{1, \max_i \left\|\check{P}_i^\dagger\right\|\right\}$. ☐

We can now proceed by constructing a unique Lipschitz continuous selector of F.

Theorem 3. *Let the assumptions of Proposition 2 hold. There exists a continuous mapping $s : \mathscr{C}(V) \to V$ defined as*

$$s(A) = m \int_{\partial\mathcal{B}_1} h_A(l) l \, \mathrm{d}\mu(l), \tag{16}$$

for $A \in \mathscr{C}(V)$, where $h_A(l) := \sup_{a \in A}\langle a, l\rangle$ is the support function of A, and μ is the Lebesgue measure. Furthermore, a unique Lipschitz selector of F, known as the Steiner selector, *is given by $s_{\text{v}} : V \to V$:*

$$s_{\text{v}}(u) := s \circ F(u), \tag{17}$$

which has Lipschitz constant

$$L_{s,\mathrm{v}} \leq 3m + 4\sqrt{m}L_F \leq 3m + 4\sqrt{m}\max\left\{1, \max_i \left\|\check{P}_i^{\dagger}\right\|\right\}. \tag{18}$$

We can now determine bounds for the error introduced by viabilizing maps in case of $u_{\mathrm{cmd}} \notin U_{\mathrm{v}}$ when projecting to the closest value in U_{v}.

B. Approximate Viabilizing Remapping of Affected Control Signals

We proceed by considering the remapping of commanded control signals $u_{\mathrm{cmd}} \in U$ to viable control signals $u_{\mathrm{v,cmd}} \in U_{\mathrm{v}}$ such that $u_{\mathrm{cmd}} = Q_{\check{\mathcal{U}}}^{-1} u_{\mathrm{v,cmd}}$. In particular, we are interested in the case where we only have access to $\tilde{F} = \tilde{Q}_{\check{\mathcal{U}}}^{-1}$. In light of this necessity, we consider the effect of remapping inviable control inputs in $U \setminus U_{\mathrm{v}}$ to their closest viable counterpart.

Proposition 3. *Consider (i) an N-mode affine c-CDM Q satisfying the hypotheses of Theorem 1, or (ii) a Lipschitz continuous CDM P with Lipschitz constant L_P satisfying the hypotheses of Theorem 2. Let $u_{\mathrm{cmd}} \notin \tilde{U}_{\mathrm{v}}$ and define viable set of control inputs $\tilde{U}_{\mathrm{v}} := Q\check{U}_{\mathrm{inner}} \cup (U \setminus \check{U}_{\mathrm{outer}})$. Then, for (i) the following projection error bound holds:*

$$\begin{aligned}
&\left\|u_{\mathrm{cmd}} - \mathrm{proj}_{\tilde{U}_{\mathrm{v}}} u_{\mathrm{cmd}}\right\| \leq d_{\mathrm{H}}\left(\check{U}_{\mathrm{outer}}, Q\check{U}_{\mathrm{inner}}\right) \\
&\leq \max_{\substack{j=1,\ldots,N \\ l \in \partial\mathcal{B}_1}} \|p_j + (\check{o}_j(l)P_j - \hat{o}(l))l\| \\
&\leq \max_i \|p_i\| + \|P_i - I\| \|\hat{o}_i\|_\infty + \|\hat{o}_i - \check{o}_i\|_\infty \\
&\leq \sigma_N + \rho_{N,N'}K + \delta_{N'}
\end{aligned} \tag{19}$$

where $\delta_{N'} = \max_i \|\hat{o}_i - \check{o}_i\|_\infty$, $\rho_{N,N'} = \max_i \|\hat{o}_i\|_\infty$, $\sigma_N = \max_i \|p_i\|$, and $K_N = \max_i \|P_i - I\|$. For (ii) the following project error bound holds:

$$\begin{aligned}
&\left\|u_{\mathrm{cmd}} - \mathrm{proj}_{\tilde{U}_{\mathrm{v}}} u_{\mathrm{cmd}}\right\| \leq d_{\mathrm{H}}\left(\check{U}_{\mathrm{outer}}, P\check{U}_{\mathrm{inner}}\right) \\
&\leq \varepsilon_N + \sigma_N + \rho_{N,N'}K_N + (1 + L_P)\delta_{N'}
\end{aligned} \tag{20}$$

where $\varepsilon_N = \max_{i,j} \varepsilon_{i,j}$, for $\varepsilon_{i,j} := \|P\mathbf{u}_i[j] - \check{P}_i\mathbf{u}_i[j]\|$, and $u[i,j] := \mathbf{u}_i[j]$, where $\mathbf{u}_i$ is an array composed of all control inputs in the i'th cluster.

Proposition 3 provides explicit bounds on the input error between the commanded input and the nearest viable input for N-mode affine c-CDMs and Lipschitz continuous CDMs. These errors bounds can be regarded as input disturbance specifications, given which robust controllers can be synthesized. We call a feedback controller *viable under CDM P* if it is robust to input disturbances produced by the error bounds given above.

We proceed by considering a particular case in which the commanded control signal reads $u = Kx$, for some fixed matrix $K \in \mathbb{R}^{m\times n}$. We show how the theory developed in this work reduces to a tractably implementable vialibizing Steiner selector in the case of fixed gain control laws. In particular, we wish to find a viabilizing transformation that takes CDM $Q = (p, P)$ and matrix K, and produces a viable gain matrix $\hat{K}$ and transformation $x' : X \to \mathbb{R}^n$, such that $Kx = Q_i\hat{K}x'(x)$ for all x.

V. Example

The following examples considers the viabilizibility of a fixed gain full-state control law. We assume the full-state is available with no error, and the CDM has been identified as an affine map $Qu = p + Pu$. This example covers a wide range of control laws, showing how viabilization can safeguard additional system properties such as robustness, without introducing additional components to the control design. The example given is a specialization of Theorem 3.

We state the following constructive theorem on producing a viable control law from a nominal fixed gain full-state control law $u = Kx$ and an affine CDM (p, P).

Theorem 4 (Viabilizing Transformation for Fixed Gain Control Laws)**.** *For a gain matrix $K \in \mathbb{R}^{m\times n}$ and an affine CDM $Qu = p+Pu$ parameterized by $p \in \mathbb{R}^m$ and $P \in \mathbb{R}^{m\times m}$, where P has* full row rank*, if $p \in \mathrm{range}(K)$, the following controller incorporates a viabilizing transformation:*

$$u_{\mathrm{v}}(x) = \hat{K}(x - q), \tag{21}$$

such that $Qu_{\mathrm{v}}(x) = Kx$, where $q \in K^{-1}p$, and

$$\hat{K} := (PP^{\mathsf{T}})^{-1}P^{\mathsf{T}}K. \tag{22}$$

We will present further applications to medical and aerospace systems in a future publication.

VI. Conclusion

This work introduces the notion of *control authority degradation maps* (CDMs) for affine-in-control nonlinear systems, proving identifiability conditions for various CDMs and developing a real-time reconstruction method with error bounds. After obtaining an approximation to the CDM, *viabilizing maps* which map commanded control signals to viabilizing control inputs, which in turn approximate the command control signal after degradation. Conditions and an efficient method for obtaining Lipschitz continuous viabilizing maps are provided, and the methods are demonstrated to improve CDM reconstruction quality over time in a controlled partial differential equation application.

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