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Enhancing Fault Diagnosis Through Robust Model Reference Adaptive Control: A Set-Based Approach

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Abstract— **A method for designing inputs in active fault diagnosis combined with a robust model reference adaptive controller is proposed in this paper. The approach ensures that the output of a plant with unknown parameters converges to a reference model during the diagnosis process. The paper proves that Lyapunov stability can be guaranteed, given a condition on the parameters of the robust adaptive law. Additionally, a novel set-membership filter, employing constrained zonotopes, is introduced. This filter is computationally efficient and does not require outer approximations. The active fault diagnosis is accomplished by generating reference inputs online, allowing the output sets to separate for different plant models. A demonstration of the proposed method on a numerical example is provided in the final.**

I. INTRODUCTION

Fault diagnosis has been a prominent topic for the past three decades in the field of control, with various approaches being explored [1], [2], [3], [4]. Typically, in the model-based category, fault diagnosis methods are classified as passive or active. The former diagnoses faults by analyzing the disparities between available input-output data and system models or historical data [5], [6]. On the other hand, the latter involves designing inputs injected into the system to gather additional information about the faults, enabling their distinction and diagnosis [7]–[13]. The challenges in active fault diagnosis (AFD) revolve around ensuring reliability and robustness despite uncertainties, such as unmodeled dynamics, disturbances, and measurement noises. To address these challenges, many researchers utilize mathematical tools, such as probability distributions and convex sets, to describe uncertainties. These tools enable the development of rational input design methods [9], [10], [14], [15].

In the realm of active fault diagnosis (AFD), the literature often neglects the stability aspect of the designed inputs. In fact, the inputs are intentionally crafted to counteract convergence and stability. The fundamental principle guiding the design of AFD inputs is to differentiate the outputs' distributions from various fault and fault-free models, whether these distributions are defined by probability distribution functions or different convex sets. Due to the alignment between input-output data and the system model, the actual output should fall within a single distribution with the highest probability, serving as evidence to diagnose the fault model. In previous studies such as [10] and [15], the separation of outputs is transformed into constraints, minimizing a performance index related to the inputs. In some online AFD methods, researchers propose analytic metrics to gauge the degree of separation among different output distributions, such as Bhattacharyya distance [9], separation tendency [11], and dispersity [12]. Subsequently, the AFD task is formulated as an optimization problem aimed at maximizing these analytic metrics. In essence, all the methods mentioned rely on solving optimization problems. However, given the initial lack of knowledge about the actual system model, these optimization problems cannot impose specific stability or convergence constraints on the inputs in line with the real system model.

To solve the above concerns, we propose a stability-assured AFD method based on model reference adaptive control (MRAC) in this paper. MRAC enables the output of a plant, potentially having unknown parameters, to converge to the output of a reference model. The adaptive controller includes adaptive gains, auxiliary signals, and reference inputs. The auxiliary signals are obtainable at each time instance, and the convergence of output is determined by the adaptive law of the gains. To address bounded disturbances and measurement noises, we employ a robust adaptive law with σ-modification, preventing parameter drift [16], [17]. We establish that uniformly ultimate boundedness can be ensured under specific conditions on the parameters of the robust adaptive law. All uncertainties are characterized using constrained zonotopes, which are closed under Minkowski sums, linear transformations, and intersections [18]. Leveraging these properties, we introduce an exact set-membership filtering method. Compared to previous works [19], [20], our method is more computationally efficient and does not require outer approximations. AFD is accomplished by designing reference inputs to maximize the distance between different prediction output sets online. This optimization task is efficiently solved by evaluating objective function values at the vertices of feasible regions.

The reminder of this paper is organized as follows. Section II presents the problem statement and related models. The robust model reference adaptive controller is provided in Section III, and then the convergence of the error model is proved by a Lyapunov function. The input design method for online AFD is proposed based on set-membership filtering in Section IV. A numerical example is given in Section V. The conclusion is drawn in Section VI.

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Notations and preliminaries: $\mathbb{R}^{n\times m}$ is the set of all real-valued matrices with the dimension of $n \times m$, and specially, $\mathbb{R}^n := \mathbb{R}^{n\times 1}$. Zero matrix is represented by 0, and I is the identity matrix. The trace of a matrix $X \in \mathbb{R}^{n\times n}$ is represented by $\mathrm{tr}(X)$. The maximum and minimum eigenvalues of a real symmetric $X \in \mathbb{R}^{n\times n}$ are represented by $\lambda_{max}(X)$ and $\lambda_{min}(X)$, respectively. For a vector or matrix $X \in \mathbb{R}^{n\times m}$, the symbol $\|X\| = \sqrt{\mathrm{tr}(X^{\mathrm{T}}X)}$ represents its Frobenius norm. When X is a vector, $\|X\|$ is its Euclidean norm. Given two sets $X, Y \subset \mathbb{R}^n$, their Minkowski sum is defined as $X \oplus Y = \{x+y \mid \forall x \in X, \forall y \in Y\}$. A constrained zonotope is described as $X = \{c+R\xi \mid \|\xi\|_\infty \leq 1, H\xi = b\}$, and represented by $\langle c, R, H, b\rangle$ for brevity. When H and b are omitted, a constrained zonotope becomes a zonotope, represented by $\langle c, R\rangle$ [18].

II. PROBLEM STATEMENT

The following discrete-time linear time-invariant (LTI) system are considered,

$$x(t+1) = A_i x(t) + B_i u(t) + w(t), \tag{1a}$$

$$y(t) = C_i x(t) + Du(t) + v(t), \tag{1b}$$

with the time index $t \in \mathbb{N}$, state $x(t) \in \mathbb{R}^{n_x}$, output $y(t) \in \mathbb{R}^{n_y}$, input $u(t) \in \mathbb{R}^{n_u}$, unknown input vector $w(t) \in \mathbb{R}^{n_w}$ (e.g., disturbances, unmodeled dynamics), and measurement noise $v(t) \in \mathbb{R}^{n_v}$. The coefficient matrices of $w(t)$ and $v(t)$ are omitted for the simplicity of main results without loss of generality. There are a total of m models with the same structure but different parameter matrices A_i, B_i, and C_i indexed by $i \in \mathbb{I} = \{1, \ldots, m\}$. All the models have the same matrix D. The index $i = 1$ represents the nominal model and the rest are faulty. The unknown input $w(t)$ and measurement noise $v(t)$ lay within constrained zonotopes, i.e., $w(t) \in \langle c_w, R_w, H_w, b_w\rangle$, and $v(t) \in \langle c_v, R_v, H_v, b_v\rangle$.

The system switches from the nominal model to one of the $m-1$ faulty models if a fault occurs. In previous work of AFD, the basic principle is designing inputs to separate the outputs of different models away from each other, so that the fault can be identified by the location of the system outputs. The central idea of these methods is to enhance the faulty manifestation of different models. However, the stability and control performance cannot be considered in these AFD formulation. In this paper, an AFD method that ensures stability is proposed based on model reference adaptive control. A robust adaptive controller is given for the plant to track the output of a reference model, whether the plant is the nominal model or any faulty model.

The reference model is given as

$$x_m(t+1) = A_m x_m(t) + B_m r(t), \tag{2a}$$

$$y_m(t) = C_m x_m(t) + D_m r(t), \tag{2b}$$

where the state $x_m(t)$, output $y_m(t)$, and reference command input $r(t)$ have the same dimensions as $x(t)$, $y(t)$, and $u(t)$, respectively, and $r(t)$ is bounded in a convex polytope Λ. The parameter matrices A_m, B_m, C_m, and D_m are known, and A_m is Schur stable, i.e., its spectral radius less than 1.

Assumption 2.1: All the C_i, $i \in \mathbb{I}$ are invertible.

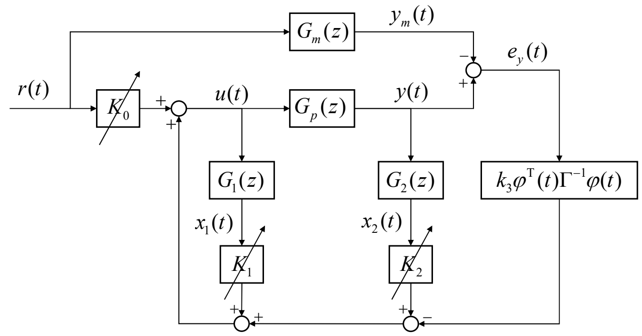

Fig. 1. Model reference adaptive controller [21].

The purpose of Assumption 2.1 is to simplify the set-membership filtering in Section IV. If Assumption 2.1 is not satisfied, we need to compute the intersections of constrained zonotopes and strips, which does not affect the contributions of this paper, but make the filtering more complicated.

Assumption 2.2: The transfer function matrix of the reference model $G_m(z) = C_m(zI - A_m)^{-1}B_m + D_m$ determined by the tuple (A_m, B_m, C_m, D_m) is strictly positive real (SPR).

III. ROBUST MODEL REFERENCE ADAPTIVE CONTROLLER

Represent the transfer matrix of the system (1) as $G_p(z)$ when $w(t)$ and $v(t)$ are zero. Then a classical model reference adaptive controller can be given as shown in the Fig. 1 [21]. The tracking error of outputs is

$$e_y(t) := y(t) - y_m(t). \tag{3}$$

Two auxiliary signals $x_1(t)$ and $x_2(t)$ are generated by

$$G_1(z):\ x_1(t+1) = A_\alpha x_1(t) + B_\alpha u(t), \tag{4a}$$

$$G_2(z):\ x_2(t+1) = A_\beta x_2(t) + B_\beta y(t), \tag{4b}$$

in which the parameters A_α, B_α, A_β, and B_β are given in advance, and for simplicity they can be chosen as $A_\alpha = A_\beta$ and $B_\alpha = B_\beta$, i.e., $G_1(z) = G_2(z)$. According to Fig. 1, the controller is designed as

$$\begin{aligned} u(t) =& K_0(t)r(t) + K_1(t)x_1(t) + K_2(t)x_2(t) \\ &- k_3\varphi(t)^{\mathrm{T}}\Gamma^{-1}\varphi(t)e_y(t), \end{aligned} \tag{5}$$

where $K_0(t)$, $K_1(t)$, and $K_2(t)$ are the adaptive gains, $\varphi(t) = [r^{\mathrm{T}}(t), x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$, and $\Gamma = \Gamma^{\mathrm{T}} > 0$. It has been proved that constant gains $K_0^\star$, $K_1^\star$, and $K_2^\star$ exist that when $K_0(t) \equiv K_0^\star$, $K_1(t) \equiv K_1^\star$ and $K_1(t) \equiv K_1^\star$, the transfer function matrix of the plant together with the controller $u(t)$ will match that of the reference model, i.e., $G_m(z)r(z) = G_p(z)u(z)$, and

$$e_y(t) = y(t) - y_m(t) \to 0, \text{ as } k \to \infty. \tag{6}$$

Represent the adaptive parameter as

$$\theta(t) = [K_0(t), K_1(t), K_2(t)]^{\mathrm{T}}. \tag{7}$$

If $\theta(t)$ is updated according to the adaptive law

$$\theta(t+1) = \theta(t) - \Gamma^{-1}\varphi(t)e_y^{\mathrm{T}}(t), \tag{8}$$

the parameter $\theta^{\mathrm{T}}(t)$ will converge to $\theta_\star^{\mathrm{T}} = [K_0^\star, K_1^\star, K_2^\star]$, and the tracking error $e_y(t) \to 0$.

However, when $w(t)$ and $v(t)$ appear, the adaptive law (8) is not suitable anymore, since it will cause the parameter drift issue. The σ-modification method proposed by [16] introduces a constant damping into the adaptive law to improve the robustness. It is widely used in adaptive control due to its simplicity. The modified adaptive law is given as

$$\theta(t+1) = \theta(t) - \Gamma^{-1}[\varphi(t)e_y^{\mathrm{T}}(t) + \sigma\theta(t)], \sigma > 0. \tag{9}$$

The robust adaptive controller is still given as the form of (5), which is rewritten as

$$u(t) = \theta^{\mathrm{T}}(t)\varphi(t) - k_3\varphi^{\mathrm{T}}(t)\Gamma^{-1}\varphi(t)e_y(t). \tag{10}$$

The stability and convergence of the robust adaptive controller are analyzed according to a Lyapunov function in the following.

A. Error Model

The parameter error is defined as

$$\tilde{\theta}(t) = \theta(t) - \theta_\star, \tag{11}$$

in which $\theta_\star$ is the matching parameter defined as before. Then the robust adaptive controller can be represented by

$$u(t) = \theta_\star^{\mathrm{T}}\varphi(t) + \tilde{\theta}^{\mathrm{T}}(t)\varphi(t) - \varepsilon(t), \tag{12}$$

where $\varepsilon(t) = k_3\varphi^{\mathrm{T}}(t)\Gamma^{-1}\varphi(t)e_y(t)$ for brevity. Then the plant combined with the adaptive controller can be described by the difference equations as follows:

$$\begin{aligned}\bar{x}(t+1) =& \bar{A}\bar{x}(t) + \bar{B}K_0^\star r(t)\\ &+ \bar{B}(\tilde{\theta}^{\mathrm{T}}(t)\varphi(t) - \varepsilon(t)) + \omega(t),\end{aligned} \tag{13}$$

where $\bar{x}(t) = [x^{\mathrm{T}}(t), x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ is the augmented state vector, $\omega(t) = [w^{\mathrm{T}}(t), 0, v^{\mathrm{T}}(t)B_\beta^{\mathrm{T}}]^{\mathrm{T}}$ is the merged unknown input. The output of the plant can be obtained by

$$\begin{aligned}y(t) =& \bar{C}\bar{x}(t) + \bar{D}K_0^\star r(t)\\ &+ \bar{D}(\tilde{\theta}^{\mathrm{T}}(t)\varphi(t) - \varepsilon(t)) + v(t).\end{aligned} \tag{14}$$

The details of parameter matrices $\bar{A}$, $\bar{B}$, and $\bar{C}$ can be derived from the control structure shown in Fig. 1, and $\bar{D} = D$.

When $w(t)$, $v(t)$, and $\tilde{\theta}(t)$ are zero, the plant together with the adaptive controller is equivalent to the reference model $G_m(z)$ exactly. In this situation, the plant's output is equal to the reference output $y_m(t)$, and then the difference equations (13) and (14) of the plant become

$$\bar{x}_m(t+1) = \bar{A}\bar{x}(t) + \bar{B}K_0^\star r(t), \tag{15a}$$

$$y_m(t) = \bar{C}\bar{x}_m(t) + \bar{D}K_0^\star r(t), \tag{15b}$$

where $\bar{x}_m$ represents the augmented state in the matching situation. The error model is obtained as the following by subtracting the ideally matched plant (15a) and (15b) from the actual plant (13) and (14),

$$e(t+1) = \bar{A}e(t) + \bar{B}(\tilde{\theta}^{\mathrm{T}}(t)\varphi(t) - \varepsilon(t)) + \omega(t), \tag{16a}$$

$$e_y(t) = \bar{C}e(t) + \bar{D}(\tilde{\theta}^{\mathrm{T}}(t)\varphi(t) - \varepsilon(t)) + v(t), \tag{16b}$$

in which $e(t) = \bar{x}(t) - \bar{x}_m(t)$.

B. Lyapunov Stability

Lemma 3.1: (Positive Real Lemma [22]) If the transfer function matrix $G(z) = C(zI - A)^{-1}B + D$ is SPR, there exist matrices $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, L and W such that

$$A^{\mathrm{T}}PA - P = -LL^{\mathrm{T}} - Q, \tag{17a}$$

$$A^{\mathrm{T}}PB = C^{\mathrm{T}} - LW, \tag{17b}$$

$$W^{\mathrm{T}}W = D + D^{\mathrm{T}} - B^{\mathrm{T}}PB. \tag{17c}$$

Because the reference model $G_m(z)$ is SPR, and transfer function matrix of (15) is equal to $G_m(z)$, based on the Lemma 3.1, we can find the corresponding matrices P, Q, L and W for the tuple $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ such that (17) are satisfied.

For the error model (16), define the Lyapunov function as

$$V(e(t), \tilde{\theta}(t)) = e^{\mathrm{T}}(t)Pe(t) + \mathrm{tr}(\tilde{\theta}^{\mathrm{T}}(t)\Gamma\tilde{\theta}(t)). \tag{18}$$

The Lyapunov difference is obtained by

$$\Delta V_t = V(e(t+1), \tilde{\theta}(t+1)) - V(e(t), \tilde{\theta}(t)). \tag{19}$$

Let $\Phi(t) = \tilde{\theta}^{\mathrm{T}}(t)\varphi(t) - \varepsilon(t)$. Based on (16a) and Lemma 3.1, we have

$$\begin{aligned}
&e^{\mathrm{T}}(t+1)Pe(t+1) - e^{\mathrm{T}}(t)Pe(t)\\
=& e^{\mathrm{T}}(t)(\bar{A}^{\mathrm{T}}P\bar{A} - P)e(t) + 2e^{\mathrm{T}}(t)\bar{A}^{\mathrm{T}}P\bar{B}\Phi(t)\\
&+ \Phi^{\mathrm{T}}(t)\bar{B}^{\mathrm{T}}P\bar{B}\Phi(t)\\
&+ 2\big(\bar{A}e(t) + \bar{B}\Phi(t)\big)^{\mathrm{T}}P\omega(t) + \omega^{\mathrm{T}}(t)P\omega(t)\\
=& -e^{\mathrm{T}}(t)Qe(t) - e^{\mathrm{T}}(t)LL^{\mathrm{T}}e(t)\\
&+ 2e^{\mathrm{T}}(t)(\bar{C}^{\mathrm{T}} - LW)\Phi(t)\\
&+ \Phi^{\mathrm{T}}(t)(\bar{D} + \bar{D}^{\mathrm{T}} - W^{\mathrm{T}}W)\Phi(t)\\
&+ 2\big(\bar{A}e(t) + \bar{B}\Phi(t)\big)^{\mathrm{T}}P\omega(t) + \omega^{\mathrm{T}}(t)P\omega(t)\\
=& -e^{\mathrm{T}}(t)Qe(t) - \|L^{\mathrm{T}}e(t) + W\Phi(t)\|_2^2\\
&+ e^{\mathrm{T}}(t)\bar{C}^{\mathrm{T}}\Phi(t) + \Phi^{\mathrm{T}}(t)\bar{D}^{\mathrm{T}}\Phi(t)\\
&+ \Phi^{\mathrm{T}}(t)\bar{C}e(t) + \Phi^{\mathrm{T}}(t)\bar{D}\Phi(t)\\
&+ 2\big(\bar{A}e(t) + \bar{B}\Phi(t)\big)^{\mathrm{T}}P\omega(t) + \omega^{\mathrm{T}}(t)P\omega(t)\\
=& -e^{\mathrm{T}}(t)Qe(t) - \|L^{\mathrm{T}}e(t) + W\Phi(t)\|_2^2\\
&+ 2\big(\bar{C}e(t) + \bar{D}\Phi(t)\big)^{\mathrm{T}}\Phi(t)\\
&+ 2\big(\bar{A}e(t) + \bar{B}\Phi(t)\big)^{\mathrm{T}}P\omega(t) + \omega^{\mathrm{T}}(t)P\omega(t)\\
=& -e^{\mathrm{T}}(t)Qe(t) - \|L^{\mathrm{T}}e(t) + W\Phi(t)\|_2^2\\
&+ 2\big(e_y(t) - v(t)\big)^{\mathrm{T}}\Phi(t)\\
&+ 2\big(\bar{A}e(t) + \bar{B}\Phi(t)\big)^{\mathrm{T}}P\omega(t) + \omega^{\mathrm{T}}(t)P\omega(t)\\
=& -e^{\mathrm{T}}(t)Qe(t) - \|L^{\mathrm{T}}e(t) + W\Phi(t)\|_2^2\\
&+ 2e_y^{\mathrm{T}}(t)\tilde{\theta}^{\mathrm{T}}(t)\varphi(t) - 2e_y^{\mathrm{T}}(t)\varepsilon(t) - 2v^{\mathrm{T}}(t)\Phi(t)\\
&+ 2\big(\bar{A}e(t) + \bar{B}\Phi(t)\big)^{\mathrm{T}}P\omega(t) + \omega^{\mathrm{T}}(t)P\omega(t).
\end{aligned} \tag{20}$$

According to the adaptive law with σ-modification (9), $\tilde{\theta}(t)$ is updated according to

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - \Gamma^{-1}[\varphi(t)e_y^{\mathrm{T}}(t) + \sigma\theta(t)]. \tag{21}$$

Therefore

$$\begin{aligned}
&\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}(t+1)\Gamma\tilde{\theta}(t+1)) - \mathrm{tr}(\tilde{\theta}^{\mathrm{T}}(t)\Gamma\tilde{\theta}(t)) \\
=&\mathrm{tr}\big(-e_y(t)\varphi^{\mathrm{T}}(t)\tilde{\theta}(t) - \tilde{\theta}^{\mathrm{T}}(t)\varphi(t)e_y^{\mathrm{T}}(t)\big) \\
&+ \mathrm{tr}(e_y(t)\varphi^{\mathrm{T}}(t)\Gamma^{-1}\varphi(t)e_y^{\mathrm{T}}(t)) \\
&+ \sigma^2\mathrm{tr}(\theta^{\mathrm{T}}(t)\Gamma^{-1}\theta(t)) - 2\sigma\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}(t)\theta(t)) \\
&+ 2\sigma\mathrm{tr}(e_y(t)\varphi^{\mathrm{T}}(t)\Gamma^{-1}\theta(t)) \\
=& -2e_y^{\mathrm{T}}(t)\tilde{\theta}^{\mathrm{T}}(t)\varphi(t) + \frac{1}{k_3}e_y^{\mathrm{T}}(t)\varepsilon(t) \\
&+ \sigma^2\mathrm{tr}(\theta^{\mathrm{T}}(t)\Gamma^{-1}\theta(t)) - 2\sigma\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}(t)\theta(t)) \\
&+ 2\sigma\mathrm{tr}(e_y(t)\varphi^{\mathrm{T}}(t)\Gamma^{-1}\theta(t)).
\end{aligned} \tag{22}$$

Combining equations (20) and (22), then

$$\begin{aligned}
\Delta V_t =& -e^{\mathrm{T}}(t)Qe(t) - \|L^{\mathrm{T}}e(t) + W\Phi(t)\|_2^2 \\
&- \frac{2k_3-1}{k_3}e_y^{\mathrm{T}}(t)\varepsilon(t) - 2v^{\mathrm{T}}(t)\Phi(t) \\
&+ 2\big(\bar{A}e(t) + \bar{B}\Phi(t)\big)^{\mathrm{T}}P\omega(t) \\
&+ \omega^{\mathrm{T}}(t)P\omega(t) + \sigma^2\mathrm{tr}(\theta^{\mathrm{T}}(t)\Gamma^{-1}\theta(t)) \\
&- 2\sigma\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}(t)\theta(t)) \\
&+ 2\sigma\mathrm{tr}(e_y(t)\varphi^{\mathrm{T}}(t)\Gamma^{-1}\theta(t)).
\end{aligned} \tag{23}$$

Note that

$$\begin{aligned}
e_y^{\mathrm{T}}(t)\varepsilon(t) =& k_3 e_y^{\mathrm{T}}(t)\big(\varphi^{\mathrm{T}}(t)\Gamma^{-1}\varphi(t)e_y(t)\big) \\
=& k_3\mathrm{tr}\big(e_y(t)\varphi^{\mathrm{T}}(t)\Gamma^{-1}\varphi(t)e_y^{\mathrm{T}}(t)\big).
\end{aligned} \tag{24}$$

Since $Q > 0$, there exists $a_0 \in (0,1)$, such that $Q - \frac{1-a_0^2}{a_0^2}LL^{\mathrm{T}} > 0$. Substitute (24) into the Lyapunov difference (23), then

$$\begin{aligned}
\Delta V_t =& -\|\frac{1}{a_0}L^{\mathrm{T}}e(t) + a_0W\Phi(t)\|^2 \\
&- e^{\mathrm{T}}(t)(Q - \frac{1-a_0^2}{a_0^2}LL^{\mathrm{T}})e(t) \\
&- (1-a_0^2)\Phi^{\mathrm{T}}(t)W^{\mathrm{T}}W\Phi(t) \\
&+ 2e^{\mathrm{T}}(t)\bar{A}^{\mathrm{T}}P\omega(t) + 2\Phi^{\mathrm{T}}(t)\bar{B}^{\mathrm{T}}P\omega(t) \\
&- 2v^{\mathrm{T}}(t)\Phi(t) + \omega^{\mathrm{T}}(t)P\omega(t) \\
&- (2k_3-2)\mathrm{tr}\big(e_y(t)\varphi^{\mathrm{T}}(t)\Gamma^{-1}\varphi(t)e_y^{\mathrm{T}}(t)\big) \\
&+ 2\sigma^2\mathrm{tr}(\theta^{\mathrm{T}}(t)\Gamma^{-1}\theta(t)) - 2\sigma\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}(t)\theta(t)) \\
&- \mathrm{tr}\big((e_y(t)\varphi^{\mathrm{T}}(t) - \sigma\theta^{\mathrm{T}}(t))\Gamma^{-1} \\
&\quad (e_y(t)\varphi^{\mathrm{T}}(t) - \sigma\theta^{\mathrm{T}}(t))^{\mathrm{T}}\big).
\end{aligned} \tag{25}$$

Note that the first and last terms of the equation (25) are negative. Select $k_3 > 1$, then the sixth term is also negative. Use $\theta(t) = \tilde{\theta}(t) + \theta_\star$ in the rest terms, then

$$\begin{aligned}
\Delta V_t \leq& -e^{\mathrm{T}}(t)(Q - \frac{1-a_0^2}{a_0^2}LL^{\mathrm{T}})e(t) \\
&- (1-a_0^2)\Phi^{\mathrm{T}}(t)W^{\mathrm{T}}W\Phi(t) \\
&+ 2e^{\mathrm{T}}(t)\bar{A}^{\mathrm{T}}P\omega(t) + 2\Phi^{\mathrm{T}}(t)\bar{B}^{\mathrm{T}}P\omega(t) \\
&- 2v^{\mathrm{T}}(t)\Phi(t) + \omega^{\mathrm{T}}(t)P\omega(t) \\
&+ 2\sigma^2\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}(t)\Gamma^{-1}\tilde{\theta}(t)) + 2\sigma^2\mathrm{tr}(\theta_\star^{\mathrm{T}}\Gamma^{-1}\theta_\star) \\
&+ 4\sigma^2\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}(t)\Gamma^{-1}\theta_\star) \\
&- 2\sigma\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}(t)\tilde{\theta}(t)) - 2\sigma\mathrm{tr}(\tilde{\theta}^{\mathrm{T}}(t)\theta_\star).
\end{aligned} \tag{26}$$

Let $a_1 = \lambda_{min}(Q - \frac{1-a_0^2}{a_0^2}LL^{\mathrm{T}}) > 0$, $a_2 = (1-a_0^2)\lambda_{min}(W^{\mathrm{T}}W) > 0$, $a_3 = \|\bar{A}^{\mathrm{T}}P\|$, $a_4 = \|\bar{B}^{\mathrm{T}}P\|$, $a_5 = \lambda_{max}(P)$, $a_6 = \|\Gamma^{-1}\|$, $a_7 = \mathrm{tr}(\theta_\star^{\mathrm{T}}\Gamma^{-1}\theta_\star)$, $a_8 = \|\Gamma^{-1}\theta_\star\|$, and $a_9 = \|\theta_\star\|$. Since $\omega(t)$ and $v(t)$ are all bounded, let $\omega_0 = \max\|\omega(t)\|$, $v_0 = \max\|v(t)\|$. Based on some inequalities, such as $\|AB\| \leq \|A\|\|B\|$ and $\mathrm{tr}(AB) \leq \big(\mathrm{tr}(AA^{\mathrm{T}})\big)^{\frac{1}{2}}\big(\mathrm{tr}(BB^{\mathrm{T}})\big)^{\frac{1}{2}} = \|A\|\|B\|$, it can be obtained that

$$\begin{aligned}
\Delta V_t \leq& -a_1\|e(t)\|^2 - a_2\|\Phi(t)\|^2 \\
&+ 2a_3\omega_0\|e(t)\| + (2a_4\omega_0 + 2v_0)\|\Phi(t)\| \\
&+ a_5\omega_0^2 + 2\sigma^2a_6\|\tilde{\theta}(t)\|^2 + 2\sigma^2a_7 \\
&+ 4\sigma^2a_8\|\tilde{\theta}(t)\| - 2\sigma\|\tilde{\theta}(t)\|^2 + 2\sigma a_9\|\tilde{\theta}(t)\|.
\end{aligned} \tag{27}$$

Therefore,

$$\begin{aligned}
\Delta V_t \leq& -a_1\left(\|e(t)\| - \frac{a_3\omega_0}{a_1}\right)^2 \\
&- a_2\left(\|\Phi(t)\| - \frac{a_4\omega_0 + v_0}{a_2}\right)^2 \\
&- (2\sigma - 2\sigma^2a_6)\left(\|\tilde{\theta}(t)\| - \frac{2\sigma a_8 + a_9}{2-2\sigma a_6}\right)^2 \\
&+ \frac{a_3^2\omega_0^2}{a_1} + \frac{(a_4\omega_0 + v_0)^2}{a_2} + \frac{\sigma(2\sigma a_8 + a_9)^2}{2-2\sigma a_6} \\
&+ a_5\omega_0^2 + 2\sigma^2a_7.
\end{aligned} \tag{28}$$

In the adaptive law (9), the parameters σ and Γ are optional. If we select them such that $\sigma\|\Gamma^{-1}\| < 1$, i.e., $2\sigma - 2\sigma^2a_6 > 0$, then from inequality (28) it can be obtained that $\Delta V_t \leq 0$ outside a compact set $\mathscr{S}$:

$$\begin{aligned}
\mathscr{S} = \Bigg\{&\left(\|e(t)\|, \|\Phi(t)\|, \|\tilde{\theta}(t)\|\right) : a_1\left(\|e(t)\| - \frac{a_3\omega_0}{a_1}\right)^2 \\
&+ a_2\left(\|\Phi(t)\| - \frac{a_4\omega_0 + v_0}{a_2}\right)^2 \\
&+(2\sigma - 2\sigma^2a_6)\left(\|\tilde{\theta}(t)\| - \frac{2\sigma a_8 + a_9}{2-2\sigma a_6}\right)^2 \leq T\Bigg\},
\end{aligned} \tag{29}$$

in which $T > 0$ is equal to the sum of the constant terms of the inequality (28). Therefore, uniformly ultimate boundedness is achieved.

IV. Framework of Online AFD

It is noticeable that the input $u(t)$ of adaptive controller (5) depends on the output error $e_y(t) = y(t) - y_m(t)$, while the output $y(t)$ depends on $u(t)$ at the same time. This formulation of the adaptive controller (5) is convenient to analyze the Lyapunov stability. Its physical realization is given in Fig. 2, where $G'(z) = G_m(z) - D_m$, and $K_3' = D_m^{-1}D$. The structure of Fig. 2 was proposed early in [21], [23].

The subsystem $G'(z)$ is given as

$$x'(t+1) = A_m x'(t) + B_m u'(t), \tag{30a}$$

$$y'(t) = C_m x'(t). \tag{30b}$$

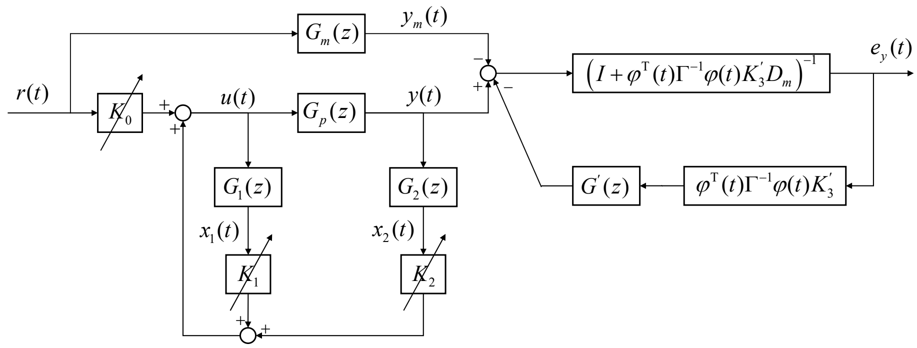

Fig. 2. Physical realization of model reference adaptive controller [21], [23].

In the whole system of Fig. 2, only the state $x_p(t)$ of plant $G_p(z)$ cannot be obtained directly, the output $y_p(t)$ and the rest states and outputs of $G_1(z)$, $G_2(z)$, $G'(z)$, and $G_m(z)$ are known exactly. The initial state $x_p(0)$ is assumed lying within a constrained zonotope, i.e., $x_p(0) \in X_0 = \langle c_0, H_0, A_0, b_0\rangle$. For any possible plant model i, the state set $X^i(t)$ can be determined iteratively by three steps: prediction, filtering, and re-prediction.

A. Prediction Step

Given the input $u(t)$ and prediction state set $\hat{X}_i(t)$, the system equation of the plant can be written in a set version as

$$\hat{\hat{X}}_i(t+1) = A_i\hat{X}_i(t) \oplus B_i u(t) \oplus W, \tag{31a}$$

$$\hat{Y}_i(t) = C_i\hat{X}^i(t) \oplus Du(t) \oplus V, \tag{31b}$$

where $\hat{\hat{X}}_i(t+1)$ is the pre-prediction set of state $x_i(t+1)$, and $\hat{Y}_i(t)$ is the prediction set of $y_i(t)$.

Moreover, given the input $u(t+1)$, we have

$$\hat{\hat{Y}}_i(t+1) = C_i\hat{\hat{X}}_i(t+1) \oplus Du(t+1) \oplus V, \tag{32}$$

where $\hat{\hat{Y}}_i(t+1)$ is the pre-prediction set of output $y_i(t+1)$.

B. Filtering Step

After injecting the input $u(t)$, the plant's output $y(t)$ can be obtained. Then we can define the consistent state set of $x_i(t)$ as

$$X_i^y(t) = \{x \in \mathbb{R}^{n_x} \mid y(t) - C_i x - Du(t) \in V\}. \tag{33}$$

Since C_i is invertible, based on the set operation of constrained zonotope, $X_i^y(t) = \langle c_i^f, R_i^f, H_v, b_v\rangle$, where

$$c_i^f = C_i^{-1}(y(t) - Du(t) - c_v), R_i^f = -C_i^{-1}R_v. \tag{34}$$

Then the exact state set $X_i(t)$ can be constructed by

$$X_i(t) = \hat{X}_i(t) \cap X_i^y(t), \tag{35}$$

which means that the prediction state set $\hat{X}_i(t)$ is filtered by the real output $y(t)$.

C. Re-prediction Step

After getting the exact state set $X_i(t)$ by (35), the re-prediction step is given by

$$\hat{X}_i(t+1) = A_i X_i(t) \oplus B_i u(t) \oplus W, \tag{36a}$$

$$Y_i(t) = C_i X_i(t) \oplus Du(t) \oplus V, \tag{36b}$$

where $\hat{X}_i(t+1)$ is the prediction state set for the next step, and $Y_i(t)$ is the filtered output set of $y_i(t)$.

D. Designing Reference Inputs for AFD

The principle of AFD is based on the consistency between model and input-output data. If the plant is in model i, then the real output $y(t)$ should always be contained inside the expected output set $Y_i(t)$. Therefore, the unmatched plant model can be excluded by checking whether

$$y(t) \notin Y_i(t). \tag{37}$$

Obviously, it will be easier to exclude the unmatched plant models, if the expected output sets $Y_i(t), \forall i \in \mathbb{I}$, can be separated from each other as much as possible. Based on (35),

$$X_i(t) \subseteq \hat{X}_i(t). \tag{38}$$

Then, according to (31a) and (36a),

$$\hat{X}_i(t) \subseteq \hat{\hat{X}}_i(t). \tag{39}$$

Thus,

$$Y_i(t) \subseteq \hat{Y}_i(t) \subseteq \hat{\hat{Y}}_i(t). \tag{40}$$

Therefore, the separation of $\hat{\hat{Y}}_i(t), \forall i \in \mathbb{I}$, can also lead to the separation of $Y_i(t), \forall i \in \mathbb{I}$.

In this paper, the online AFD is achieved by separating pre-prediction output sets $\hat{\hat{Y}}_i(t+1), \forall i \in \mathbb{I}$, asymptotically. At time index t, the prediction state sets are given by

$$\hat{X}_i(t) = \langle \hat{c}_i, \hat{R}_i, \hat{H}_i, \hat{b}_i\rangle, \forall i \in \mathbb{I}. \tag{41}$$

Based on (31a), the pre-prediction state sets can be represented as $\hat{\hat{X}}_i(t+1) = \langle \hat{\hat{c}}_i, \hat{\hat{R}}_i, \hat{\hat{H}}_i, \hat{\hat{b}}_i\rangle, \forall i \in \mathbb{I}$, where

$$\hat{\hat{c}}_i = A_i\hat{c}_i + B_i u(t) + c_w, \hat{\hat{R}}_i = [A_i\hat{R}_i \ R_w], \tag{42a}$$

$$\hat{\hat{H}}_i = \begin{bmatrix} \hat{H}_i & 0 \\ 0 & H_w \end{bmatrix}, \hat{\hat{b}}_i = \begin{bmatrix} \hat{b}_i \\ b_w \end{bmatrix}. \tag{42b}$$

Then based on (32), the pre-prediction output sets at $t+1$ can be obtained by $\hat{\hat{Y}}_i(t+1) = \langle \hat{\hat{c}}_i^y, \hat{\hat{R}}_i^y, \hat{\hat{H}}_i^y, \hat{\hat{b}}_i^y \rangle$, where

$$\hat{\hat{c}}_i^y = C_i \hat{\hat{c}}_i + Du(t+1) + c_v, \hat{\hat{R}}_i^y = [C_i \hat{\hat{R}}_i \; R_v], \tag{43a}$$

$$\hat{\hat{H}}_i^y = \begin{bmatrix} \hat{\hat{H}}_i & 0 \\ 0 & H_v \end{bmatrix}, \hat{\hat{b}}_i^y = \begin{bmatrix} \hat{\hat{b}}^i \\ b_v \end{bmatrix}. \tag{43b}$$

In fact, separating the output sets which are described by some convex set representations is a tricky task. The proposed method attempts to provide an efficient solution. For the constrained zontopes $\hat{\hat{Y}}_i(t+1) = \langle \hat{\hat{c}}_i^y, \hat{\hat{R}}_i^y, \hat{\hat{H}}_i^y, \hat{\hat{b}}_i^y \rangle$, it is obviously that $\hat{\hat{Y}}_i(t+1) \subseteq \tilde{Y}_i(t+1) = \langle \hat{\hat{c}}_i^y, \hat{\hat{R}}_i^y \rangle$, which means that the zonotope $\tilde{Y}_i(t+1)$ is an outer approximation of $\hat{\hat{Y}}_i(t+1)$. Thus, separating $\tilde{Y}_i(t+1)$ asymptotically can also lead to the separation of $\hat{\hat{Y}}_i(t+1)$. It is very direct to separate some zonotopes by maximizing the distance between their centers.

For two possible models $i, j \in \mathbb{I}$, the distance between the centers of $\hat{\hat{Y}}_i(t+1)$ and $\hat{\hat{Y}}_j(t+1)$ can be measured as

$$\|\hat{\hat{c}}_i^y - \hat{\hat{c}}_j^y\|_2^2 = \|Q_{ij}u(t) + J_{ij}\|_2^2, \tag{44}$$

where $Q_{ij} = C_i B_i - C_j B_j$, and $J_{ij} = C_i A_i \hat{c}_i + C_i c_w - C_j A_j \hat{c}_j - C_j c_w$. Since the input $u(t)$ is generated by the adaptive controller with an external reference input $r(t)$,

$$u(t) = K_0(t)r(t) + K_1(t)x_1(t) + K_2(t)x_2(t). \tag{45}$$

Therefore, substituting (45) into (44), then the reference input $r(t)$ can be designed by

$$\max_{r(t)\in\Lambda} \sum_{i,j\in\mathbb{I}, i\neq j} \|Q'_{ij}(t)r(t) + J'_{ij}(t)\|_2^2. \tag{46}$$

The expressions of the parameters $Q'_{ij}(t)$ and $J'_{ij}(t)$ are omitted. Since the objective of (46) is a convex quadratic function and the feasible region $r(t) \in \Lambda$ is a convex polytope, the optimal solution of (46) must be the vertex of Λ. Therefore, the optimal $r^\star(t)$ can be determined by

$$r^\star(t) = \operatorname*{argmax}_{r(t)\in\mathcal{V}(\Lambda)} \sum_{i,j\in\mathbb{I}, i\neq j} \|Q'_{ij}(t)r(t) + J'_{ij}(t)\|_2^2, \tag{47}$$

where $\mathcal{V}(\Lambda)$ represents the set of the vertices of Λ. Since Λ has finite vertices, (47) can be solved very efficiently.

V. NUMERICAL EXAMPLE

The following example is provided to demonstrate the proposed method. The reference model is given as

$$A_m = \begin{bmatrix} 0.61 & -0.24 \\ 0.16 & 0.53 \end{bmatrix}, B_m = \begin{bmatrix} 1.00 & -0.26 \\ 0.33 & 0.74 \end{bmatrix},$$

$$C_m = \begin{bmatrix} 1.05 & 0.12 \\ -0.31 & 0.87 \end{bmatrix}, D_m = \begin{bmatrix} 0.84 & -0.20 \\ -0.41 & 0.96 \end{bmatrix}.$$

Three possible models of the plant are given as

$$A_1 = \begin{bmatrix} 0.61 & -0.12 \\ 0.08 & 0.53 \end{bmatrix}, B_2 = \begin{bmatrix} 0.81 & -0.26 \\ 0.33 & 0.74 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1.05 & 0.12 \\ -0.31 & 0.55 \end{bmatrix},$$

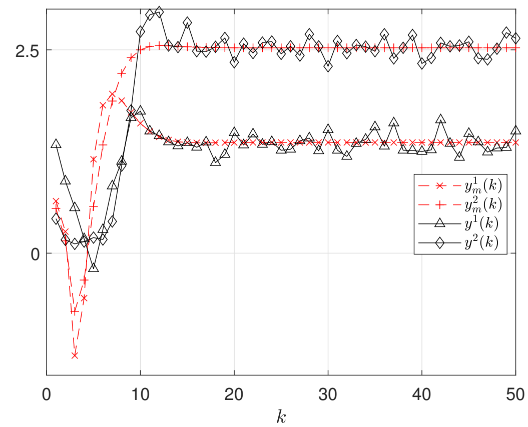

Fig. 3. Output curves of the plant and reference model.

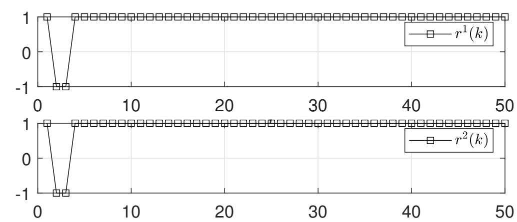

Fig. 4. The designed reference inputs.

where the rest parameter matrices of model $i = 1, 2, 3$, if not indicated, are the same as reference model. The auxiliary systems $G_1(z)$ and $G_2(z)$ are given as

$$A_\alpha = A_\beta = \begin{bmatrix} 0.59 & 0.13 \\ -0.11 & 0.72 \end{bmatrix}, B_\alpha = B_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The parameters of robust adaptive law are $\sigma = 0.01$, $\Gamma^{-1} = 0.1I$. The plant is in model $i = 1$. The initial states of plant is generated randomly by $x_0 \in \langle [1, 1]^{\mathrm{T}}, 0.3I \rangle$. The initial states of reference model, auxiliary systems, and subsystem $G'(z)$ are all set as 0. Both $w(t)$ and $v(t)$ are generated randomly all the time within a zonotope $\langle 0, 0.1I \rangle$. The reference input is bounded by $r(t) \in \langle 0, I \rangle$.

Fig. 3 presents the output of plant and reference model, denoted by $y(t)$ and $y_m(t)$, and the superscripts represent components. The designed reference inputs are shown in Fig. 4. When the model of plant is diagnosed, the reference inputs is kept unchanged. The process of filtering and fault diagnosis is presented in Fig. 5, where the filtered output set of model i is represented as $Y_i(t)$, and the point $y(t)$ is the real output of plant. It indicates that models $i = 3$ and $i = 2$ are excluded at time instance $t = 3$ and $t = 9$, respectively.

VI. CONCLUSIONS

A novel approach to active fault diagnosis is introduced, utilizing model reference adaptive control to ensure stability throughout the diagnostic process. The method employs a robust adaptive controller, guaranteeing stability in active fault diagnosis. Additionally, a set-membership filter employing constrained zonotopes is proposed. In future research, efforts will be directed towards enhancing the set separation metric for constrained zonotopes.

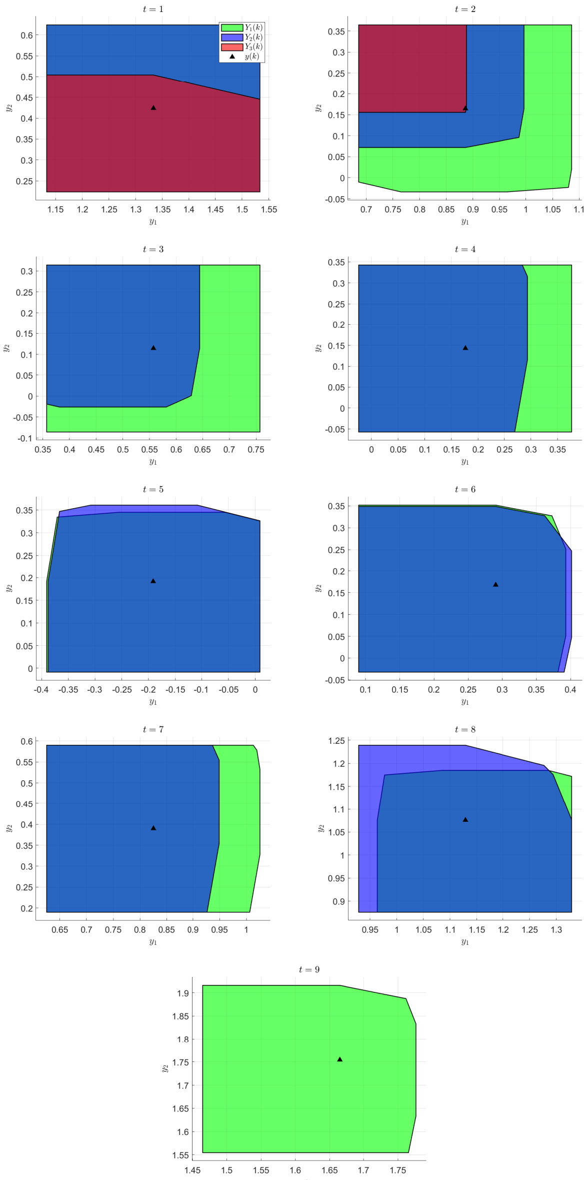

Fig. 5. The process of filtering and fault diagnosis (The sets are plotted with the help of MPT3 [24]).

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